Control of Multiagent Systems Using Linear Cyclic Pursuit With Heterogenous Controller Gains

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In this paper, behavior of a group of autonomous mobile agents under cyclic pursuit is studied. Cyclic pursuit is a simple distributed control law, in which the agent \( i \) pursues agent \( i+1 \mod n \). The equations of motion are linear, with no kinematic constraints on motion. Behaviorally, they are identical but may have different controller gains. We generalize existing results in the literature, which consider only homogenous gains, to the case where controller gains are heterogenous. We show that, by selecting suitable controller gains, collective behavior of agents can be controlled significantly to obtain not only point convergence but also directed motion. In particular, we obtain analytical results that relate the controller gains to the direction of movement of the agents when the system is unstable. Invariance results with respect to the sequence of pursuit between agents and finite switching of connections. Simulation experiments are given in support of the analytical results.

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1 Introduction

Multiagent systems are groups of autonomous mobile agents used for collaborative operations. The system operates without a centralized controller or global information. Linear cyclic pursuit is a decentralized control law, used in such multiagent systems, that has a simple structure and uses simple local interaction between agents to obtain desired global behavior. Bruckstein et al. [1] modeled the behavior of ants, crickets, and frogs with continuous and discrete pursuit laws and examined the possible evolution of global behavior. Convergence to a point in linear pursuit and achievable global formation among a group of autonomous mobile agents are discussed by Lin et al. [2] and Marshall et al. [3]. A survey of the related consensus and agreement problem is given in Ren et al. [4]. The present work is an extension of the analysis done by Sinha and Ghose [5] on the stability and rendezvous conditions of cyclic pursuit laws. A behavior of interest in such systems is to obtain directed motion. In this paper, we obtain conditions under which the system is unstable, but all the agents converge to a single asymptote, thus achieving directed motion. Finally, we show several interesting invariance properties of that generalized linear cyclic pursuit with heterogenous gains with respect to the sequence of pursuit between agents and finite switching. Some preliminary results on these have earlier appeared in Ref. [6].

2 Generalized Linear Cyclic Pursuit

There are \( n \) agents, numbered \( 1 \rightarrow n \), in a general \( d \) dimensional space, such that an agent \( i \) pursues agent \( i+1 \mod n \). The position of agent \( i \) at any time \( t \geq 0 \) is given by

\[
Z_i(t) = [y_i^1(t) \ y_i^2(t) \ \cdots \ y_i^d(t)]^T \in \mathbb{R}^d \quad i = 1, 2, \ldots, n
\]

(1)

The kinematics of agent \( i \) is given by

\[
\dot{Z}_i = u_i
\]

(2)

where \( u_i \) is the control of agent \( i \). The control \( u_i \), for linear pursuit, is obtained as

\[
u_i = k_i(Z_{i+1} - Z_i) \quad \forall i
\]

(3)

where \( k_i \) is the controller gain of agent \( i \). The controller gains \( k = [k_i]_{i=1}^n \) need not be the same for all agents.

For every agent \( i \), each coordinate \( y_i^\delta \), \( \delta = 1, \ldots, d \), of \( Z_i \), evolves independently in time. Hence, Eq. (2) can be decoupled into \( d \) identical linear system of equations of the form

\[
\dot{X} = AX
\]

(4)

where

\[
A = \begin{bmatrix}
-k_1 & k_2 & 0 & \cdots & \cdots & 0 \\
0 & -k_2 & k_3 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -k_{n-1} & k_n \\
k_n & 0 & \cdots & 0 & -k_1
\end{bmatrix}
\]

(5)

Here, \( X = [x_1, x_2, \ldots, x_n] \), and each \( x_i \) represents a \( y_i^\delta \) for some \( \delta = 1, \ldots, d \). Equation (4) represents the dynamics in each direction. If gains \( k_i \) are identical for all agents, then the problem reduces to that addressed by Marshall et al. [3]. The characteristic equation of \( A \) is
\[ \rho(s) = \prod_{i=1}^{n} (s + k_i) - \prod_{i=1}^{n} k_i \tag{6} \]

If \( k_i \neq 0, \forall i \), then there is one and only one root of \( \rho(s) \) at the origin. The solution of Eq. (4) in the frequency domain is \( X(s) = (sI-A)^{-1}X(0) \). The \( i \)th component of \( X(s) \) is

\[ x_i(s) = \frac{1}{\rho(s)} \sum_{q=1}^{n} b^i_q(s)x_q(0) \quad i = 1, \ldots, n \tag{7} \]

where

\[ b^i_q(s) = \begin{cases} \prod_{i=1}^{n} (s + k_i) & q = i \\ \prod_{j=1, j \neq q}^{n} k_j \prod_{i=1}^{n} (s + k_i) & q > i \\ \prod_{j=1, j \neq q}^{n} k_j \prod_{i=1}^{n} (s + k_i) & q < i \end{cases} \tag{8} \]

Let the nonzero roots of Eq. (6) be \( (\sigma_p + j\omega_p)_p, p = 1, \ldots, \tilde{n} \), where \( \tilde{n} \) is the number of distinct roots of Eq. (6). Taking the inverse Laplace transform of \( x_i(t) \), we get

\[ x_i(t) = \sum_{p=1}^{\tilde{n}} \sum_{q=1}^{n} n_p \sum_{r=1}^{\sigma_p + j\omega_p} \frac{a^p_{qr}}{\rho(s)} x_q(0) e^{\sigma_r t + j\omega_p t} + x_f \tag{9} \]

where \( n_p \) is the algebraic multiplicity of the \( p \)th root, and

\[ a^p_{qr} = \left[ \frac{d}{r!} \frac{d}{ds} \left( x - (\sigma_p + j\omega_p) x \right) \right]_{s=\sigma_p + j\omega_p} \]

and

\[ x_f = \sum_{q=1}^{n} \left( \frac{x_q}{k_q} \right) \sum_{q=1}^{n} \left( \frac{1}{k_i} \right) \]

Here, \( x_f \) corresponds to the root \( s = 0 \). The trajectory \( x_i(t) \) of agent \( i \) depends on the eigenvalues \( (\sigma_p + j\omega_p), p = 1, \ldots, \tilde{n} \), of \( A \) which, in turn, are functions of the gains \( k_1, i = 1, \ldots, n \). If all the eigenvalues of \( A \) have negative real parts, then the system (4) is stable. We will relax this condition further in the next section and find the combination of the gains for which system (4) is stable.

### 3 Stability Analysis

If the system (4) is stable, as \( t \to \infty, \dot{x}_i(t) \to 0, \forall i \). This implies that eventually all the agent will converge to a point. Here, we state the stability and rendezvous results without proofs (which are available in Ref. [5]).

**Theorem 1.** The system of \( n \) agents, given by Eq. (4), will converge to a point if and only if (a) at most one \( k_i \) is negative or zero and (b) \( \sum_{i=1}^{n} k_i > 0 \).

**Theorem 2. (Reachable Point) If a system of \( n \) agents have initial positions at \( Z_{i0} \) and gains \( k = [k_{i1}, \ldots, k_{in}]^T, \forall i \), which satisfy Theorem 1, then they converge to a point \( Z_f \) given by

\[ Z_f = \sum_{i=1}^{n} \left( \frac{1/k_i}{\sum_{j=1}^{n} 1/k_j} \right) Z_{i0} \tag{10} \]

where \( Z_f \) is called a reachable point for this system of \( n \) agents.

From Eqs. (2) and (3), we also observe that

\[ \sum_{i=1}^{n} Z_i(t)/k_i = 0 \Rightarrow \sum_{i=1}^{n} Z_i(t)/k_i = \text{const} \quad \forall t \tag{11} \]

### 4 Directed Motion

When the system (4) is not stable, we can obtain directed motion under certain conditions. For this paper, we define the most positive eigenvalue of a linear system as the eigenvalue with the largest real part. Note that this is different from the notion of dominant eigenvalue [7], which is the eigenvalue with the highest absolute value.

**Theorem 3.** Consider a system of \( n \) agents with kinematics given by Eq. (4). The trajectory of all the agents converge to a straight line as \( t \to \infty \) if and only if the most positive eigenvalue of Eq. (4) is real and positive.

**Proof.** If the most positive eigenvalue is positive, then Eq. (4) is unstable. Let the unit vector along the velocity vector of agent \( i \) at time \( t \) be \( v_i(t) = [1/\bar{v}_i(t)]v_i^1(t) \cdots v_i^n(t) \|

where, \( v_i^j(t) = y^j_i(t)/\bar{v}_i(t), \forall j, \delta \), and \( \bar{v}_i(t) = \sqrt{[y^1_i(t)]^2 + \cdots + [y^n_i(t)]^2} \). If all the agents have to converge to a straight line as \( t \to \infty \), then

\[ \lim_{t \to \infty} [v_i^j(t)/\bar{v}_i(t)] = \lim_{t \to \infty} [v_i^k(t)/\bar{v}_i(t)] \quad \forall i, j, \delta \tag{12} \]

Equivalently, for all \( i, j \in \{1, \ldots, n\} \) and \( \delta, \gamma \in \{1, \ldots, d\} \)

\[ \lim_{t \to \infty} [v_i^j(t)/v_i^\gamma_i(t)] = \lim_{t \to \infty} [v_i^k(t)/v_i^\delta_i(t)] = \theta_{\delta \gamma} \tag{13} \]

where \( \theta_{\delta \gamma} \) is a constant independent of time and agent identity. To prove Eq. (14), consider any one of the \( d \) dimensions, represented by \( x_i(t) \). Differentiating Eq. (9), we get

\[ \dot{x}_i(t) = \sum_{p=1}^{\tilde{n}} \sum_{q=1}^{n} n_p \sum_{r=1}^{\sigma_p + j\omega_p} \frac{a^p_{qr}}{\rho(s)} \left( x_q(0) e^{\sigma_r t + j\omega_p t} \right) + x_f \tag{14} \]

Let \( V_i = \dot{X} \), then

\[ V_i = \dot{X} = \dot{A}X = \dot{A}V_i \quad V_i(0) = AX(0) \tag{15} \]

Thus, \( V_i \) has the same dynamics as Eq. (4) and \( v_i^j(t) \) can be obtained similar to Eq. (9) as

\[ v_i^j(t) = \sum_{p=1}^{\tilde{n}} \sum_{q=1}^{n} n_p \left( \sum_{r=1}^{\sigma_p + j\omega_p} \frac{a^p_{qr}}{\rho(s)} \right) x_q(0) e^{\sigma_r t + j\omega_p t} \tag{16} \]

where \( x_q^0(t) = k_{q1}x_q(t) - x_q(t) \). From Eqs. (15) and (17), for all \( p, q, \) and \( r = n_p \), we get

\[ a^p_{qr}(\sigma_p + j\omega_p) = k_{q1}a^p_{(p+1)q} - k_{q1} \tag{17} \]

Let \( R_p = \sigma_p + j\omega_p \). Then, from Eq. (18)

\[ a^p_{qr}(\sigma_p + j\omega_p) = \prod_{i=2}^{q} [k_{qi}(R_p + k_i)] = M_{pq} \tag{18} \]

for \( q > 1 \) and \( M_{q1} = 1 \). We can rewrite \( \dot{x}_i(t) \) from Eq. (15) as

\[ \dot{x}_i(t) = \sum_{p=1}^{\tilde{n}} \sum_{q=1}^{n} \frac{a^p_{qr}}{\rho(s)} \left( M_p|Q^0_p| x_q(0) \right) e^{\sigma_r t + j\omega_p t} \tag{19} \]

where

\[ \sum_{i=1}^{n} Z_i(t)/k_i = 0 \Rightarrow \sum_{i=1}^{n} Z_i(t)/k_i = \text{const} \quad \forall t \tag{20} \]
\[ Q_{pq}(t) = \sum_{r=1}^{n} \alpha_{pq}^r [(r - 1) t^{r-1} + (\sigma_p + j \omega_p) t^{r-\nu}]. \]  

Therefore, the instantaneous slope of the trajectory of agent \( i \) in the \((y, \hat{y})\) plane is

\[ \frac{\partial}{\partial t} \frac{\dot{y}_i(t)}{\dot{y}_i(t)} = \sum_{r=1}^{\nu} \alpha_{pq}^r \sum_{q=1}^{n} M_{pq} Q_{pq}(t) y_q^0(0) \left[ \epsilon(\sigma_p - \sigma_n + j \omega) t^{\nu-1} \right]. \]

(22)

Let \( \sigma_n > \sigma_p, \forall p, p \neq m \). Thus, \( \sigma_n \) is the real part of the most positive eigenvalue of \( A \), and

\[ y_i^0(t) = \sum_{r=1}^{\nu} \alpha_{pq}^r \sum_{q=1}^{n} M_{pq} Q_{pq}(t) y_q^0(0) \left[ \epsilon(\sigma_p - \sigma_n + j \omega) t^{\nu-1} \right]. \]

(23)

As \( t \to \infty \), \( Q_{pq}(t) \) will be dominated by \( \epsilon^{-\sigma_n} t^{\nu-1}, \forall p, p \neq m \). Therefore, all the terms will go to zero as \( t \to \infty \) except for \( p=m \). Now, if \( \omega_m = 0 \) (which implies that the most positive eigenvalue is real), then Eq. (23) simplifies to

\[ \lim_{t \to \infty} [\dot{y}_i(t) / \dot{y}_i(t)] = \sum_{q=1}^{n} M_{mq} Q_{mq}(t) y_q^0(0) / \sum_{q=1}^{n} M_{mq} Q_{mq}(t) y_q^0(0). \]

(24)

From Eq. (21), \( \lim_{t \to \infty} Q_{pq}(t) = R_p = (\sigma_p + j \omega_p) \) for all \( p \). So, \( \lim_{t \to \infty} Q_{pq}(t) = R_m = \sigma_m, \) since \( \omega_m = 0 \). Hence, dividing the two numerator and denominator of Eq. (24) by \( R_m = \sigma_m \), we get

\[ \lim_{t \to \infty} [v_i(t) / u_i(t)] = \sum_{q=1}^{n} M_{mq} y_q^0(0) / \sum_{q=1}^{n} M_{mq} y_q^0(0) \]

(25)

where \( \theta_{\delta} \) is independent of time and the agent identity \( i \). It is a constant and a function of \( k_i, i = 1, \ldots, n \), and the initial positions of the agents, \( Z_0, i = 1, \ldots, n \). Similarly, we have

\[ \lim_{t \to \infty} [y_i(t) / y_i(t)] = \sum_{q=1}^{n} M_{mq} y_q^0(0) / \sum_{q=1}^{n} M_{mq} y_q^0(0). \]

(26)

Therefore, as \( t \to \infty \), \( y_i^0(t)/y_i(t) \to (y_i^0(t) - y_i^0(0))/y_i(t) \). Hence, \((y_i^0, y_i)\) is on the straight line along which the agents move. Now, to prove the converse, let \( \omega_m = 0 \), then

\[ \lim_{t \to \infty} \frac{\dot{y}_i}{\dot{y}_i} = \sum_{q=1}^{n} M_{mq} (a_{m1p} R_m e^{j \omega_m} + a_{m1p}^* R_m e^{-j \omega_m}) y_q^0(0). \]

(27)

where \( R_m^* \) is the conjugate of \( R_m \) and

\[ a_{m1p}^* = \left[ x - (\sigma_m + j \omega_m) \right] y_{i}^{0}(s) / \rho(s) \]

where \( \rho(s) = i \sigma_m + j \omega_m \). Therefore, Eq. (27) can be written as

\[ \lim_{t \to \infty} \frac{\dot{y}_i}{\dot{y}_i} = \sum_{q=1}^{n} M_{mq} R_m e^{j \omega_m} (\phi_q + \omega_m t) y_q^0(0) \]

(28)

where \( a_{m1p} R_m = R_m e^{j \omega_m} \). From the above, we see that if \( \omega_m \neq 0 \), the agents will not converge to a straight line.

Remark 1. The straight line asymptote of the trajectories (after sufficiently large time) passes through \( Z_j = [y_j^0, y_j, \ldots, y_j^T] \in \mathbb{R}^d \). We call this point the asymptote point.

Remark 2. When \( \omega_m = 0 \), the agents do not converge to a straight line. However, the direction in which the \( i \)th agent moves, after a sufficiently large \( t \), is given by Eq. (28).

Remark 3. Even though the agents converge to a straight line, the direction of motion of all the agents need not be the same. In fact, if the gain of only one agent is negative, all the agents move in the same direction; otherwise they move in two opposite directions along the straight line. The direction in which an agent \( i \) will move is determined from the sign of the coefficient, \( a_{m1p} R_m \), of \( x_i(t) \) in Eq. (9), where the subscripts and superscript have the same interpretation as in Theorem 3.

Next, we simplify the condition in Theorem 3. Instead of finding the eigenvalues of Eq. (4), we find the condition on the gains of the agents under which the agents will converge to a straight line. The direction \( \theta_{\delta} \) in Eq. (25) is a function of \( k_i, i = 1, \ldots, n \). For fixed initial positions \( Z_0, \forall i \), we can find the combinations of \( k_i, \forall i \), such that the agents ultimately move in a straight line. For this, we need the following results.

Lemma 1. If \( \alpha \neq j \beta \) is a complex conjugate root of \( f(s) \), then \( f(s) = \sum_{i=1}^{n} (s + k_i) = K \), where \( K \neq 0 \), then \( f(s) > 0 \) if \( K < 0 \) and \( f(s) < 0 \) if \( K > 0 \).

Proof. Since \( \alpha \neq j \beta \) is a complex conjugate pair of roots of \( f(s) \), \( f(\alpha \neq j \beta) = 0 \), from which \( (1/K)[(\alpha^2 + \beta^2)]^{-1} = 1 \). Now, if \( \beta \neq 0 \), we have \( (1/K)[(\alpha^2 + \beta^2)]^{-1} \rightarrow 1 \) as \( \beta \rightarrow 0 \). If \( K > 0 \), then \( K < 1 \) and \( (\alpha^2 + \beta^2) \) is a root of \( f(s) = 0 \). Hence \( f(s) = 0 \) for \( K < 0 \). For \( K > 0 \), it can be similarly shown that \( f(s) > 0 \).

If \( k = \Pi_{i=1}^{n} k_i \), \( f(s) = 0 \) and when \( K > 0 \), \( f(s) = 0 \). Let us now order the gains in an increasing sequence as \( K_i \leq K_2 \leq \cdots \leq K_n \).

Theorem 4. The \( n \) agents will converge to a straight line asymptotically if and only if the system of \( n \) agents is unstable (violates Theorem 1) and any one of the following conditions are satisfied: (i) \( \Pi_{i=1}^{n} k_i > 0 \); (ii) \( k_i > 0 \), and \( k_i > 0, i = 2, \ldots, n \); (iii) There exists \( \exists k_i \leq k_i \leq k_i \) such that \( \Pi_{i=1}^{n} k_i > 0 \) and satisfies \( q(\xi) < \Pi_{i=1}^{n} k_i < 0 \).

Proof. Since the system is unstable, at least the most positive eigenvalue has a non-negative real part. It remains to be shown that the most positive eigenvalue is also real.

Case 1. If \( \Pi_{i=1}^{n} k_i > 0 \) and \( \alpha \neq j \beta \) \( (\beta \neq 0) \) is a root of \( \rho(s) \), then \( \rho(s) = 0 \) (Lemma 1). Since \( (\alpha^2 + \beta^2) \rightarrow 0 \) as \( \alpha \rightarrow \infty \), its most positive eigenvalue has to be real and positive.

Case 2. This is a special case of Case 3 and will be proved after Case 3.

Case 3. Consider the root locus of \( f(s) = \Pi_{i=1}^{n} (s + k_i) \) parametrized by \( K \) where \( K \rightarrow 0 \). The breakaway points of \( f(s) \) is the solution of \( dq(s)/ds = d^2q(s)/ds^2 = 0 \) and let \( \xi \) be a real solution. If \( \Pi_{i=1}^{n} k_i > 0 \), the roots of \( f(s) \) that are approaching the point \( \xi \) will remain real when \( K = \Pi_{i=1}^{n} k_i \)

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The root locus exists between \((-k_2, -k_1)\). Let the breakaway point in the region \((-k_2, -k_1)\) be \(s = \xi_m\). If \(q(\xi_m) < \Pi_{i=1}^n k_i\), then \(\rho(s)\) has a real root, say, \(\xi_r\), where \(\xi_r > \xi_m\) and \(\xi_r\) is the most positive eigenvalue of \(\rho(s)\). Now, let \(\xi_1\) and \(\xi_2\) be the two real roots of \(f(s)\) approaching the breakaway point \(\xi_m\). Since \(f(s) < 0\) for \(s \in (\xi_1, \xi_2)\), from Lemma 1, \(f(s)\) cannot have any complex conjugate roots whose real part lies in \((\xi_1, \xi_2)\). Hence, the locus of complex conjugate roots cannot cross \(\xi_m\) for \(0 < K < \Pi_{i=1}^n k_i\). Therefore, \(\xi_r\) remains the most positive root of \(\rho(s)\).

**Case 2.** Here, \(\tilde{k}_1 < 0\) and \(\tilde{k}_2 > 0\). From \(f(s) = \Pi_{i=1}^n (s + k_i) - K\), the root locus will exist in the region \((-\tilde{k}_2, -\tilde{k}_1)\), which contains the root of \(\rho(s)\) at the origin. Therefore, when \(K = \Pi_{i=1}^n k_i\), \(f(s)\) will have two real roots in the region \((-\tilde{k}_2, -\tilde{k}_1)\), one at the origin and the other positive (say, \(\xi_r > 0\)), since the system is unstable. This is the only real positive root of \(f(s)\) when \(K = \Pi_{i=1}^n k_i\). As proved in Case (iii), no other complex conjugate roots will have a real part more than \(\xi_r\).

The converse is proved by contradiction. Let the agents converge to a straight line as \(t \to \infty\), i.e., the most positive eigenvalue of \(\rho(s)\) is real and positive but none of the three conditions hold. This means \(\Pi_{j=1}^n k_j < q(\xi_m) < 0\) and more than one gain is negative. Following the arguments in Case (iii) above, we conclude that the most positive eigenvalue cannot be real. Hence, the agents cannot converge to a straight line.

### 5 General Pursuit Sequence and Switching Invariance

We assumed a particular sequence in which an agent pursues another. We now show that even where the connection among the agents is changed, certain properties of the system remain unchanged, so long as the pursuit is cyclic and the gains are unchanged. The sequence in which an agent pursues another is called the pursuit sequence. The initial pursuit sequence is defined as \(C_0 = \{1, 2, \ldots, n\}\), where the first agent follows the second, the
second follows the third, and so on until the last agent follows the first. The set of all pursuit sequences is given by $Q$, which has elements that are permutations of $C_0$. When there is a change in the pursuit sequence of the agents from $C_i$ to $C_j$, it is called switching. If switching occurs a finite number of times during the process, then we call it a finite switching case.

**THEOREM 5.** The stability of the linear cyclic pursuit is pursuit sequence invariant.

**Proof.** The stability of Eq. (6) depends on the roots of $\rho(s) = \prod_{i=1}^{n} (s + k_i) - \prod_{i=1}^{n} k_i = 0$. With cyclic pursuit, $\rho(S)$ remains unchanged even when the pursuit sequence among the agents is different. Thus, stability of the system is pursuit sequence invariant.

**THEOREM 6.** Reachable point of a stable linear cyclic pursuit is pursuit sequence invariant.

**Proof.** Consider Eq. (10), which gives the coordinates of the rendezvous point. This equation is independent of the sequence of connection among the agents. The only requirement is that the pursuit should be cyclic. Hence, the rendezvous point is independent of the connectivity of the agents so long as the cyclic pursuit condition is satisfied.

**THEOREM 7.** Asymptote point of an unstable linear cyclic pursuit system, satisfying Theorem 3, is pursuit sequence invariant.

**Proof.** For the agent $i$, as $t \to \infty$, the unit velocity vector can be written from Eq. (25) as $v_i(t) = \left(\frac{v_i}{\|v_i\|}\right)[1, \theta_{i1}, \theta_{i2}, \ldots, \theta_{id}]^T$. Since $\theta_{i\delta}, \forall \gamma, \delta; \gamma \neq \delta$ depends on the connection of the agents, the unit velocity vector varies as the connection is changed. Thus, for different connections, the asymptote to which the agents converge is different. However, from Eq. (26), all the asymptotes pass through the point $Z_f$ (the asymptote point), which is independent of the connection.

The pursuit sequence invariance property discussed so far is based on the assumption that the pursuit sequence between the agents remained constant throughout time. Now, we show that even if the pursuit sequence between the agents switch during the process, some of the properties remain unchanged.

**THEOREM 8.** (Stability With Switching) Stability of the linear cyclic pursuit is invariant with finite pursuit sequence switching.

**Proof.** From Theorem 5, $\rho(S)$ in Eq. (6) is the same for all the pursuit sequences. Thus, for finite number of switchings, $\rho(S)$ will remain unchanged. Hence, the stability of the system is determined by the stability of the system after the last switching and hence remains invariant with finite switching of pursuit sequences.

**THEOREM 9.** (Reachability With Switching) Reachable point of a stable linear cyclic pursuit system is invariant with finite pursuit sequence switching.

**Proof.** Let the switching of connections between the agents occur at $t_1, \ldots, t_m$, $m < \infty$ such that $0 < t_1 < \cdots < t_m < \infty$, and the connection during $t_j < t < t_{j+1}$ is $C_j$, where $C_j \in Q$, $\forall j$. We prove the
switching invariance property by showing that the reachable point remains the same after a switch. At \( t = t_i \), let the connection among the agents be \( C_i \) and their position be \( Z_i^j, \forall i \). If there is no further switching, then the rendezvous point, from Eq. (10), is \( Z_f = (\sum_{i=1}^{n} Z_i^j/k_i)/\sum_{i=1}^{n} 1/k_i \). Let the next switching occur at \( t_j k_i \), when the position of agent \( i \) is \( Z_i^{j+1} \). For \( t = t_j k_i \), the connection is \( C_{j+1} \). If there are no more switching of connections, let the reachable point be \( Z_f \). Now, from Eqs. (10) and (11), we can write

\[
Z_f \sum_{i=1}^{n} \frac{1}{k_i} = \sum_{i=1}^{n} Z_i^{j+1} = \sum_{i=1}^{n} \frac{Z_f}{k_i} = Z_f \sum_{i=1}^{n} \frac{1}{k_i} \tag{29}
\]

This shows that \( Z_f = Z_f \). Hence, when there is one switching of connection, the rendezvous point does not change. This can be extended to a finite number of switchings to show that the reachable point remains unchanged after the final switch \( t_f \).

**Theorem 10.** (Directed Motion With Switching) Asymptote point of an unstable linear cyclic pursuit system, satisfying Theorem 3, is invariant with finite pursuit sequence switching.

Proof: From Theorem 9, we see that the point \( Z_f \) remains invariant even after a finite number of switching of connection among the agents. Theorem 7 shows that irrespective of the connection between the agents, the asymptote, along with the agents converge, passes through the point \( Z_f \). Hence, even after a finite number of switching of connections, the agents will converge to a line that will pass through \( Z_f \).

Note that the trajectory of the agents may change due to switching but the stability, reachable point, and the asymptote point remain unchanged.

**6 Simulation Results**

A system of five agents are considered with pursuit sequence \( C_0 \). The initial positions of the agents are \( S = \{ (10, -1), (7, 2), (0, 10), (-7, 5), (4, -8) \} \).

**Case 1.** Consider \( k = \{ -3, 6, 8, -10, 12 \} \), which satisfies Condition (i) of Theorem 4. The agents converge to a straight line (Fig. 1(a)). The eigenvalues of this system are \( \{-11.3096 + j2.2893, -11.3096 - j2.2893, -0.7314, 0, 10.3506\} \) and the most positive eigenvalue is real. The slope of the asymptote, calculated from Eq. (25), is \(-33.46 \text{ deg} \), which matches with the simulation result.

Moreover, it can be seen that some agents move in one direction while the others in the opposite direction.

**Case 2.** Consider \( k = \{ -3, 6, 8, 10, 12 \} \), which satisfies Condition (ii) of Theorem 4. The agents asymptotically converge to a straight line (Fig. 1(b)). The eigenvalues of this system are \( \{-15.0187, -9.6509 + j5.5039, -9.6509 - j5.5039, 0, 1.3205\} \) and the most positive eigenvalue is real. The slope of the asymptote, calculated from Eq. (25), is \(-34.9238 \text{ deg} \), which matches with the simulation.

**Case 3.** Consider \( k = \{ -3, -6, 8, -15, 12 \} \). This system does not satisfy Conditions (i) and (ii) of Theorem 4. Since \( q(\xi_m) = -7.8 \times 10^4 < \Pi_m k_i = -2.5 \times 10^4 \), Condition (iii) of Theorem 4 is satisfied and the agents converge asymptotically to a straight line (Fig. 2(a)). The eigenvalues are \( \{-12.6825, -6.1105, 0.82297, 14.5632\} \) and the slope, calculated from Eq. (25), is \(-36.08 \text{ deg} \), which matches with the simulation.

**Case 4.** Consider \( k = \{ -3, -6, -8, -10, -12 \} \). Here, none of the conditions of Theorem 4 are satisfied as \( q(\xi_m) = -132.15 > \Pi_m k_i = -1.7 \times 10^5 \). Hence, the agents do not converge asymptotically to a straight line (Fig. 2(b)). The eigenvalues are \( \{0, 5.4627 + j6.0269, 5.4627 - j6.0269, 14.0373 + j3.7501, 14.0373 - j3.7501\} \). The most positive eigenvalue is not real and hence, the system also violates the condition of Theorem 3.

**Case 5.** Consider \( k = \{ 4, 6, 8, 10, 12 \} \). The simulation results are shown in Figs. 3(a) and 3(b) for different pursuit sequences and we find that in both cases the agents converge to the same point. This is the connection invariance property.

**References**


