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On Persistency of Excitaiton
[stability of adaptive systems]

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Outline

- **Introduction**
  - ⁎ Preliminaries: motivations, definitions
  - ⁎ Recall on (basic) adaptive control
- **Linear time-varying systems**
  - ⁎ Uniform persistency of excitation
  - ⁎ A result on convergence rates
  - ⁎ MRAC-type systems (SPR)
- **Nonlinear systems**
  - ⁎ Uniform δ-persistency of excitation
  - ⁎ Necessary and sufficient conds. for stability
- **General conclusions**

Introduction

[Preliminaries]

Theorem (KYP) Let $Z(s) = C^\top [sI - A]^{-1} B$ be a $p \times p$ transfer function s.t.:

- the pair (A, B) is completely controllable;
- the pair (A, C) is completely observable.

Then, $Z(\cdot)$ is strictly positive real if and only if there exists a positive definite matrix P such that

$$PA + A^\top P = -Q$$
$$PB = C^\top.$$

Theorem The matrix A is Hurwitz (its eigen-values have strictly negative real parts) if and only if for any $Q = Q^\top$, positive definite, there exists $P = P^\top > 0$ s.t.

$$PA + A^\top P = -Q$$

Definition 1 (Persistency of excitation)

A locally integrable function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m \times n}$ *is said to be persistently exciting if there exist* T *and* $\mu > 0$ *such that*

$$\int_t^{t+T} \Phi(s)\Phi(s)^\top ds \geq \mu \qquad \forall\, t \geq 0 \tag{1}$$

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Remarks

- Φ, in the definition, is a function of time, only
- Typically, $m \geq n$ hence, $\Phi(t)\Phi(t)^\top$ is rank deficient for each $t \geq 0$ however, (1) may still hold; it is a lowerbound on the "average" of $\Phi(t)\Phi(t)^\top$

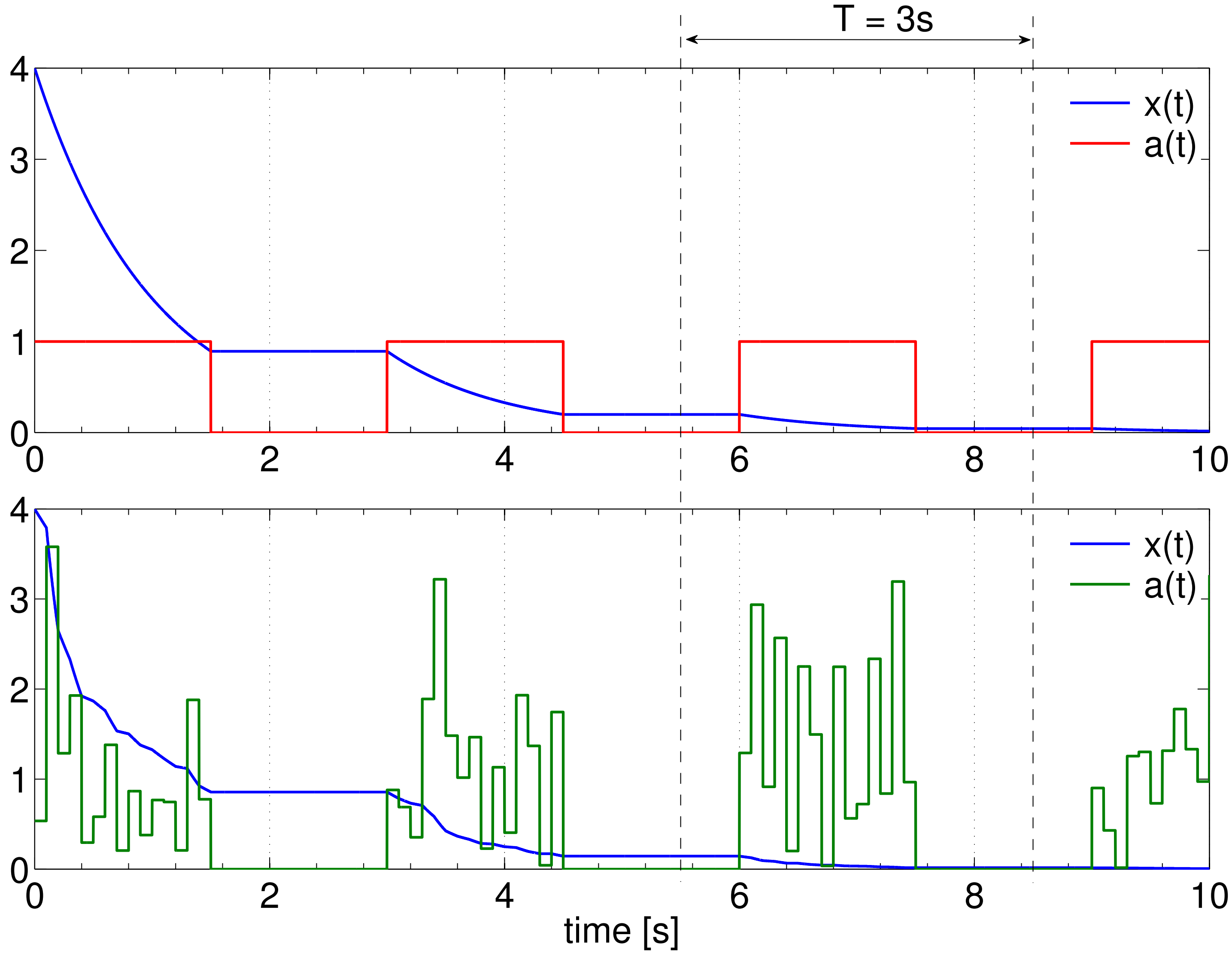
In dynamical systems:

e.g., $\dot{x} = Ax$ "is GES" if A is Hurwitz (full rank and $\lambda_{i\,\mathsf{R}}(A) < 0$)

$\dot{x} = -\Phi(t)\Phi(t)^\top x$ is still GES iff Φ is PE, even if $\lambda_{i\,\mathsf{R}}(-\Phi(t)\Phi(t)^\top) \not< 0$

Illustration of persistency of excitation

Consider the system $\dot{x} = -a(t)x$. Seemingly, $\exists\, \mu$, $\mu > 0$: $\displaystyle\int_t^{t+3} a(s)^2 ds \geq \mu$



[Preliminaries]

Fact. Consider the system

$$\dot{x} = -a(t)^2 x,$$

with $a(t)$, $\dot{a}(t)$ bounded.

The origin is globally exponentially stable iff there exist $\mu, T > 0$ such that

$$\int_t^{t+T} a(s)^2 ds \geq \mu \qquad \forall t \geq 0.$$

Gradient systems. Consider the system

$$\dot{x} = -\Phi(t)\Phi(t)^\top x, \quad \Phi(t) \in \mathbb{R}^{m \times n}, \quad m \geq n$$

with $\Phi(t)$, $\dot{\Phi}(t)$ bounded.

The origin is globally exponentially stable iff Φ is persistently exciting.

—see *e.g.*, [Anderson et al; Narendra & Annaswamy; Sastry & Bodson; ...]

Rmk. Convergence rates: [Sukumar *et al*; Loria & Panteley; Brocket ...]

Lemma 1 *(linear MRAC). Consider the linear time-varying (LTV) system*

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A & B\phi(t)^\top \\ -\phi(t)C & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix},$$

– $e \in \mathbb{R}^n$ *is the tracking error*

– $\tilde{\theta} \in \mathbb{R}^m$ *is the parameter estimation error*

– $\phi : \mathbb{R} \to \mathbb{R}^m$ *is the* regressor *function.*

Assume that:

- *the triple* (A, B, C) *is strictly positive real (satisfies the KYP lemma):*

$$V := z^\top P z > 0 \implies \dot{V} = -|e|^2 \leq 0;$$

- ϕ *is absolutely continuous;* ϕ *and* $\dot{\phi}$ *are bounded almost everywhere;*

Then, the origin is uniformly globally exponentially stable if and only if ϕ *is PE.*

Introduction

[Basics on adaptive control]

Consider the linear autonomous system

$$\begin{aligned} \dot{x} &= \mathsf{A}x + \mathsf{B}u \\ y &= \mathsf{C}x \end{aligned}$$

in canonical form.

- Let (A,B) be controllable and (A,C) be observable.
- Because (A,B) is controllable, we can perform pole placement:
  [there exists (a row vector) K such that $(\mathsf{A} - \mathsf{B}K)$ is Hurwitz]
- However, if there is uncertainty in A we cannot compute the appropriate K
- Let $u = -\hat{K}x$ where K is an estimate of (the ideal) K;
  let $\tilde{K} := \hat{K} - K$ then,

$$\begin{aligned} \dot{x} &= (\mathsf{A} - \mathsf{B}K)x - \mathsf{B}\tilde{K}x \\ y &= \mathsf{C}x \end{aligned}$$

Analysis.

- Let $A := \mathsf{A} - \mathsf{B}K$. By design, this matrix is Hurwitz
- Also, the pair (A, C) is controllable and $P\mathsf{B} = \mathsf{C}^\top$ therefore, let

$$V = \frac{1}{2}x^\top P x + \frac{1}{2\gamma}\tilde{K}\tilde{K}^\top$$

$$\Longrightarrow \qquad \dot{V} \;=\; -x^\top\big[A^\top P + PA\big]x - x^\top P\mathsf{B}x^\top\tilde{K}^\top + \frac{1}{\gamma}\dot{\tilde{K}}\tilde{K}^\top$$

- We use the (passivity-based) update law: $\dot{\tilde{K}} = \gamma x^\top \mathsf{C}^\top x^\top$

  Then: $\qquad \dot{V} = \; -x^\top Q x$

Claim. [after adaptive control texts]: $x \to 0$ and $\tilde{K}$ is bounded.

Proof: After ch. III-Lemma 1, if a once continuously differentiable function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfies

$$\varphi, \dot{\varphi} \in \mathcal{L}_\infty, \qquad \varphi \in \mathcal{L}_2.$$

Then, necessarily $\lim\limits_{t\to\infty} \varphi(t) = 0.$

Rmk. Does $\tilde{K} \to 0$?

[Basics on adaptive control]

Fact: Adaptive control systems are, in general, nonlinear time-varying

The closed-loop system has the (familiar) form

$$\begin{aligned} \dot{x} &= Ax + B(t)\tilde{\theta}, \\ \dot{\tilde{\theta}} &= -\gamma C(t) x, \end{aligned}$$

We have: $\tilde{\theta} \in \mathcal{L}_\infty$, $x \to 0$

$$\begin{aligned} B(t) &:= -\mathsf{B}x(t)^\top \in \mathbb{R}^{n\times n} \\ C(t) &:= -x(t)\mathsf{B}^\top P \in \mathbb{R}^{n\times n} \\ A &:= (\mathsf{A} - \mathsf{B}K) \\ \tilde{\theta} &= \tilde{K}^\top \end{aligned}$$

Rmk. The notations on the right are convenient, but, at best, ambiguous!

- For a start, the matrix $B(t)$ depends on state trajectories hence, on the initial conditions (uniformity ...)
- If we the goal is to stir $x(t) \to 0$, how to pretend to use persistency of excitation? ($x \equiv 0 \implies B \equiv 0$ convergence of $\tilde{\theta}$...)

Problem: How do we ensure (uniform) stability and convergence?

[Model Reference Adaptive Control]

- Consider now the tracking control problem, to stir $x \to x^*$, for a pair of systems:

Plant:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \Phi(x)^\top \theta + g(x) u \end{aligned}$$

Reference model:

$$\begin{aligned} \dot{x}_1^* &= x_2^* \\ &\vdots \\ \dot{x}_{n-1}^* &= x_n^* \\ \dot{x}_n^* &= f(x^*) \end{aligned}$$

- Let $u := g(x)^{-1}\left[f(x^*) - \Phi(x)^\top \hat{\theta} - K(\cdot)e\right]$ and $\dot{\tilde{\theta}} = \gamma \Phi(x) e_n$

Then, define the error $e := x - x^*$. Its dynamics corresponds to

$$\begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & \\ \vdots & & \ddots & \\ & & & 1 \\ -k_1 & & \cdots & -k_n \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \Phi(x)^\top \tilde{\theta}$$

$$\dot{\tilde{\theta}} = \gamma \Phi(x) [0 \ \cdots \ 1] e$$

Common mistake.

Such closed-loop system, is commonly written in the compact form:

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A & B\Phi^\top \\ -\Phi C & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix}, \quad z := \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix}$$

Then, global exponential stability is some times claimed invoking Lemma 1; converse theorems are used to establish statements on robust stability, ... !

Rmk. The function Φ depends on x and, since $x := e + x^*(t)$, the system dynamics is, actually, nonlinear:

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A & B\phi(t,z)^\top \\ -\phi(t,z)C & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix}, \quad \phi(t,z) := \Phi(e + x^*(t))$$

while the system in Lemma 1 is *linear*!!

Problem statement

How do we infer the (asymptotic) stability of the origin of

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A & B\phi(t,z)^\top \\ -\phi(t,z)C & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \qquad x := \begin{bmatrix} e \\ \theta \end{bmatrix}$$

with A Hurwitz, (A, B) controllable, and (A, C) observable?

What is more, how to guarantee the stability of the origin for

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A(\cdot) & B(\cdot) \\ C(\cdot) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where A, B and C are, generally speaking, functions of time and the states but have "certain structural properties" ?

Rmk. We do not want to assume that $B(\cdot)$ is full rank

[Model-Reference-Adaptive-Control]

- Consider the case-study:

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A & B\phi(t,z)^\top \\ -\phi(t,z)C & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \quad z := \begin{bmatrix} e \\ \theta \end{bmatrix}$$

and assume that we know P such that, defining,

$$V := e^\top P e + \frac{1}{2}|\theta|^2 > 0,$$

we obtain

$$\dot{V} = -|e|^2 \leq 0.$$

- Inspired by Lemma 1, can we conjecture that some **boundedness** conditions on $\phi(t,z)$ in addition to **persistency of excitation** should suffice for UGAS (UGES?).

Problem: What does PE mean for the **state-dependent** function $\phi(t,z)$?

- Some authors use:

$$\int_t^{t+T} \phi(\tau, z(\tau, t_\circ, z_\circ))\phi(\tau, z(\tau, t_\circ, z_\circ))^\top d\tau \geq \mu I \quad \forall t \geq t_\circ \ .$$

[Model-Reference-Adaptive-Control]

- The solutions are bounded (UGS). Hence, we (re)consider the system as parameterized linear time-varying:

$$\begin{bmatrix} \dot{\bar{e}} \\ \dot{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} A & B\phi(t, z(t, t_\circ, z_\circ))^\top \\ -\phi(t, z(t, t_\circ, z_\circ))C & 0 \end{bmatrix} \begin{bmatrix} \bar{e} \\ \bar{\theta} \end{bmatrix}$$

with i.c.: $(t_*, \bar{z}_*)$ $z(t)$ are solutions of the original NL system

Then, we observe the following:

- If we assume that $\phi(t, z(t, t_\circ, z_\circ))$ is persistently exciting, *i.e.*,

$$\int_t^{t+T} \phi(\tau, z(\tau, t_\circ, z_\circ))\phi(\tau, z(\tau, t_\circ, z_\circ))^\top d\tau \geq \mu\, I \quad \forall t \geq t_\circ \ .$$

  (and if it is also bounded with bounded derivative) then, the origin is globally exponentially stable uniformly in the initial conditions $(t_*, \bar{z}_*)$.

- **Iff** the initial conditions $(t_*, \bar{z}_*) = (t_\circ, z_\circ)$ then, $\bar{z}(t, t_*, \bar{z}_*) = z(t, t_\circ, z_\circ)$,

[Model-Reference-Adaptive-Control]

- The solutions are bounded (UGS). Hence, we (re)consider the system as parameterized linear time-varying:

$$\begin{bmatrix} \dot{\bar{e}} \\ \dot{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} A & B\phi(t, z(t, t_\circ, z_\circ))^\top \\ -\phi(t, z(t, t_\circ, z_\circ))C & 0 \end{bmatrix} \begin{bmatrix} \bar{e} \\ \bar{\theta} \end{bmatrix}$$

with i.c.: $(t_*, \bar{z}_*)$ $z(t)$ are solutions of the original NL system

However, in

$$\int_t^{t+T} \phi(\tau, z(\tau, t_\circ, z_\circ))\phi(\tau, z(\tau, t_\circ, z_\circ))^\top d\tau \geq \mu I \quad \forall t \geq t_\circ,$$

[Q1] μ, and T depend on the initial conditions that generate the trajectories of the original **nonlinear system** hence, we loose uniformity in $(t_\circ, z_\circ)$

[Q2] What if $\phi(t, 0) \equiv 0$? ... the **PE** property is **lost** near the origin!

Rmk. We cannot claim global exponential stability for the nonlinear system

Linear parameterised time-varying systems
[Q1: Problem statement]

Let $\mathcal{D}$ be a closed set and let $\lambda \in \mathcal{D}$ be *a parameter*
(e.g. $\lambda := (t_\circ, z_\circ)$, $\mathcal{D} := \mathbb{R}_{\geq 0} \times \mathbb{R}^n$)

We shall study systems of the form

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A(t,\lambda) & B(t,\lambda)^\top \\ -C(t,\lambda) & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \quad z := \begin{bmatrix} e \\ \theta \end{bmatrix} \tag{LTV}$$

where $e \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$, $A(t,\lambda) \in \mathbb{R}^{n\times n}$, $B(t,\lambda) \in \mathbb{R}^{n\times p}$, $C(t,\lambda) \in \mathbb{R}^{n\times p}$ are uniformly bounded.

We aim at establishing uniform exponential stability of the origin, *i.e.*, that there exist r, k and $\gamma > 0$ such that for all $t \geq t_\circ$, all $t_\circ \geq 0$ and all $\lambda \in \mathcal{D}$,

$$|z_\circ| < r \;\Rightarrow\; |z(t,\lambda,t_\circ,z_\circ)| \leq k\,|z_\circ|\,e^{-\gamma(t-t_\circ)} \; .$$

Linear parameterised time-varying systems
[The essential tools]

Definition 2 (λ-uniform persistency of excitation) *Let $\phi : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n \times m}$, $\phi(t,\lambda)$ be absolutely continuous in both arguments. We say that $\phi(t,\lambda)$ is λ-uniformly persistently exciting (λ-uPE) if there exist μ and $T > 0$ such that*

$$\int_t^{t+T} \phi(\tau,\lambda)\phi(\tau,\lambda)^\top d\tau \geq \mu I, \qquad \forall\, t \geq 0, \lambda \in \mathcal{D}.$$

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Lemma 2 (Measure Lemma) *Consider a function $\phi : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}$. Assume that there exists ϕ_M such that $|\phi(t,\lambda)| \leq \phi_M$ for all $t \geq 0$ and all $\lambda \in \mathcal{D}$. Assume further that $\phi(\cdot,\cdot)$ is λ-uPE. Then, for any $t \geq 0$ the measure of the set*

$$I_{\mu,t} := \left\{ \tau \in [t,\, t+T] \,:\, |\phi(\tau,\lambda)| \geq \frac{\mu}{2T\phi_M} \right\} \tag{1}$$

satisfies

$$\textit{meas}[I_{\mu,t}] \geq \sigma_\mu := \frac{T\mu}{2T\phi_M^2 - \mu}\,. \tag{2}$$

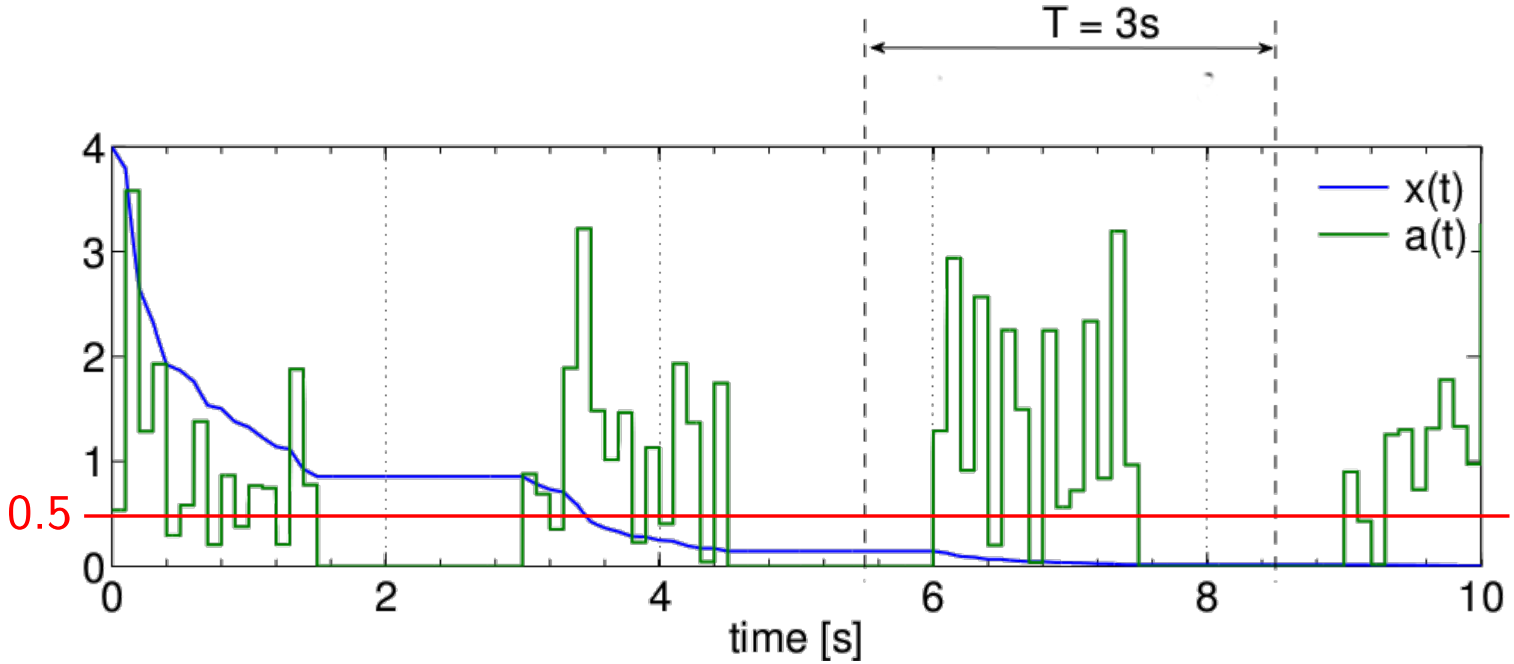
Linear parameterised time-varying systems

[Example]

Claim. The origin of $\dot{x} = -\phi(t,\lambda)^2 x$ is uniformly globally exponentially stable

Idea: Let $V(x) := \frac{1}{2}|x|^2$ so that

$$\dot{V} = -\phi(t,\lambda)^2 x^2 \leq 0 \qquad (\Rightarrow \text{UGS}).$$



Rmk. On each window $[t, t+T]$ there is a collection of intervals $I_{\mu,t}$ during which $\phi(t,\lambda)^2 \geq 0.5$ and $V(x(t))$ takes a "good" decrease

Linear parameterised time-varying systems
[The essential tools]

Lemma 3 (Integration lemma for UGES) *Assume that there exist constants r, c, $p > 0$ such that the solution $x(\cdot\,; \lambda, t_\circ, x_\circ)$ of $\dot{x} = f(t, \lambda, x)$ satisfies*

$$\max\left\{ |x|_\infty \,,\, |x|_p \right\} \leq c\,|x_\circ| \tag{3}$$

for all $x_\circ \in B_r$ and all $t_\circ \geq 0$. Then, the system is λ-ULES with $k_\lambda := ce^{1/p}$ and $\gamma_\lambda := [p\,c^p]^{-1}$. Moreover, if $c > 0$ exists for all $x_\circ \in R^n$, the system λ-UGES (GES unif. in the i.c. and in λ).

Lemma 4 (Output injection) *Let $A : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n\times n}$, $C : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{m\times n}$, and $K : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n\times m}$ be continuous and bounded on their domains.*

- *Assume that the origin of the system $\dot{\bar{x}} = A(t,\lambda)\bar{x}$ is λ-UGES.*
- *Then, the system $\dot{x} = A(t,\lambda)x + K(t,\lambda)y$ where $y := C(t,\lambda)x$, is λ-UGES if there exists $c > 0$ such that*

$$\int_{t_\circ}^{\infty} |y(s)|^2\, ds \;\leq\; c^2\,|x_\circ|^2 \qquad\qquad \forall\,(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n\,. \tag{4}$$

Linear parameterised time-varying systems

Lemma 5 (Speed-gradient systems) *For the system*

$$\dot{x} = -\phi(t,\lambda)\phi(t,\lambda)^\top x, \qquad \phi(t,\lambda) \in \mathbb{R}^{m\times n}$$

assume that $\phi(t,\lambda)$ is λ-uPE with parameters T and $\mu > 0$ and there exists a constant $\phi_M > 0$ such that, for almost all $t \geq 0$ and all $\lambda \in \mathcal{D}$

$$\max\left\{ |\phi(t,\lambda)|, \left| \frac{\partial \phi(t,\lambda)}{\partial t} \right| \right\} \leq \phi_M \ . \tag{5}$$

Then the system is λ-UGES with

$$k = 1, \qquad \gamma \ \geq \ \frac{\mu}{\mathsf{e}^2\, T[1 + \phi_M^4 T^2]}$$

That is,

$$|x(t)| \leq k|x_\circ|\mathsf{e}^{-\lambda(t-t_\circ)} \qquad \forall\, t \geq t_\circ,\ t_\circ \geq 0,\ \lambda \in \mathcal{D}$$

Linear parameterised time-varying systems

[Passive-interconnected systems]

Theorem 1 (UGES of LTV) *The origin of the system*

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A(t,\lambda) & B(t,\lambda)^\top \\ -C(t,\lambda) & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \; z := \begin{bmatrix} e \\ \theta \end{bmatrix}, \tag{6}$$

under Assumptions 1 and 2, is λ-UGES if and only if $B(t,\lambda)$ is λ-uPE.

Assumption 1 *there exists $b_M > 0$ such that, for almost all $t \geq 0$ and all $\lambda \in \mathcal{D}$*

$$\max\left\{ |A(t,\lambda)|, |B(t,\lambda)|, \left| \frac{\partial B(t,\lambda)}{\partial t} \right| \right\} \leq b_M \; . \tag{7}$$

Assumption 2 *There exist symmetric matrices $P(t,\lambda)$ and $Q(t,\lambda)$ such that*

$$P(t,\lambda)B(t,\lambda)^\top = C(t,\lambda)^\top$$
$$-Q(t,\lambda) := A(t,\lambda)^\top P(t,\lambda) + P(t,\lambda)A(t,\lambda) + \dot{P}(t,\lambda)$$

There exist p_m, q_m, p_M, and $q_M > 0$ such that, for all $(t,\lambda) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$,

$$p_m I \leq P(t,\lambda) \leq p_M I, \qquad q_m I \leq Q(t,\lambda) \leq q_M I$$

Proof of Theorem 1. We split the system and use output injection:

First, consider the globally invertible change of coordinates:

$$\xi := \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -B(t,\lambda) & I \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}$$

so $\{z = 0\}$ is λ-UGES for (6) if and only if so is $\{\xi = 0\}$ for the system

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A(t,\lambda) & B(t,\lambda)^\top \\ -R_1(t,\lambda) & -B(t,\lambda)B(t,\lambda)^\top \end{bmatrix}}_{\mathcal{A}(t,\lambda)} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B(t,\lambda)^\top B(t,\lambda) \\ R_1(t,\lambda) - R_2(t,\lambda) \end{bmatrix}}_{\mathcal{K}(t,\lambda)} \xi_1 \,,$$

We establish that:

1) the origin of $\dot{\xi} = \mathcal{A}(t,\lambda)$ is λ-UGES,

2) the solutions $\xi(t,\lambda)$ are uniformly bounded,

3) ξ_1 is square integrable (uniformly in λ), and

4) $\mathcal{K}(t,\lambda)$ is bounded.

Linear parameterised time-varying systems

Corollary. The solutions satisfy the bound:

$$|x(t,\lambda)| \leq t_{\mathrm{M}} t_{\mathrm{M}}^{\mathsf{inv}} \left(\frac{\pi\,\mathbf{e}}{\rho}\right)^{1/2} |x_\circ|\,\mathbf{e}^{-\dfrac{\rho}{2\pi}(t-t_\circ)} \qquad \forall t \geq t_\circ\,.$$

where:

$$\pi := c_{32} + (c_* t_{\mathrm{M}}^{\mathsf{inv}})^2 \left[\frac{(c_{32}k_{\mathrm{M}})^2}{4(1-\rho)}\right], \qquad 0 \;<\; \rho \;\leq\; \min\left\{p_m,\, \frac{1}{2b_{\mathrm{M}}^2}\right\}$$

$$c_{32} := \max\left\{p_{\mathrm{M}},\, \frac{1}{2\gamma_x}\right\}, \qquad \gamma_x := \frac{\mu}{T(1+b_{\mathrm{M}}^2 T)}$$

- γ_x is the convergence rate for $x(t,\lambda)$ in $\dot{x}(t,\lambda) = -B(t,\lambda)B(t,\lambda)^\top x(t,\lambda)$
- c_* is a bound on $|e|_2 = \left(\int_{t_\circ}^{\infty} |e(t,\lambda)|^2\right)^{1/2}$
- t_{M}, $t_{\mathrm{M}}^{\mathsf{inv}}$ are bounds on coordinates transformations
- k_{M} is a bound on an *output injection* term
- b_{M} is the bound on $B(t,\lambda)$ and its derivative

Problem statement
[Model-Reference-Adaptive-Control]

- "Since the solutions are bounded (UGS) one can consider the LTV system" :

$$\begin{bmatrix} \dot{\bar{e}} \\ \dot{\bar{\theta}} \end{bmatrix} = \begin{bmatrix} A & B\phi(t, z(t, t_\circ, z_\circ))^\top \\ -\phi(t, z(t, t_\circ, z_\circ))C^\top & 0 \end{bmatrix} \begin{bmatrix} \bar{e} \\ \bar{\theta} \end{bmatrix}$$

$$z(t) = [e(t)^\top,\ \theta(t)^\top]^\top \qquad \text{(solutions of the original NL system)}$$

However, in

$$\int_t^{t+T} \phi(\tau, z(\tau, t_\circ, z_\circ))\phi(\tau, z(\tau, t_\circ, z_\circ))^\top d\tau \geq \mu I \quad \forall t \geq t_\circ \ .$$

[Q1] μ, and T depend on the initial conditions that generate the trajectories of the original **nonlinear system** hence, we loose uniformity in $(t_\circ, z_\circ)$

[Q2] What if $\phi(t, 0) \equiv 0$? ... the **PE** property is **lost** near the origin!

Persistency of excitation for nonlinear systems
[Q2: what if $\phi(t, 0 \equiv 0)$?]

Example 1 *Consider the system* $\dot{z} = -\sin(t)^2 z^3$ *or, equivalently,*

$$\dot{x} = -\sin(t)^2 z(t,\lambda)^2 x\,, \qquad x_\circ = z_\circ, \quad t_\circ^x = t_\circ^z := t_\circ$$

- *Assume that, given any* $\delta > 0$, $\exists \mathcal{I}_\delta \subset \mathbb{R}_{\geq 0}$, *such that*

  $$|z(t,\lambda)| \geq \delta \quad \forall\, t \in \mathcal{I}_\delta$$

  then, defining $v(t) := {}^1\!/_2\, x(t)^2$, *we have*

  $$\dot{v}(t) = -\delta^2 \sin(t)^2 v(t) \quad \forall\, t \in \mathcal{I}_\delta$$

  On the other hand,

  $$\int_t^{t+\pi} \sin(\tau)^2 \delta^2 d\tau = \frac{\pi}{2}\delta^2$$

  that is, $\dot{v}(t) = -\varphi(t)^2 v(t)$, *where* $\varphi(t) := \sin(\tau)\delta$ *is PE.*

- *We conclude that:* $|z(t,\lambda)| \geq \delta \implies |z(t,\lambda)| \to 0$ *exponentially fast!*

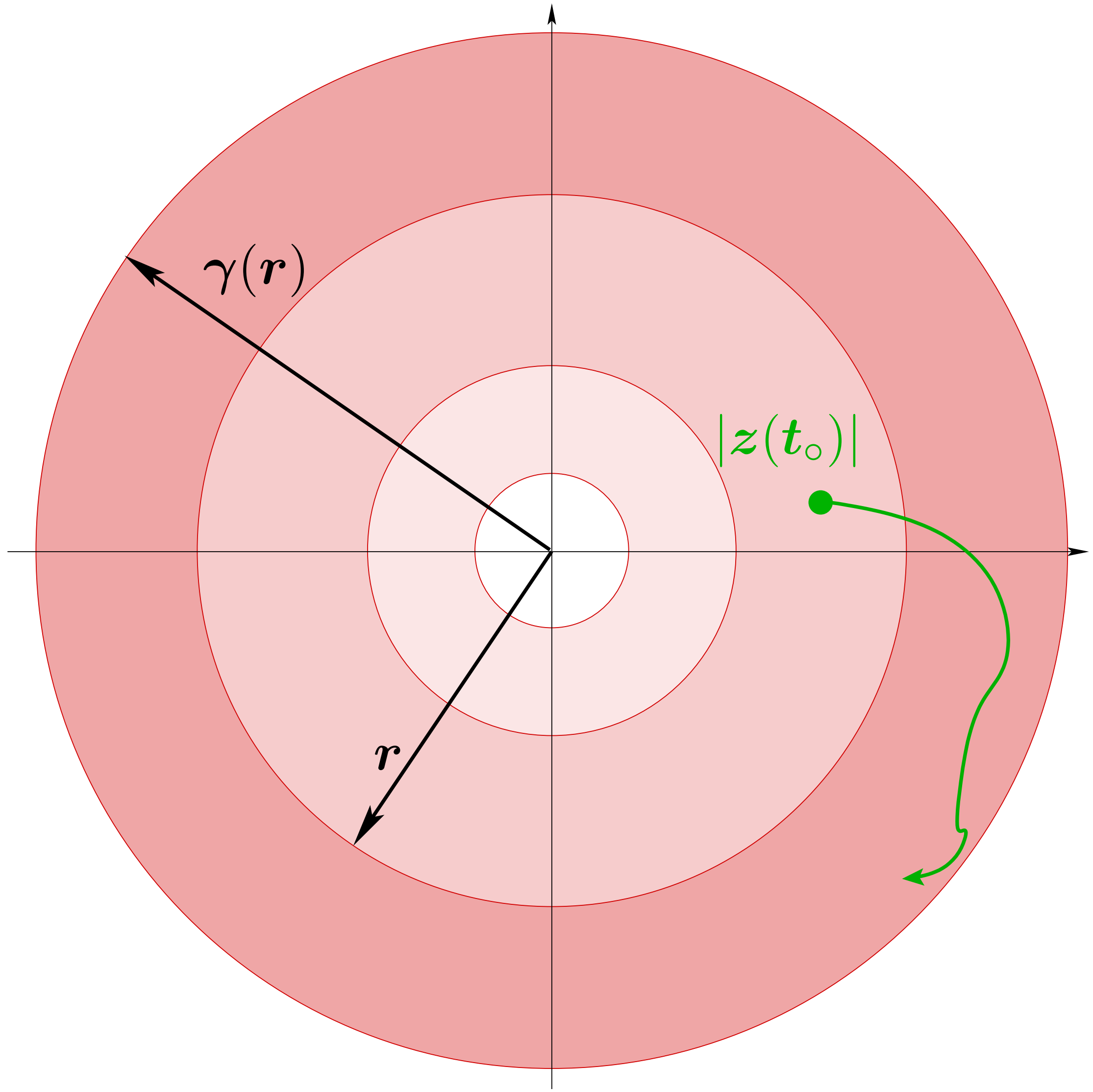
- *If this holds for any* $\delta > 0$ *we recover uniform attractivity*

Persistency of excitation for nonlinear systems
[Rationale]

- The origin is UGS, *i.e.*

$$\exists \gamma \in \mathcal{K}_\infty \; : \; \sup_{t \geq t_\circ} |z(t)| \leq \gamma\left(|z(t_\circ)|\right)$$

- Trajectories δ-far from the origin $\Rightarrow$ PE



Persistency of excitation for nonlinear systems
[Rationale]

- The origin is UGS, *i.e.*,

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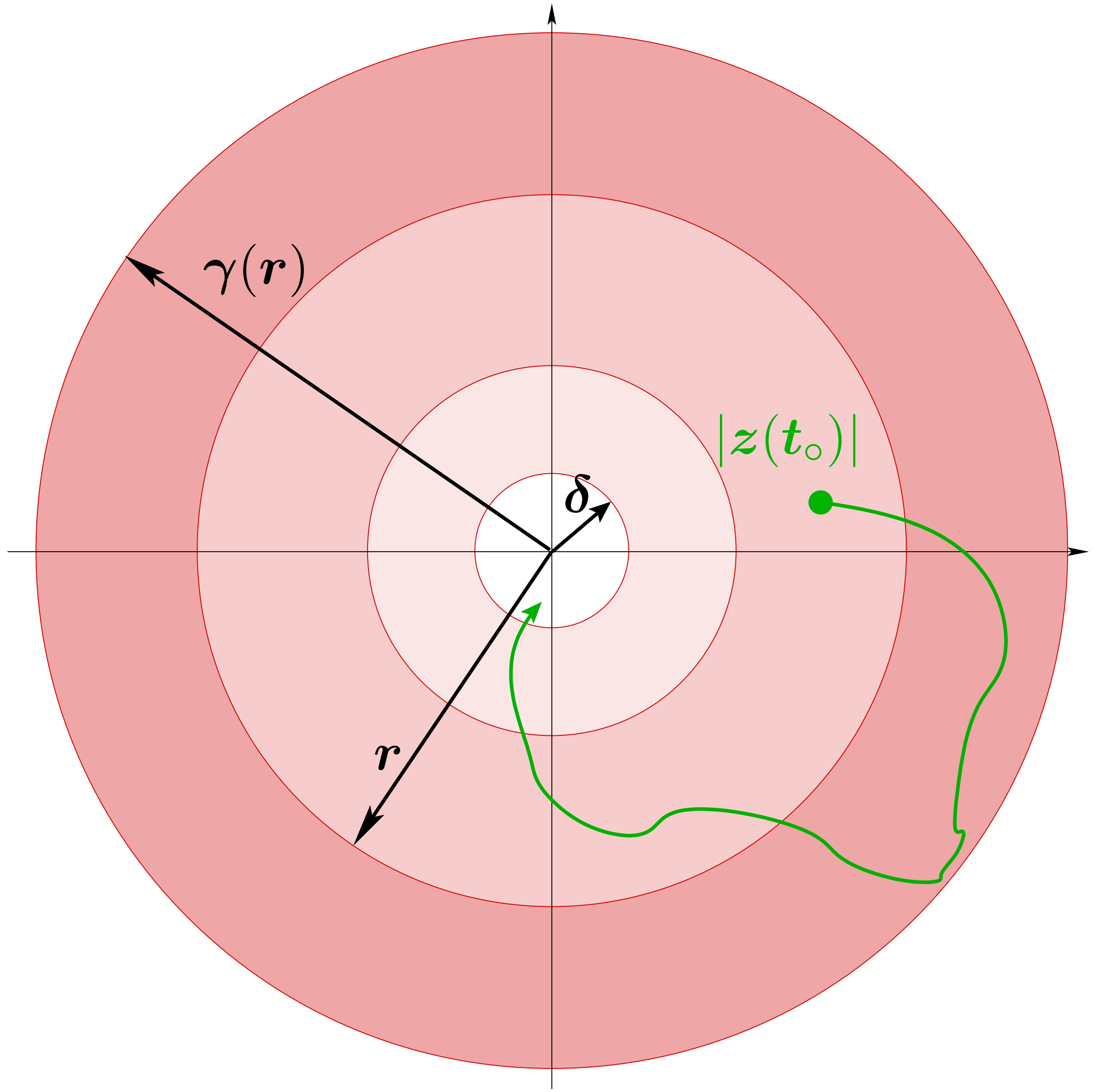

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For each $\varepsilon > 0$ and $r > 0$, $\exists\, T > 0$ s.t.

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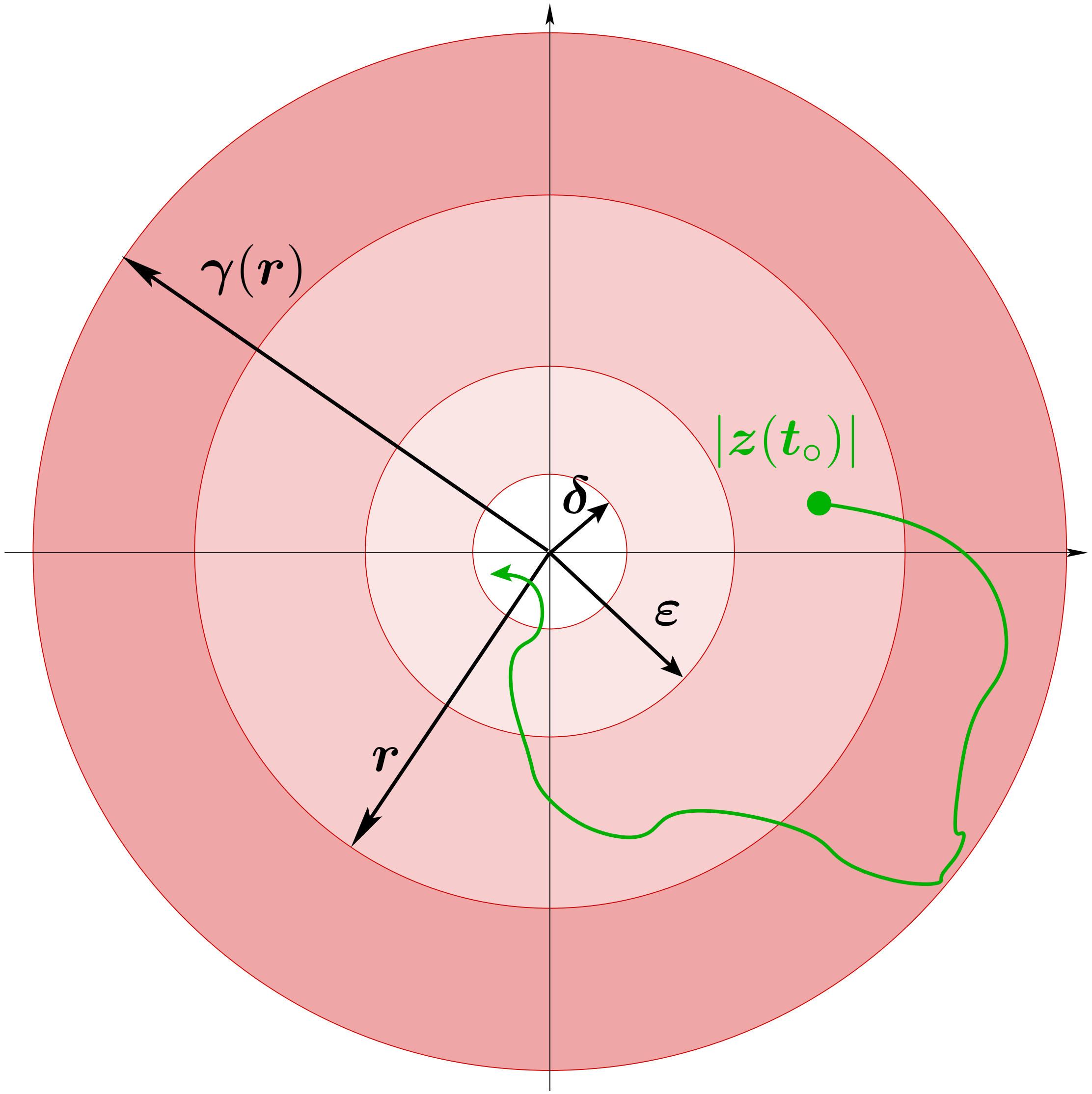

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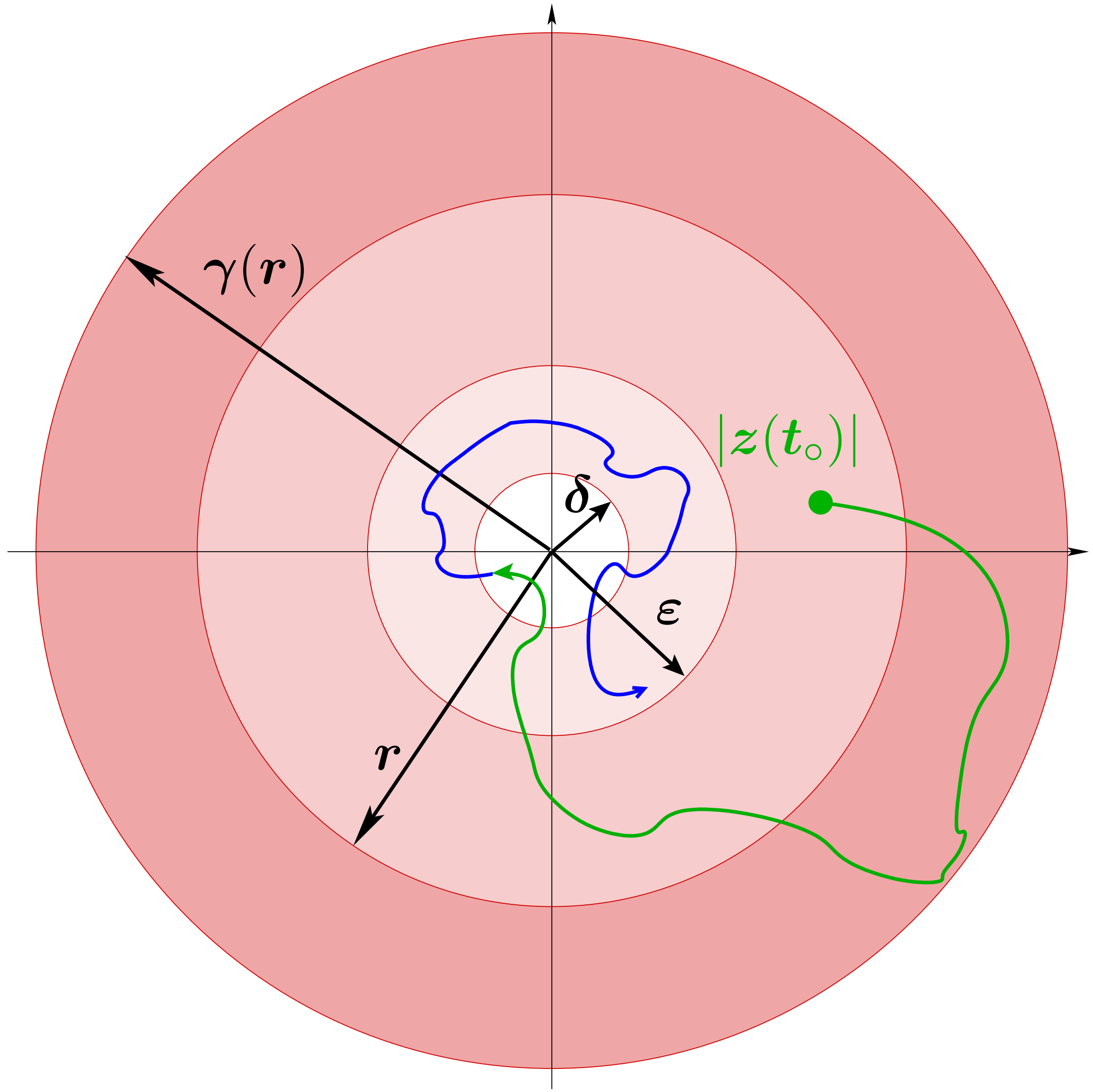

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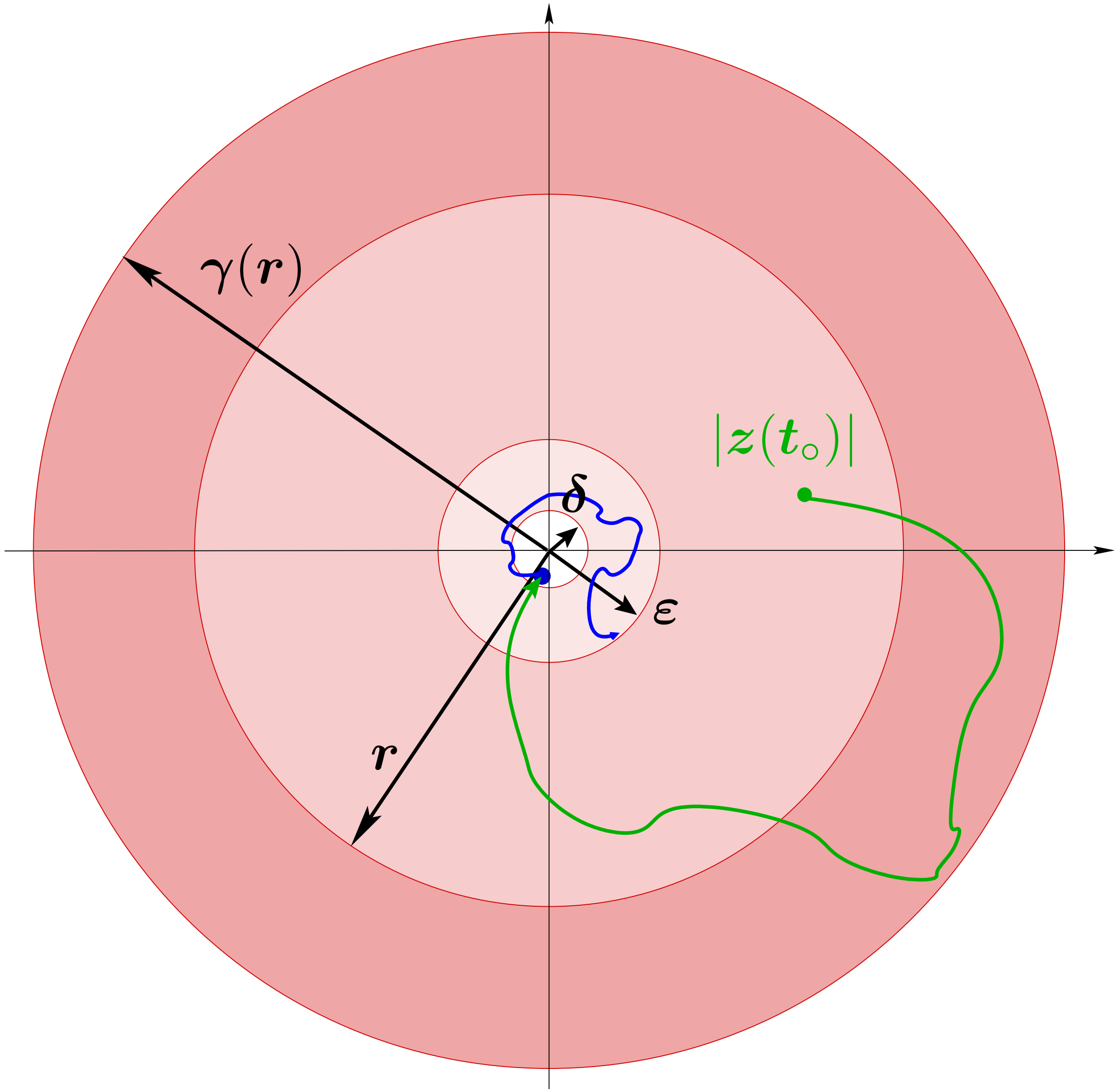

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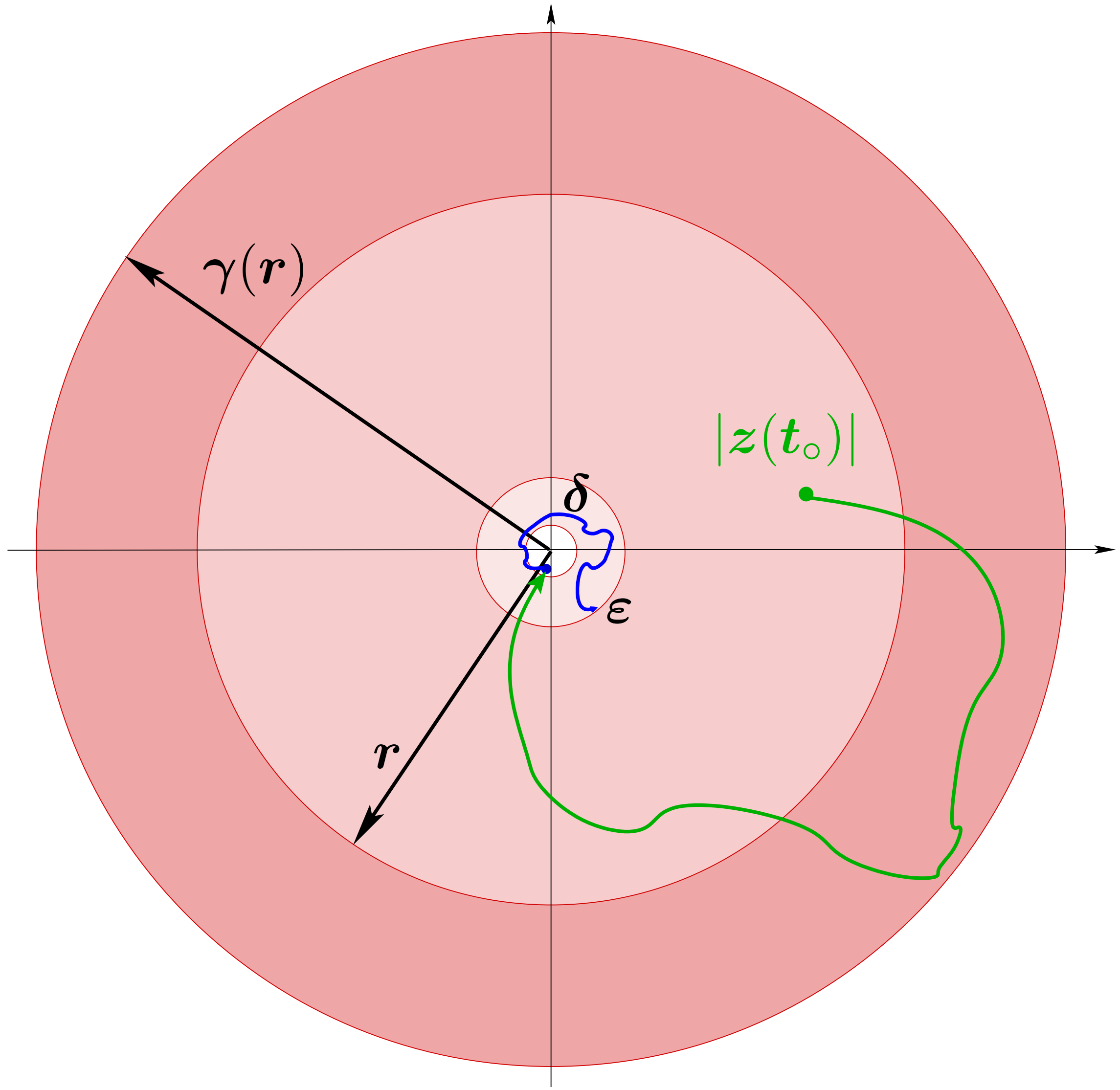

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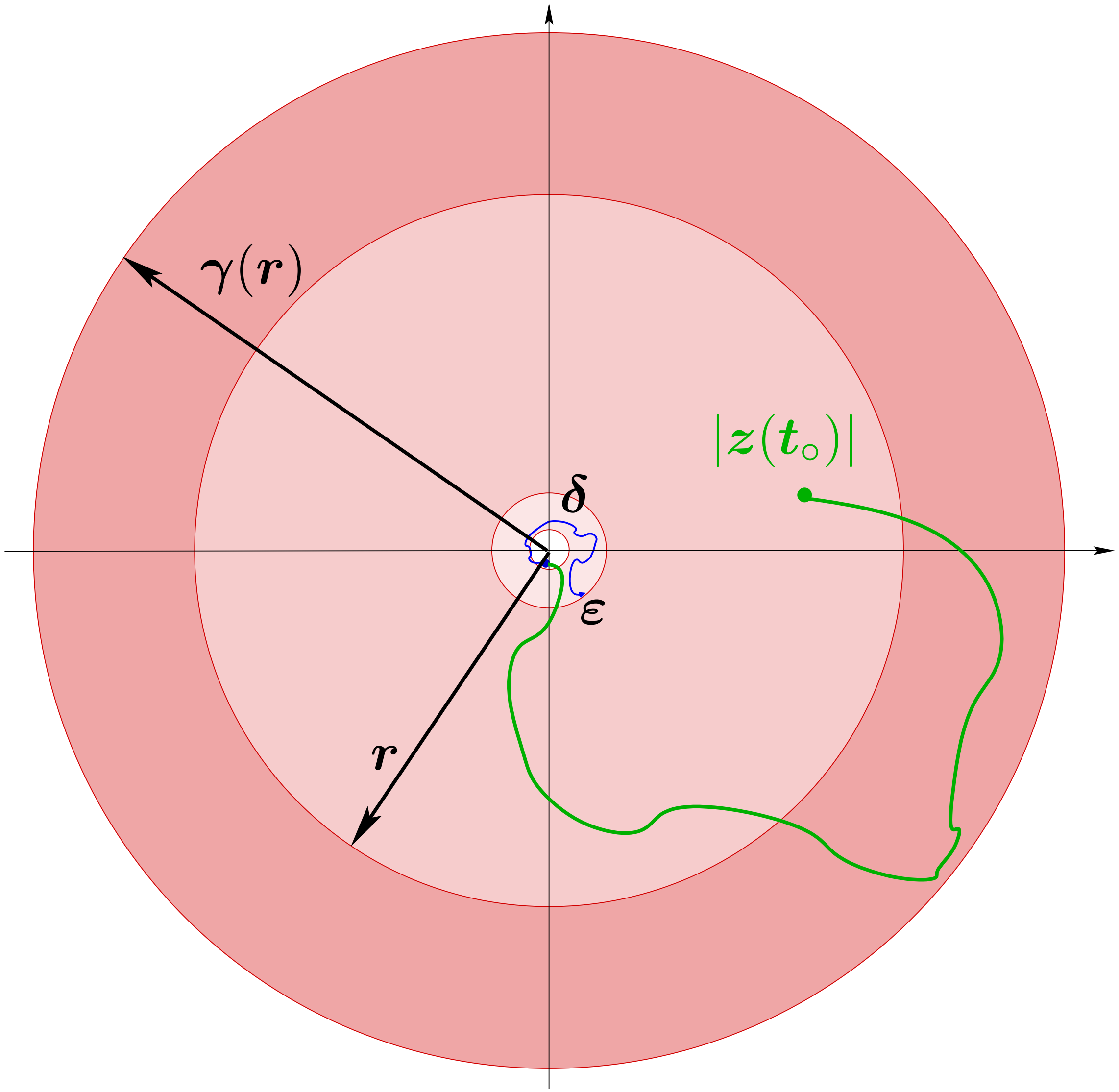
$$\left|z(t_\circ')\right| \leq \delta \Longrightarrow |z(t)| \leq \varepsilon \quad \forall t \geq t_\circ'$$



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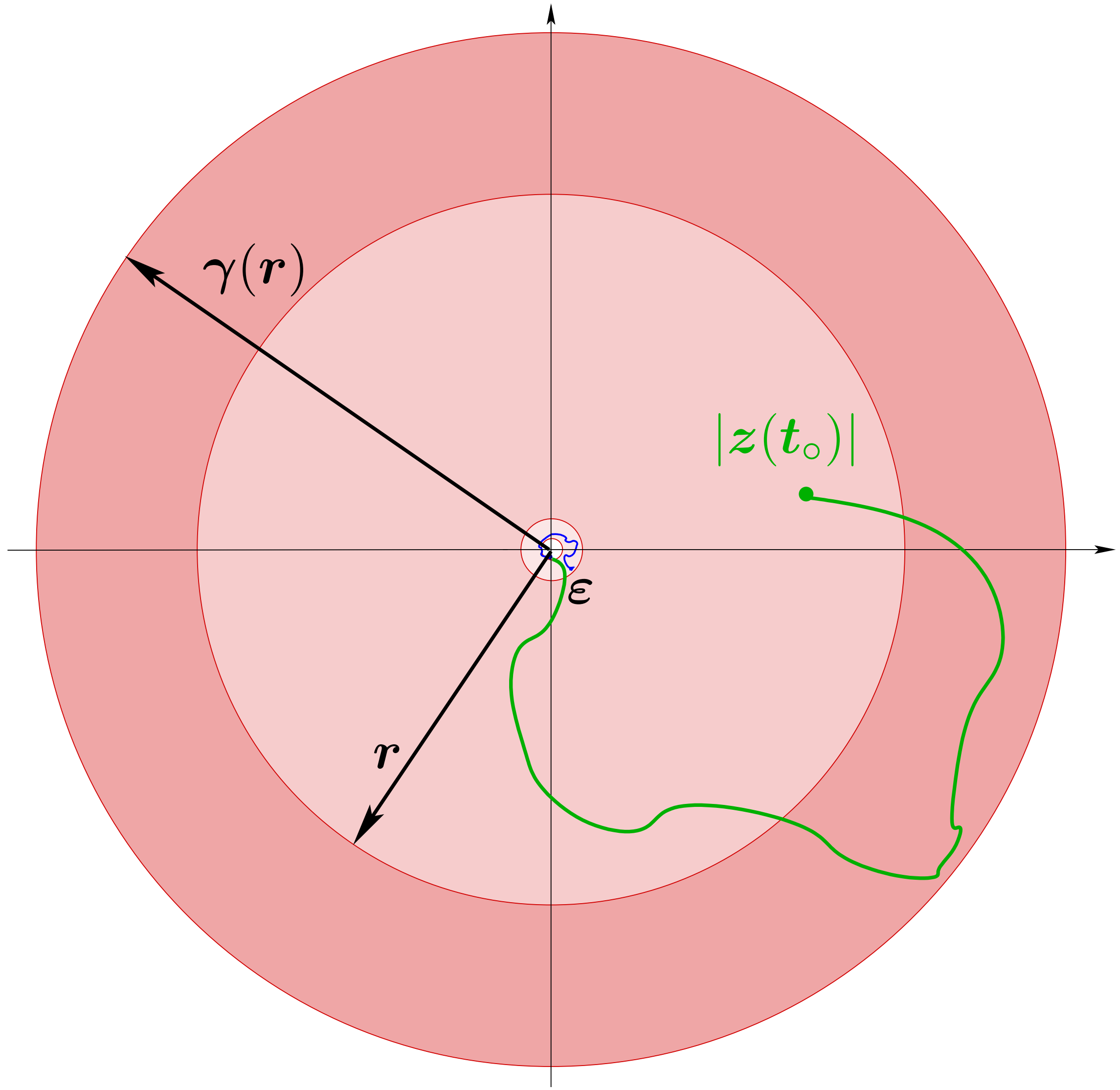

Persistency of excitation for nonlinear systems
[Uniform δ-Persistency of excitation]

Consider nonlinear time-varying systems:

$$\dot{x} = F(t, x)$$

where $F(\cdot, \cdot)$ is such that solutions exist (locally) and are unique.

Let $x \in \mathbb{R}^n$ be partitioned into $x := [x_1^\top, x_2^\top]^\top$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Define the *column vector* function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ to be such that $\phi(\cdot, x)$ is locally integrable for each $x \in \mathbb{R}^n$.

Definition 3 (Uδ-PE) *A function $\phi(\cdot, \cdot)$ is said to be uniformly δ-persistently exciting with respect to x_1 if for each $x \in (\mathbb{R}^{n_1} \setminus \{0\}) \times \mathbb{R}^{n_2}$ there exist $\delta > 0$, $T > 0$ and $\mu > 0$ s.t.*

$$|z - x| \leq \delta \quad \Longrightarrow \quad \int_t^{t+T} |\phi(\tau, z)|\, d\tau \geq \mu \quad \forall\, t \in \mathbb{R}. \tag{8}$$

•

Persistency of excitation for nonlinear systems
[Characterizations of Uδ-PE]

Lemma 6 *The function $\phi(\cdot,\cdot)$ is Uδ-PE with respect to x_1 if and only if*

> **(A)** *for each $\delta > 0$ and $\Delta \geq \delta$ there exist $T > 0$ and $\mu > 0$ such that, for all $t \in \mathbb{R}$,*

$$|x_1| \in [\delta,\, \Delta]\,,\ |x_2| \in [0,\, \Delta] \implies \int_t^{t+T} |\phi(\tau, x)|\, d\tau \geq \mu \quad \forall\, t \in \mathbb{R}.$$

•

Example 2 *Remember the system $\dot{x} = -\sin(t)^2 x^3$; for the function $\phi(t,x) := \sin(t)^2 x^2$ we have:*

$$x \in [\delta, \Delta] \implies \int_t^{t+\pi} \sin(\tau)^2 \delta^2 d\tau = \frac{\pi}{2}\delta^2$$

That is, $\phi(t,x) := \sin(t)^2 x^2$ is Uδ-PE. •

Persistency of excitation for nonlinear systems
[Characterizations of Uδ-PE]

Lemma 7 *If $(t, x) \mapsto \phi$ is continuous in x uniformly in t then $\phi(\cdot, \cdot)$ is Uδ-PE with respect to x_1 if and only if*

(B) for each x such that $x_1 \neq 0$ there exist $T > 0$ and $\mu > 0$ such that,

$$\int_t^{t+T} |\phi(\tau, x)| d\tau \geq \mu \quad \forall\, t \in \mathbb{R}$$

•

Example 3 *Let $\phi(t, x) := \Phi(t)^\top x$. Then, $\phi(t, x)$ is Uδ-PE with respect to x if and only if there exist T and $\mu > 0$ such that*

$$\int_t^{t+T} \Phi(\tau)\Phi(\tau)^\top d\tau \geq \mu I \qquad \forall\, t \in \mathbb{R}\,. \tag{9}$$

Persistency of excitation for nonlinear systems

[Characterizations of Uδ-PE]

Example 4 *Consider once more the function* $\phi(t,x) := \sin(t)^2 x^2$ *which is uniformly continuous. We see that*

$$x \neq 0 \implies \int_t^{t+\pi} \sin(\tau)^2 x^2 d\tau = \frac{\pi}{2} x^2 =: \mu(x)$$

Actually, in general, we also have the following:

Lemma 8 *The function* $\phi(\cdot,\cdot)$ *is U*δ*-PE with respect to* x_1 *if and only if*

> **(C)** *for each* $\Delta > 0$ *there exist* $\mu_\Delta \in \mathcal{K}$ *and* $\theta_\Delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *continuous strictly decreasing such that, for all* $t \in \mathbb{R}$*,*
>
> $$\{\, |x_1| \neq 0,\ |x_2| \in [0,\ \Delta] \,\} \quad \implies \quad \int_t^{t+\theta_\Delta(|x_1|)} |\phi(\tau,x)|\, d\tau \geq \mu_\Delta(|x_1|)\,.$$

Rmk. It is clear that, in general, for nonlinear functions, the "PE bound" depends on the "parameter" x

Persistency of excitation for nonlinear systems
[Uδ-PE: A sufficient and necessary condition]

Theorem 2 (UGAS $\Rightarrow$ Uδ-PE) *The origin of the system*

$$\dot{x} = F(t, x)$$

where $F(\cdot, \cdot)$ is Lipschitz in x uniformly in t, is UGAS only if $F(\cdot, \cdot)$ is Uδ-PE with respect to $x \in \mathbb{R}^n$. •

Rmk. Sufficiency also holds under extra conditions.

Proposition 1 *The origin of the system*

$$\dot{z} = -v(t)^2 z^3$$

is UGAS if and only if $v(t)$ is persistently exciting (in the usual sense). •

Sketch of proof: The origin is UGS because $V = |z|^2$ yields $\dot{V} = -v(t)^2 z^4 \leq 0$. The function $\phi(t, z) = v(t)^2 z^3$ is Uδ-PE:

$$\int_t^{t+T} v(\tau)^2 d\tau \geq \mu \quad \forall\, t \geq 0,\ z \neq 0 \qquad \Longrightarrow \qquad \int_t^{t+T} \left| v(\tau)^2 z^3 \right| d\tau \geq \mu \left| z \right|^3$$

Persistency of excitation for nonlinear systems

[Passive-interconnected systems]

Theorem 3 [11] *Consider the system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A(t,x) & B(t,x) \\ C(t,x) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

under the following assumptions:

- *We have a Lyapunov function V such that*

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$$
$$\dot{V}(t,x) \leq -\alpha_3(|x_1|) \quad \textit{a.e.}.$$

- *The functions A, B and C are locally Lipschitz in x uniformly in t, uniformly bounded in t, B is once differentiable with partial derivatives in t, and $A(t,x)|_{x_1=0} = C(t,x)|_{x_1=0} = 0$.*

The origin is UGAS if and only if $B(t,x)x_2|_{x_1=0}$ is Uδ-PE with respect to x_2

Persistency of excitation for nonlinear systems
[Slotine & Li adaptive controller]

Proposition 2 *Consider the lossless Lagrangian system (without friction)*

$$D_\theta(q)\ddot{q} + C_\theta(q,\dot{q})\dot{q} + g_\theta(q) = u$$

in closed loop with the certainty-equivalence controller

$$\begin{array}{rcl} u & = & D_{\hat{\theta}}(q)\ddot{q}_r + C_{\hat{\theta}}(q,\dot{q})\dot{q}_r + g_{\hat{\theta}}(q) - k_d s \\ \dot{\hat{\theta}} & = & -\Gamma\Phi(t,s,\tilde{q})^\top s \\ \dot{q}_r & := & \dot{q}_d - \lambda\tilde{q}, \qquad s \;:=\; \dot{q} - \dot{q}_r \end{array}$$

Then, the origin if uniformly globally asymptotically stable for any $\boldsymbol{\lambda},\ \boldsymbol{k_d} > \mathbf{0}$ **if and only if** $\Phi_\circ(t) := \Phi(t,0,0)$ *is persistently exciting, that is*

$$\int_t^{t+T} \Phi_\circ(\tau)\Phi_\circ(\tau)^\top d\tau \geq \mu, \quad \forall\, t \geq 0.$$

Rmk. Note that $\Phi_\circ(t)$ is such that

$$\Phi_\circ(t)^\top\theta \;=\; D_\theta(q_d(t))\ddot{q}_d(t) + C_\theta(q_d(t),\dot{q}_d(t))\dot{q}_d(t) + g_\theta(q_d(t))$$

Persistency of excitation for nonlinear systems
[Slotine & Li adaptive controller]

Analysis of the closed-loop system.–

The closed-loop dynamics, for which we have $x_1 \to 0$, is

$$\begin{bmatrix} \dot{\tilde{q}} \\ \dot{s} \end{bmatrix} = \underbrace{\begin{bmatrix} -\lambda I & I \\ 0 & -D_\theta^{-1}(\cdot)\left[C_\theta(\cdot) + k_d I\right] \end{bmatrix}}_{\text{"}A(t,x_1)\text{"}} \underbrace{\begin{bmatrix} \tilde{q} \\ s \end{bmatrix}}_{x_1} + \underbrace{\begin{bmatrix} 0 \\ D_\theta^{-1}(\cdot)\Phi(t,\tilde{q},s)^\top \end{bmatrix}}_{\text{"}B(t,x_1)\text{"}} \tilde{\theta}$$

$$\dot{\tilde{\theta}} = -\Gamma^{-1}\left[0 \qquad \Phi(t,\tilde{q},s)D_\theta(\cdot)\right] \underbrace{\begin{bmatrix} \lambda k_d I & 0 \\ 0 & D_\theta^{-1}(\cdot) \end{bmatrix}}_{P_\theta(\cdot)} \underbrace{\begin{bmatrix} \tilde{q} \\ s \end{bmatrix}}_{x_1}$$

The result follows, directly, from Theorem 3, by recognizing that system has the interconnected passive-systems form

$$\begin{aligned} \dot{x}_1 &= A(t,x_1)x_1 + B(t,x_1)x_2 \\ \dot{x}_2 &= -B(t,x_1)^\top P x_1 \end{aligned}$$

δ-PE controllers

[Stabilization of nonholonomic systems]

Consider the system:

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_1 x_3 \\ \dot{x}_3 &= u_2\end{aligned}$$

Rmk. [Brocket] The origin is not stabilizable via smooth autonomous state-feedback

Proposition: Let $u_2(t,x) := -ax_3 - u_1(t,x)x_2$ then,

$$\begin{aligned}\dot{x}_1 &= u_1(t,x) \\ \dot{x}_2 &= u_1(t,x)x_3 \\ \dot{x}_3 &= -ax_3 - u_1(t,x)x_2\end{aligned}$$

Rmk. The general n-dimensional case is also solvable similarly

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Rmk. We need u_1 to stabilize the x_1-equation and to excite the x_2, x_3-equations

We use: $u_1 := -k_1 x_1 + h(t, x_2, x_3)$. For instance

$$u_1 := -k_1 x_1 + \sin(t) \left[\, |x_2|^2 + |x_3|^2 \right]$$

Bibliographical remarks

- Persistency of excitation was originally introduced by K. J. Åström from LTH, Sweden, in [1], in a discrete-time context
- It has been thoroughly developed by authors that include: Narendra, Anderson, Annaswamy, Iannou, to mention a few:
  - In [9] the authors give a very detailed account of persistency of excitation and uniform asymptotic stability. See also [7, 8]
  - "Classical" theory of (linear) adaptive control systems is extensively explained in [2]; in particular, output injection.
- The material on linear parameterized systems is taken from [3]
- The concept of uniform δ-PE was originally introduced in [6]. More elaborated definitions and tools appeared in [4, 10, 5]