A note on sensor alignment in a minimal sensing and coarse actuation problem

Soumya Ranjan Sahoo, Ravi N. Banavar and Arpita Sinha

Abstract—In our work [23], we have considered the rendezvous of agents using minimal sensing and quantized control in three-dimensional space. In the present work we focus on the effect of an offset in the sensor alignment for achieving rendezvous in a two-dimensional space. A quantized control law has been proposed which allows the agents to yaw in the required direction and track its target agent. The measurement required for the proposed control law is the side from which the target moves out of the field-of-view of the pursuing agent. A Lyapunov function is chosen to find an angular range for the field-of-view and its offset, and angular speed of the agent which would guarantee rendezvous under the proposed control law. The report also includes the simulation results which match well with the predicted results.

I. INTRODUCTION

Autonomous vehicle systems have found potential applications in military operations, search and rescue, environment monitoring, commercial cleaning, material handling, and homeland security. While single vehicles performing solo missions have yielded some benefits, greater benefits will arise from the cooperation of a team of vehicles. A multi-agent system is robust to failure compared to a single agent, more efficient than individual agents in certain cases, and it is also possible to reduce the size of the individual agents and operational cost and increase system reliability. This has aroused interest in the control community in cooperative control and consensus algorithms. In [20] authors have mentioned various consensus algorithms in multi-agent coordination.

The rendezvous problem is one of the various consensus problems where all agents of a multi-agent system converge to a point at the same time. This problem has been pursued actively in the past few decades. In [1] authors have proposed a memoryless algorithm that ensures point convergence of the agents with limited visibility. This has been extended in [14] and [15] by using stop-and-go local control strategies based on relative position measurement which ensure convergence without any active communication between agents. Finite-time rendezvous algorithms with certain communication range have been presented in [13]. Cyclic pursuit problem is closely related to the rendezvous problem wherein each agent pursues its immediate neighbor on a directed cyclic graph. Pursuit curves, cyclic pursuit and stable polygons of cyclic pursuit have been addressed in [2], [5] and [21]. In [24] authors have presented a generalization of the existing cyclic pursuit results and shows that by selecting the controller gain of the agents the point of convergence can be controlled. In [18] and [19] the authors have considered cyclic pursuit with the agent following its target with an offset to its line-of-sight. The offset angle determines the type of formation. In [11] and [12] authors have discussed constant bearing cyclic pursuit of \( n \) agents in two- and three- dimensions respectively.

Sometimes the information flow between the plant and the controller gets restricted due to availability of limited communication bandwidth, security reasons or design limitations. This has motivated the use of quantized control and coarse quantized measurement of plant outputs (states). The stabilization of linear systems using quantized control and quantized measurement have been discussed in [6], [4] and [16]. In [8] the authors have presented the coarsest, least dense quantizers for state-feedback controller and estimator to stabilize a single-input-single-output linear time-invariant system. In [10], the authors have derived the coarsest quantization densities for stabilization for multiple-input-multiple-output systems in both state feedback and output feedback cases.

Minimalism means given an objective to achieve by a group of autonomous agents what is the minimum information needed to achieve the objective. Minimalism in the context of navigation in unknown environment has been explored in [26], [27], [25], [17]. In [22] authors have addressed a pursuit-evasion problem where the sensors of the agent cannot make exact depth measurements. In [9] the authors have experimentally shown that a formation of multi-agents can be achieved with local sensing and limited communication between the agents. Various sensorless manipulation tasks have been explored in [3].

In the references cited above information regarding relative position, angle or velocity is required to achieve the objective. Sometimes tasks have to be performed with minimal available data. Minimal availability of data may be due to lower bandwidth, security reasons, compact design of agents or availability of less sensors due to failure of other sensors of the agents. Focusing on minimal data helps increase the robustness of the system, results in simpler and compact design of agents, and lowers the production cost. Both quantized control and reduced sensing focus on minimal data. From the references cited above there has been a lot of work in the field of quantised control and design of controllers based on minimal data available. In [28] and [23], the authors have designed quantised control laws which
depend on minimal data available from sensors of the agents. The authors have considered agents that can move on a plane and space respectively. The agents have limited field-of-view called the windshield. Each agent tries to maintain its target in this windshield. The sensors detect whether the target is within the field-of-view or has moved out from which side of the windshield. Depending on the sensor output the agents maneuver to keep the target in the field-of-view. It has been shown that with these minimal data rendezvous can be achieved.

The agents as considered in [28] and [23] have their windshield aligned symmetrically about their forward (linear) velocity. Sometimes, providing an offset to the windshields with respect to the velocity of agents helps in achieving rendezvous in a lesser time as compared to agents in a system without any offset. In the work presented we have considered agents with an offset in the windshield with respect to their velocity. We have found sufficiency conditions on windshield angle, offset angle and angular speed of agents which guarantee rendezvous. It was observed from simulations that for a small range of offset given to the windshield of the agents under certain conditions resulted in convergence of the system in the least time.

The paper is organised as follows: Section II presents the problem addressed in the present work. Section III presents an overview of the main results discussed in this paper. Section IV and V present the sufficiency conditions on windshield angle and offset, and angular speed of agents that guarantee rendezvous for homogeneous and heterogeneous systems respectively. In Section VI simulation result for the problem has been presented and discussed. In Section VII the present work has been summarised.

II. PROBLEM FORMULATION

We assume a system of $n$ agents. The agents are assumed to be Dubin’s car vehicles [7]. They move on a plane. Each agent can move in the forward direction, and can turn left or right. There is no lateral motion. Each agent has a sensor with limited angular field-of-view with infinite range which is termed as windshield. The agent tries to maintain its target within the windshield. The windshield can sense whether the target moved out from the left or right side of the windshield. The target for the $i^{th}$ agent is the agent $(i+1)\text{ modulo } n$. The control is applied only when the target moves out of the windshield. We assume that initially all the agents have their target within their windshield. We have proposed a quantized control law based on this minimal sensing and quantized control model of the agents which guarantees rendezvous when the angular speed and windshield angle ($\phi$) of the agents satisfy certain conditions. The quantized control law takes one of the three values at all time. The vehicle model, sensors and control are now presented.

A. Vehicle Model

Let $p_i = (x_i,y_i)$ be the position of the $i^{th}$ agent in the earth-fixed frame and $\psi_i$ be its orientation.

Center line of windshield

![Image of windshield and agent](https://example.com/windshield_agent.png)

Fig. 1: Schematic of $i^{th}$ agent with offset angle $\alpha$

- Each agent has forward (linear) velocity along the body $X_b$ axis and its magnitude $v_i$ remains constant.
- The agent can turn left or right (yaw) about the body $Z_b$ axis with an angular speed $\omega$ remains constant.
- The offset of the winshield with respect to $X_b$ is the same for all the agents.

The output of the sensors actuate the controllers for

$$
\dot{x}_i = v_i \cos \psi_i , \\
\dot{y}_i = v_i \sin \psi_i , \\
\psi_i = u_i ,
$$

where

$$u_i \in \{-\omega, 0, \omega\} .$$

B. Sensors

The agents have a limited angular field-of-view called the

windshield. It is assumed here that

- The centre of the windshield has an offset of $\alpha \in (0, \pi)$ with respect to $X_b$, either to its left or right.
- For all the agents the offset is in the same direction.

The windshield has a span of $(-\phi, \phi)$ where $\phi$ is the half-angle of the windshield and $\phi \in (0, \pi)$. The sensor does not estimate any state nor give any information on the distance between agents; it just gives a discrete output based on the side of the windshield from which the target agent moves out. Let $O_i$ be the set of outputs given by the sensor of agent $i$.

$$O_i = \{-1\} \quad \text{if agent } i+1 \text{ escapes from the left side}$$

$$= \{0\} \quad \text{if agent } i+1 \text{ is in or brought back into the windshield from either side}$$

$$= \{1\} \quad \text{if agent } i+1 \text{ escapes from the right side}$$

C. Control Law

The output of the sensors actuate the controllers for necessary action. From (2) and (3) the control law $u_i$ can be expressed as

$$u_i = \omega \phi o_i .$$
As seen, the control law does not involve any history of the states nor any state estimation. In section IV-A and IV-B we find conditions on the windshield angle and angular speed of the vehicle such that the agents are able to achieve rendezvous using the proposed control law.

D. Cyclic pursuit and merging

The multi-agent system, as considered here, is represented by a directed graph $G = (\mathcal{V}, \mathcal{E})$ with the agents being at the nodes $\in \mathcal{V}$. The directed edge $e_{ij} \in \mathcal{E}(G)$ exists if agent $i$ is pursuing or communicating with or can sense agent $j$. In our case, if agent $j$ is assigned to agent $i$ then the edge $e_{ij}$ is created. Assignment of agent $i$ means the target agent which $i$ is supposed to pursue. A graph defining the assignments of the agents in the multi-agent system is called an assignment graph. For simplicity and without any loss of generality we assume that $(i + 1) \mod n$ is assigned to agent $i$. This type of assignment results in the formation of a cyclic graph. A pursuit defined by a cyclic graph is called a cyclic pursuit. Let the distance between agents $i$ and $i+1$ be denoted by $l_{i,i+1}$. Initially there are $n$ agents and hence $G$ has $n$ vertices. As agent $i$ catches up with agent $i+1$, $i$ and $i+1$ move as one entity. This is called merging. The merging operation is triggered when the distance between the pursued and the pursuer reduces to the merging radius $\rho > 0$ or less. The merging radius is the distance between the pursued and the pursuer after which they merge and move as one entity. Merging occurs if

$$e_{ii+1} \in \mathcal{E}(G), \quad l_{i,i+1} \leq \rho$$

After merging, agent $i-1$ which was pursuing $i$ starts pursuing $i+1$. The node $i$ is deleted and the edges $e_{i-1,i}$ and $e_{i+1,i}$ are deleted from $\mathcal{E}(G)$ and a new edge $e_{i-1,i+1}$ comes into effect. The number of nodes is also reduced.

E. A Lyapunov-like function

Let $V : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be a function which is defined as

$$V = \sum_{e_{ij} \in \mathcal{E}(G)} l_{ij}. \quad (5)$$

Since $V$ is the sum of distances, it will always be positive and will go to zero only when edges do not exist. $V$ chosen here is piecewise-continuous on the entire time interval. It is continuous till an agent merges with its target.

III. AN OVERVIEW OF MAIN RESULTS

We have analysed two types of systems - homogeneous and heterogeneous. In a homogeneous system, all the agents (vehicles) have identical linear speed. In a heterogeneous system, the agents have different bounded linear speeds. The agents are assumed to be in a cyclic pursuit. When the system converges $V = 0$. We thus have to ensure that agent $i$ is able to track its target and $V$ decreases with time. As shown in Fig. 2,

- $p_{i-1}$, $p_i$, $p_{i+1}$ are the positions of agents $i-1$, $i$ and $i+1$ respectively in the earth-fixed frame.
- $l_{i,i+1}$ is the line-of-sight of agent $i$.

Fig. 2: Two consecutive agents in cyclic pursuit

- $\phi_i$ is the angle between the centre of the windshield of agent $i$ and $\hat{l}_{i,i+1}$.
- $\alpha$ is the offset of the centre of the windshield of agent $i$ with respect to the velocity $\vec{v}_i$.
- $\delta_i$ is the angle between the velocity vector $\vec{v}_i$ and $\hat{l}_{i,i+1}$.
- $|\delta_i| < \phi + \alpha, i = 1, \ldots, n$.
- The angle between $l_{i-1,i}$ and $l_{i,i+1}$ is $\theta_i$.

Thus $l_{i-1,i}$ can be expressed as

$$l_{i-1,i} = -v_i \cos(\theta_i + \delta_i) - v_{i-1} \cos(\delta_{i-1}) \quad (6)$$

Summing (6) over all $i$ we get

$$V = \sum_{i=1}^{n} -v_i \left(\cos(\delta_i) + \cos(\theta_i + \delta_i)\right) \quad (7)$$

Assuming all agents to have unit speed, (7) becomes

$$V = -\sum_{i=1}^{n} \left(\cos(\delta_i) + \cos(\theta_i + \delta_i)\right) \quad (8)$$

As mentioned earlier $V$ is piece-continuous on the entire time interval. For $V$ to decrease we have to ensure that $V < 0$ when $V$ is continuous, and $V$ decreases at the points of discontinuity. For both cases, we have found sufficient conditions on $\phi + \alpha$ and $\omega$ which when satisfied together guarantees rendezvous. The angular speed has a lower bound and is independent of the number of agents in the system. It, however, depends on the speed of the agents in the system and the merging radius $\rho > 0$. The windshield angle $\phi$ has an upper bound and depends on the number of agents in the systems and the offset angle $\alpha$. With the increase in the number of agents in the system, the upper bound on $\phi$ drops and the windshield becomes narrower. The conditions on $\phi + \alpha$ and $\omega$ have been discussed in the following sections.

IV. CONDITION FOR RENDEZVOUS: HOMOGENEOUS CASE

In this section we have discussed the conditions on windshield angle and angular speed of agents. All the agents in the system have same time-invariant linear speed.

A. Condition on the windshield angle

A necessary condition for rendezvous to occur is that $V$ is strictly less than zero. Note that $V$ is discontinuous at the instants of merging. Let the instants when merging occurs be $t_k$, $k = 1, 2, \ldots$. These instants are called switching instants. As the number of agents in the multi agent system is finite the switching sequence $\{t_k\}$ is also finite. For a $n$ agent system
the maximum number of switchings that we can have is \( n - 1 \). However, \( V \) is continuous between two consecutive instants of merging \([t_k,t_{k+1})\). Thus, \( V \) is a piecewise continuous function. Hence, for \( V \) to be monotonically decreasing

- The switching sequence \( \{t_k\} \) should be finite.
- \( V(t) \) has to be strictly less than zero in the interval \( t \in [t_k,t_{k+1}) \).
- \( V(t_k+1-\varepsilon) > V(t_{k+1}+\varepsilon) \), where \( \varepsilon \) is a very small positive number in \( \mathbb{R}^+ \).

**Theorem IV.1.** Unit speed cyclic pursuit of \( n \) agents with kinematics given by (1) will rendezvous if the agents maintain their targets within the windshield and the windshield angle \( \phi \) satisfies

\[
0 < \phi + \alpha < \left\{ \begin{array}{l}
\frac{\pi}{2} \\
\operatorname{min}\{\frac{\pi}{n},\cos^{-1}\left(\frac{n-1}{n}\right)\}
\end{array} \right\} \text{ for } n \geq 3.
\]

(9)

**Proof.** The proof of this theorem is divided into four parts. First we find a condition on \( \phi \) such that \( V < 0 \) within \([t_k,t_{k+1})\). Next we prove \( V(t_k+1-\varepsilon) > V(t_{k+1}+\varepsilon) \), where \( \varepsilon \) is a very small positive number in \( \mathbb{R}^+ \). To prove \( V < 0 \) within \([t_k,t_{k+1})\) we rule out the condition \( \phi \in [\pi/n,\pi] \).

Next we find a condition on \( \phi \) for \( n = 2 \). Then we find the condition on \( \phi \) for \( n \geq 3 \), thus proving the whole theorem.

(1) **Rule out** \( \phi + \alpha \in [\pi/n,\pi] \)

**Lemma IV.2.** For any integer \( n \geq 2 \), the windshield angle \( \phi + \alpha = \pi/n \) permits trajectories for which \( V = 0 \).

**Proof.** Refer [23].

(2) **Condition on** \( \phi + \alpha \) **for** \( n = 2 \)

Now we find a condition on \( \phi \) such that \( V < 0 \). When \( n = 2 \), \( \delta_1 = \delta_2 \). For \( V < 0 \), as in [23], \( \phi + \alpha < \pi/2 \).

(3) **Condition on** \( \phi + \alpha \) **for** \( n \geq 3 \)

From **Lemma IV.2** and the fact that \( |\delta_i| < \phi + \alpha \) we have,

\[
0 < |\delta_i| < \phi + \alpha < \frac{\pi}{n}.
\]

(9)

So,

\[
-\frac{\pi}{n} < -\phi + \alpha < \delta_i < \phi + \alpha < \frac{\pi}{n}.
\]

(10)

For any closed polygon we can always consider the smaller angle as interior angle. So

\[
0 < \theta_i < \pi.
\]

(11)

From (10) and (11) we have

\[
-\frac{\pi}{n} < -\phi + \alpha + \theta_i < \delta_i + \theta_i < \phi + \alpha + \theta_i < \pi + \frac{\pi}{n}.
\]

(12)

\( V \) can be written as the sum of two functions, \( f \) and \( g \) as follows

\[
f := -\sum_{i=1}^{n} \cos(\theta_i + \delta_i)
\]

(13)

\[
g := -\sum_{i=1}^{n} \cos \delta_i
\]

(14)

Note that \( -n \leq f, g \leq n \). We need \( V \) to be negative definite. From (10) we have \( g < 0 \). Now for \( V \) to be negative definite \(-g > f \) must be true for all values of \((\theta_i + \delta_i) \forall i\). \( f \) can be positive or negative depending on the value of \((\theta_i + \phi_i)\). For negative value of \( f \) it is guaranteed that \( V < 0 \). So we need find conditions such that \( V < 0 \) even when \( f > 0 \). Consider these two mutually disjoint sets that satisfy (12).

\[
\Theta^+ = \{ (\theta_i, \ldots, \theta_n) \mid i, (\theta_i + \delta_i) \in (-\pi/n, \pi/2) \}
\]

(15)

\[
\Theta^- = \{ (\theta_i, \ldots, \theta_n) \mid i, (\theta_i + \delta_i) \in [\pi/2, \pi + \pi/n] \}
\]

(16)

As shown for the three-dimensional case [23],

\[
f < n - 1
\]

(17)

Now the behavior of \( f \) has to be analysed in \( \Theta^- \).

**Lemma IV.3.** Unit speed cyclic pursuit of \( n \) agents satisfying (12) has the property that the function \( f(\Theta, \Delta) \) has a single stationary point in \( \Theta^- \).

**Proof.** Refer [23].

**Lemma IV.4.** The stationary point is a point of maxima in \( \Theta^- \) and

\[
f_{\max} \leq -n \cos \left( \frac{(n-2)\pi + n(\phi + \alpha)}{n} \right)
\]

(18)

**Proof.** Refer [23].

**Lemma IV.5.** For \( n \) agents in cyclic pursuit with unit speed, \( f(\Theta, \Delta) \) satisfies

\[
f(\Theta, \Delta) \leq \max \left\{ n - 1, -n \cos \left( \frac{(n-2)\pi + n(\phi + \alpha)}{n} \right) \right\}
\]

(19)

**Proof.** \( f \) is a continuous function in \( \Theta^+ \cup \Theta^- \). From (17) and (18), (19) follows. In continuation of the proof of **Theorem IV.1**, for \( V < 0 \), \( -g \) has to be greater than \( f \). To ensure this \( -g \) has to be greater than max \( \{ n - 1, -n \cos \left( \frac{(n-2)\pi + n(\phi + \alpha)}{n} \right) \} \). As for the three-dimensional case discussed in [23], for \( n \geq 3 \), and \( V \) to be less than zero, we should have

\[
\phi + \alpha < \min \left\{ \cos^{-1} \left( \frac{n-1}{n} \right), \frac{\pi}{n} \right\}
\]

(20)

As \( V < 0 \) between two consecutive switching instants, \( V \) will always decrease. So the interagent distances decrease. At some time \( t_{k+1} \), there will be at least two agents, \( j \) and \( j + 1 \), the distance within which will be less than or equal to the merging radius, \( \rho \). Now we prove that \( V \) decreases when merging occurs to complete the proof.

(4) **V decreases after merging occurs:**

Let \( t_{k+1} \) be the instant when agents \( i \) and \( i + 1 \) merge. Consider a small interval \([t_{k+1} - \varepsilon, t_{k+1} + \varepsilon] \), where \( \varepsilon \) is a
positive real number and $\varepsilon << 1$. At $t_k+1 - \varepsilon$

$$V(t_{k+1} - \varepsilon) = \sum_{j=1, (j \neq i-1)}^n l_{j,i}(t_{k+1} - \varepsilon) + l_{i,j}(t_{k+1} - \varepsilon)$$

At $t_{k+1} + \varepsilon$

$$V(t_{k+1} + \varepsilon) = \sum_{j=1, (j \neq i-1)}^n l_{j,i}(t_{k+1} + \varepsilon) + l_{i,j}(t_{k+1} + \varepsilon).$$

From the triangle inequality we have,

$$l_{i,j}(t_{k+1} - \varepsilon) + l_{j,i}(t_{k+1} - \varepsilon) > l_{i,j}(t_{k+1} - \varepsilon). \quad (21)$$

$$\lim_{\varepsilon \to 0^+} (l_{i,j}(t_{k+1} - \varepsilon) + l_{j,i}(t_{k+1} - \varepsilon)) = l_{i,j}(t_{k+1}) + \rho$$

$$\lim_{\varepsilon \to 0^+} l_{i,j}(t_{k+1} + \varepsilon) = l_{i,j}(t_{k+1}) \quad (22)$$

Hence, $V(t_{k+1} + \varepsilon) < V(t_{k+1} - \varepsilon)$. Thus, $V$ decreases when the agents merge.

**B. Condition on angular speed**

**Theorem IV.6.** For the $n$ agent system with kinematics given in (1), $p > 0$ and $\nu \neq 0$ and $\omega > \frac{\nu}{p}$ is sufficient for each agent to track its target.

**Proof.** The proof of the lemma is similar to the proof presented in [23]. The small time interval $\Delta t$ can be expressed as

$$\Delta t \geq \frac{\rho}{2\nu} \cos (\phi + \alpha) \quad (25)$$

In the time interval $\Delta t$ the distance travelled by agent $i+1$ is maximum when it moves along $AP$ as compared to the distance moved by it in any other direction in the same time.

In such a scenario, to bring back agent $i+1$ into the field-of-view of agent $i$ the windshield boundary $OP$ should coincide with $OP'$ in a time less than or equal to time interval $\Delta t$. So

$$\omega \Delta t \geq \frac{\pi}{2} - (\phi + \alpha) \quad (26)$$

To satisfy (26) and from (25) the following must be true.

$$\frac{\omega \Delta t}{2\nu} \cos (\phi + \alpha) \geq \frac{\pi}{2} - (\phi + \alpha)$$

$$\omega \geq \frac{\pi - (\phi + \alpha)}{(2\nu \cos (\phi + \alpha))} \quad (27)$$

The numerator of $\frac{\pi - (\phi + \alpha)}{(2\nu \cos (\phi + \alpha))}$ decreases at a rate faster than the denominator for $(\phi + \alpha)$ varying from $0$ to $\pi/2$. So $\frac{\pi - (\phi + \alpha)}{(2\nu \cos (\phi + \alpha))}$ is maximum when $(\phi + \alpha) = 0$. So

$$\omega \geq \frac{\pi \nu}{\rho} \quad (28)$$

If the agents have their windshield angle $\phi$ and angular speed $\omega$ such that condition (9) and (28) are satisfied then rendezvous is guaranteed.

**Remark:** When the offset angle $\alpha$ is set to zero, the bounds on the windshield angle $\phi$ is same as the bounds in [28]. When a non-zero offset is given to the windshield, the range over which $\phi$ can vary decreases. With larger offset angle the windshield becomes narrower. It has been observed from simulations that over a certain range of offset angles the time taken for rendezvous is less than the time taken with no offset.

**V. CONDITION FOR RENDEZVOUS: HETEROGENEOUS AGENTS**

Let each agent have a different speed. Consider the kinematic model of agent $i$ to be (1) where $v_i \neq v_j$ where $j$ is any other agent of the system. However, $v_i$ is bounded. So

$$0 < v_{\text{min}} \leq v_i \leq v_{\text{max}} < \infty. \quad (29)$$

We consider the idea of merging in a slightly different way than the homogeneous multi-agent system. When the distance between agents $i$ and $i+1$ becomes less than the merging radius, $\rho$, the agent with higher speed merges with the other agent. Consider (5) once again as the Lyapunov function for this multi-agent system.
A. Condition on windshield angle

The rate at which the distance between agent $i-1$ and $i$ changes is given by

$$l_{i-1,i} = -v_i \cos \delta_i - v_i \cos (\theta_i + \delta_i)$$  

(30)

Summing (30) over all $i$ we get

$$V = -\sum_{i=1}^{n} v_i (\cos (\delta_i) + \cos (\theta_i + \delta_i))$$  

(31)

**Theorem V.1.** Unit speed cyclic pursuit of $n$ agents with kinematics given by (1) will rendezvous if the agents maintain their targets within the windshield and the windshield angle $\phi$ satisfies

$$0 < \phi + \alpha < \cos^{-1} \left( \frac{n - 1 + \cos \frac{\pi}{n}}{n} \right).$$  

(32)

Using Lemma IV.2 we can rule out $(\phi + \alpha) \in [\pi/n, \pi]$. Now we find a condition on $(\phi + \alpha)$ such that $V < 0$. $V$ can be written as the sum of following functions

$$f := -\sum_{i=1}^{n} v_i \cos (\theta_i + \delta_i)$$  

(33)

$$g := -\sum_{i \neq k} v_i \cos \delta_i$$  

(34)

Note that $-f, g \leq n$. We need $V$ to be negative definite. From (10) we have $g < 0$. Now for $V$ to be negative definite $-g > f$ must be true for all values of $(\theta_i + \delta_i) \forall i$. Consider two mutually disjoint sets $\Theta^+$ and $\Theta^-$ as described in (15) and (16). Now we find the maximum value that $f$ can achieve in these two sets.

(1) $f$ in the set $\Theta^+$

Let for $i = k$, $(\theta_k + \delta_k) \in (-\pi/n, \pi/2)$. So

$$\cos (\theta_k + \delta_k) > 0$$

$$-v_i \cos (\theta_k + \delta_k) < 0$$

Now,

$$f = \sum_{i=1}^{n} v_i \cos (\theta_i + \delta_i)$$

(35)

$$-\sum_{i \neq k} v_i \cos (\theta_i + \delta_i) - v_k \cos (\theta_k + \delta_k)$$

< $\sum_{i=1}^{n} v_i + 0 \left[ \sum_{i=1}^{n} v_i \cos (\theta_i + \delta_i) < \sum_{i \neq k}^{n} v_i \right]$

< $(n - 1)v_{\max}$

So, in the set $\Theta^+$

$$f < (n - 1)v_{\max}$$  

(35)

Now we find the maximum value of $f$ in the set $\Theta^-$. We state a lemma similar to Lemma IV.3.

**Lemma V.2.** A bounded speed cyclic pursuit of $n$ agents satisfying (12) has the property that the function $f(\Theta, \Delta)$ has a single stationary point in $\Theta^-$. 

**Proof.** The proof is similar to the proof stated for Lemma III.3 [23]. However, $f_{\text{mod}} = f + \lambda H$

(36)

where $\lambda$ is the Lagrange multiplier.

$$H := -\sum_{i=1}^{n} \theta_i - (n - 2)\pi + \beta^2$$

where $\beta \in \mathbb{R}$ is termed a slack variable.

$$v_i \sin (\theta_i + \delta_i) = \lambda \forall i$$  

(37)

Also we have that there exists a $i$ such that $(\theta_i + \delta_i) \in \left[ \frac{\pi}{2}, \frac{\pi - 1}{n} \pi \right]$. So, we have

$$v_i \sin (\theta_i + \delta_i) = \lambda > 0 \forall i$$

(38)

There is one $(\theta_i + \delta_i)$ for each $i$ such that $\sin (\theta_i + \delta_i) = \frac{\lambda}{v_i}$ where $(\theta_1, \ldots, \theta_n) = \Theta^-$ the stationary point. Thus, $f(\Theta, \Delta)$ has a single stationary point in $\Theta^-$. 

**Lemma V.3.** The stationary point is a point of maxima and

$$f_{\text{max}} < (n - 1 + \cos \frac{\pi}{n})v_{\max}$$  

(38)

**Proof.** The first derivative of $f_{\text{mod}}$ is given by

$$\nabla_{\theta_1, \ldots, \theta_n, \beta, \lambda} f_{\text{mod}} =$$

$$v_1 \sin (\theta_1 + \delta_1) - \lambda, \ldots, v_n \sin (\theta_n + \delta_n) - \lambda, -2\lambda \beta,$$

$$-\sum_{i=1}^{n} \theta_i + \beta^2 - (n - 2)\pi \right)$$

We now compute the second derivative of $f_{\text{mod}}$ at the stationary point to characterize a maxima or minima.

$$\nabla_{\theta_1, \ldots, \theta_n, \beta, \lambda}^2 f_{\text{mod}} (\theta_i + \delta_i, \nu_i) =$$

$$
\begin{bmatrix}
 v_1 \cos (\xi_1) & 0 & \cdots & 0 & 0 & 0 & -1 \\
 0 & v_2 \cos (\xi_2) & \cdots & 0 & 0 & 0 & -1 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \cdots & v_n \cos (\xi_n) & 0 & 1 & -1 \\
 -1 & -1 & \cdots & -1 & 0 & 0 & 1
\end{bmatrix}
$$

where $\xi_i = (\theta_i + \delta_i), \forall i = 1, \ldots, n$. All the principal minors of $-\nabla_{\theta_1, \ldots, \theta_n, \beta, \lambda}^2 f_{\text{mod}} (\theta_i + \delta_i, \nu_i)$ are positive. So $-\nabla_{\theta_1, \ldots, \theta_n, \beta, \lambda}^2 f_{\text{mod}} (\theta_i + \delta_i, \nu_i)$ is positive definite and hence, $\nabla_{\theta_1, \ldots, \theta_n, \beta, \lambda}^2 f_{\text{mod}} (\theta_i + \delta_i, \nu_i)$ is a negative definite matrix. Thus, the stationary point is a point of maxima. Now,

$$f = -\sum_{i=1}^{n} v_i \cos (\theta_i + \delta_i)$$

$$= -\sum_{i \neq k}^{n} v_i \cos (\theta_i + \delta_i) - v_k \cos (\theta_k + \delta_k)$$

< $-\sum_{i \neq k}^{n} v_i \cos (\theta_i + \delta_i) - v_k \cos (\frac{n - 1}{n} \pi)$$

< $\sum_{i \neq k}^{n} v_i + v_k \cos (\frac{\pi}{n})$

< $(n - 1)v_{\max} + v_{\max} \cos (\frac{\pi}{n})$.

**Lemma V.4.** For $n$ agents in cyclic pursuit with bounded speed, $f(\Theta, \Delta)$ satisfies

$$f(\Theta, \Delta) < (n - 1 + \cos (\pi/n))v_{\max}$$  

(39)
Proof. $f$ is a continuous function in $\Theta^+ \cup \Theta^-$. From (35) and (38), we have

$$f < \max \left\{ (n-1)v_{\max}, (n-1 + \cos \frac{\pi}{n})v_{\max} \right\} \quad (40)$$

From (40), (39) follows. ■

In continuation of the proof of Theorem V.1, for $\dot{V} < 0$, $-g$ has to be greater than $f$. To ensure this $-g$ has to be greater than $(n-1 + \cos \frac{\pi}{n})$. So,

$$-g > (n-1 + \cos \frac{\pi}{n})$$

From (34) we have,

$$\sum_{i=1}^{n} v_i \cos (\delta_i) > (n-1 + \cos \frac{\pi}{n}). \quad (41)$$

From (10) $-\pi/n < -\,(\phi + \alpha) < \phi_i < (\phi + \alpha) < \frac{\pi}{n}$. So, $\cos (\phi + \alpha) < \cos \delta_i$. So, to satisfy (41) for all $\delta_i$’s the following must be true.

$$ncos \,(\phi + \alpha) > (n-1 + \cos \frac{\pi}{n}) \quad \Rightarrow \quad \cos (\phi + \alpha) > (n-1 + \cos \frac{\pi}{n})$$

Hence, for $\dot{V}$ to be less than zero, we should have

$$\phi + \alpha < \cos^{-1} (n-1 + \cos \frac{\pi}{n}). \quad (42)$$

■

B. Condition on angular speed

Now we find a condition on $\omega$ such that the agents are able to track their targets.

Theorem V.5. Given a system of $n$ agents with kinematics (1), $\rho > 0$ and $v \neq 0$, a sufficient for each agent to track its target is $\omega > \frac{v_{\max}}{\rho}$.

Proof. Consider an agent with linear speed $v_{\max}$ which is pursuing a target with linear speed $v_{\max}$. The proof in this case is similar to the proof stated for Theorem IV.6. ■

VI. SIMULATIONS FOR RENDEZVOUS WITH OFFSET

We considered a “five-agent” system. The agents start from random points on a plane with the respective target agent inside the field-of-view. The forward speed for each agent is 10 units/second and $\rho = 0.15$. We assume that initially each agent has been assigned its target in order and has the target in its field-of-view. Since $n = 5$, the limits on $\phi$ and $\omega$ are $\frac{\pi}{5}$ and $209.4$ respectively. Consider $\phi = 0.2 \frac{\pi}{5}$ and $\omega = 210$ rad/second. We find the agents converge to a point for $\alpha = 0.3 \frac{\pi}{5}$ with the offset of the windshield with respect to the agent’s velocity in anti-clockwise and clockwise directions. Fig. 5 and 6 illustrate the simulation result for clockwise offset of the windshield.

Fig. 5: Trajectory of five agents with $\phi = 0.2 \frac{\pi}{5}$, $v = 10$, $\omega = 210$, $\alpha = 0.3 \frac{\pi}{5}$ and clockwise offset.

Fig. 6: Total distance ($V$) and individual distance between consecutive agents.

Fig. 7 and 8 illustrate the simulation result for anti-clockwise offset of the windshield.

Fig. 7: Trajectory of five agents with $\phi = 0.2 \frac{\pi}{5}$, $v = 10$, $\omega = 210$, $\alpha = 0.3 \frac{\pi}{5}$ and anti-clockwise offset.

Fig. 8: Total distance ($V$) and individual distance between consecutive agents.

Fig. 9 and 10 illustrate the divergence of agents when (9) is violated.
The offset angle for which the time taken is less is dependent on initial conditions of the agents of the system. It has also been observed that for the same initial conditions of the agents while keeping the value of $\rho$, $v$, $\omega$ same, the range of $\alpha$ for which the time taken for rendezvous is less is the same for the whole range of allowable $\phi$. See Fig. 12

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig12.png}
\caption{The range of $\alpha$ remains same for different values of $\phi$.}
\end{figure}

VII. SUMMARY

In the present work, agents with simple kinematics has been considered. They can cruise with constant linear speed, and can turn left and right with constant angular speed. There is no linear motion in the lateral direction. Each agent has an angular field-of-view and tries to maintain its target within its view. The windshield of each agent has an offset with respect to the linear velocity. The windshield is not symmetric about the velocity vector of the agent. We have shown that these agents with constraints on their motion, having minimal sensor data, quantized control, and an offset in the windshield, can achieve rendezvous without any coordinates information, state estimation or communication between them.

By analysing the geometry of the cyclic pursuit we have obtained sufficient conditions on the windshield angle and angular speed of the agents which ensure the convergence of the multi-agent system. Our simulations support the bounds derived in the main theorems. It was also observed that given the initial position and orientation of agents, linear speed, angular speed, and windshield angle there exists a small range of $\alpha$ (offset) that ensures the convergence of the multi-agent system happen in least time.

REFERENCES


