A Control Scheme to Achieve Coverage Using Unicycles

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Abstract—In this paper, a simple range-based control strategy has been proposed for a unicycle based mobile robot. Under the control strategy, the robot covers the area surrounding a stationary target in an annular pattern. The conditions on the control strategy have been analysed such that starting from any initial condition, the robot can always achieve the coverage of any annulus when either the maximum or the minimum radius or both have been specified. Simulations have been performed for the proposed scheme and the mathematical results have been validated.

Index Terms—Area coverage, unicycle robot.

I. INTRODUCTION

The problem of autonomous coverage of an area by a mobile robot has drawn wide interest among researchers around the world. Autonomous area coverage has several applications like farm irrigation, vacuum cleaning, foraging, minefield de-mining and so on. Apart from these, it also finds more critical applications like surveillance of a target point which can be a building, tower, etc.

The main aim in coverage problem is to guide a mobile robot such that it covers a desired area in an autonomous manner. The desired area could be known or unknown depending on the application. A vast literature exists which addresses the coverage problem. There are several algorithms which are used to cover unknown areas. A lot of work has been focussed on map based coverage techniques (see [1] - [4]) in which the area of interest is divided into cells and then values are assigned to the cells based on the presence of free space or obstacles.

Zelinsky et al.[1] use maps for coverage but with a different approach. The solution ensures complete coverage by the use of a distance transform path planning methodology. The solution was simulated and implemented on an autonomous robot called Yamabico to give satisfactory performance. Wong et al.[2] use topological maps to address the problem and their results are verified by simulation showing that over 99% of the surface is covered. In topological maps, the environment is represented as graphs where landmarks are nodes and edges represent the connectivity between the landmarks. The algorithm achieves coverage with single mobile agent. Rutishauser et al.[3] solve the problem of collaborative coverage using a swarm of networked miniature robots again by the use of grid based methods. Stachniss et al.[4] introduced the concept of coverage maps where each cell of a given grid corresponds to the amount of a cell which is covered by an obstacle; the coverage maps are improvement of occupancy grids which are based on the assumption that each cell is either occupied or free. The model presented in the paper allows updation of the coverage maps upon input obtained from sensors. For a multi-agent system, the problem of coverage, addressed by Batavia et al.[5], has been solved with high accuracy for a semi-structured environment. The approach has the advantage of an operator driving the outline of a desired coverage area as input to a coverage generation algorithm.

Acar et al.[6] have solved the coverage problem by the decomposition of the environment, by the use of voronoi diagrams.

There exist several surveillance based systems which make the use of the wireless sensor networks to achieve coverage of an area. The sensor nodes are programmed to behave autonomously. Lai et al.[7] divide the deployed sensors into disjoint subsets of sensors, or sensor covers such that each sensor covers every target and works in turn. Dhillon et al.[8] and Wang et al.[9] have achieved average coverage, but more emphasis has been laid on the placement of sensors for effective coverage; in [8] the sensor nodes have been assumed to be fixed. A surveillance system based on wireless sensor networks has been discussed by Yan et al.[10]. The system provides a differentiated surveillance system which deploys sensors nodes in appropriate areas depending on the degree of surveillance required. But, the wireless sensor networks suffer from the problem of energy requirements which increase with the number of sensor nodes used in the network.

In this paper, we have proposed control scheme which uses only range information in order to achieve area coverage. The
The target has been assumed to be stationary. The coverage area is chosen to be annular denoted by maximum and minimum radii, $R_{\text{max}}$ and $R_{\text{min}}$ as shown on Fig. 1. This is because even any irregular shaped boundary can be covered by the annulus. Preliminary work has been done by Tripathy et al. in [11] and [12] where it has been shown that the mobile robot can always achieve coverage starting from any initial conditions. In this paper, we have shown that,

- There always exist controller gains in order to reach a desired $R_{\text{max}}$ from specified initial conditions.
- There always exist controller gains in order to reach a desired $R_{\text{min}}$ from specified initial conditions.
- Any desired annular with specified $R_{\text{max}}$ and $R_{\text{min}}$ can be covered using appropriate controller gains.

The paper has been organised as follows. Section II gives the mathematical analysis of control scheme. In Section III, the constraints on the controller gains have been specified in order to generate a desired pattern with either $R_{\text{max}}$ or $R_{\text{min}}$ or both specified. Section IV validates the theoretical results obtained and finally, Section V concludes the paper.

II. ANALYSIS

The paper discusses the problem of guiding a mobile robot so that it can cover the area surrounding a stationary target. The mobile robot has been assumed to be a unicycle with the kinematics given as:

$$\dot{x} = v \cos \alpha(t), \quad \dot{y} = v \sin \alpha(t), \quad \dot{\alpha} = u(t)$$

(1)

where $(x(t), y(t))$ are the instantaneous position coordinates of the mobile robot, $v$ is the speed of the robot which has been to be constant, $\alpha(t)$ is the heading angle of the robot and finally, $u(t)$ is the control strategy which is input to the robot and is given by,

$$u(t) = \eta r.$$

(2)

Here, $\eta \in \mathbb{R} \setminus \{0\}$ is the controller gain and $r$ is the instantaneous line of sight (LOS) distance of the robot from a virtual target point. $\eta > 0$ always makes the robot turn counterclockwise while $\eta < 0$ makes it turn clockwise. The two dimensional engagement geometry between the robot and the virtual target point has been shown in Fig. 2. The target point is denoted as $T$ and $\theta$ is the LOS angle with the reference. The control strategy given in (2) generates annular patterns which is shown in Fig. 1. The pattern can be bounded by maximum and minimum radii, denoted as $R_{\text{max}}$ and $R_{\text{min}}$, respectively, throughout the paper. Consider Fig. 2, the LOS between the agent and the target can be characterised as $v_r = \dot{r} = -v \cos(\alpha - \theta)$ and $v_\theta = \dot{\theta} = -v \sin(\alpha - \theta)$. Let, $\phi = \alpha - \theta$, then, $\dot{\phi} = \eta r + \frac{3v_r \sin \phi}{\eta}$. Now, by computing $\frac{d}{dr}$ and re-arranging gives: $(v_r^2 - v_\theta) dr + v_r \cos \phi \, d\phi = 0$. On solving the integration, we establish a relationship between $r$ and $\phi$ as,

$$r^3 + \frac{3v_r \sin \phi}{\eta} = K$$

(3)

where $K$ is determined from the robot’s initial position coordinates, $r_0(= r(0))$ and $\phi_0(= \phi(0))$ as,

$$K = \left(\frac{3v_r \sin \phi_0}{\eta} + r_0^3\right).$$

(4)

**Proposition 1:** For any initial condition, $K \in [-\sqrt{\frac{3v_r}{\eta}}, \infty)$.

**Proof:** The range of $K$ can be calculated from (4).

Assuming $r_0 > 0$ always.

$$\frac{\partial K}{\partial r_0} = 3r_0^2 + \frac{2v_r}{\eta} r_0 \sin \phi_0 = 0$$

$$\frac{\partial K}{\partial \phi_0} = -\frac{3v_r}{\eta} r_0 \cos \phi_0 = 0$$

gives two inflexion points as:

1) $r_0 = 0$

2) $r = \sqrt{\frac{2}{\eta}}$ and $\phi_0 = \frac{3\pi}{2}$ for $\eta > 0$ and $\phi_0 = \frac{\pi}{2}$ for $\eta < 0$.

From the Hessian matrix is given by,

$$H = \begin{pmatrix}
6r_0 & \frac{3v_r \cos \phi_0}{\eta} \\
\frac{3v_r \cos \phi_0}{\eta} & -\frac{3v_r}{\eta} r_0 \sin \phi_0
\end{pmatrix}$$

we find that $r_0 = 0$ corresponds to a saddle point. For the second inflexion point, $H$ is positive definite and hence, $K$ is minimum. The minimum value of $K = -2\sqrt{\frac{3v_r}{\eta}}$. The maximum value of $K$ tends to infinity as $r_0$ goes to infinity. Hence, proved.

**Lemma 1:** $R_{\text{max}}$ and $R_{\text{min}}$ occur when $\phi$ is

$$\begin{array}{c|c|c|c|c}
\phi & \eta > 0 & \eta > 0 & \eta < 0 & \eta < 0, \\
R_{\text{max}} & 3\pi/2 & 3\pi/2 & \pi/2 & \pi/2 \\
R_{\text{min}} & \pi/2 & 3\pi/2 & 3\pi/2 & \pi/2
\end{array}$$

**Proof:** Since $\dot{r} = -v \cos \phi$, $r$ will have its maximum and minimum value when $\dot{r} = 0$. This implies $\phi \in \{\pi/2, 3\pi/2\}$ since $\phi \in [0, 2\pi)$. Consider the following cases:

Case 1 ($c > 0, \phi = 3\pi/2$ or $c < 0, \phi = \pi/2$): LHS of (3) can be represented as

$$f(r) = r^3 + \frac{3v_r}{|c|}$$

(5)
which is a depressed cubic equation with $f(0) = 0$, $\lim_{r \to \infty} f(r) = \infty$ and $f'(0)$ changes sign at $r = \sqrt{\frac{v}{|c|}}$. A plot of $f(r)$ is given in Fig. 3. For $f(r) = K$, there is one real positive root (say $r_f$), when $K > 0$ and two real positive roots ($r_{f1}, r_{f2}$), when $0 \geq K \geq K_{\text{min}}$. Using $f'(r) = 0$, $K_{\text{min}} = -2\sqrt{\frac{v^3}{|c|^2}}$. This is the minimum value $K$ can have for any initial condition which can be validated using (4).

Case 2 ($c > 0, \phi = \pi/2$ or $c < 0, \phi = 3\pi/2$): Similarly, the LHS of (3) can be represented as

$$g(r) = r^3 + \frac{3v r}{|c|}$$ \hspace{1cm} (6)

with $g(0) = 0$, $g'(0) > 0$ and $\lim_{r \to \infty} g(r) = \infty$, which is plotted in Fig. 3. $g(r) = K$ has only one real positive root ($r_g$), when $K > 0$ and no real positive roots when $K \leq 0$. It is trivial to show that $r_f > r_g$ and hence the proof follows.

Theorem 1: For a given initial condition, $(r_0, \phi_0)$, a robot with kinematics (1) and control (2) covers an annular pattern with

$$R_{\text{max}} = 2\sqrt{\frac{v}{|c|}} \cos \left(\frac{1}{3} \cos^{-1} \left(\frac{1}{2} \sqrt{\frac{|c|^2}{v^2}}\right)\right)$$ \hspace{1cm} (7)

$$R_{\text{min}} = 2\sqrt{\frac{v}{|c|}} \cos \left(\frac{1}{3} \cos^{-1} \left(\frac{1}{2} \sqrt{\frac{|c|^2}{v^2}} + \frac{4\pi}{3}\right)\right)$$ \hspace{1cm} (8)

for $K \in [-2\sqrt{\frac{v^3}{|c|^2}}, 0]$.

$$R_{\text{max}} = 2\sqrt{\frac{v}{|c|}} \cos \left(\frac{1}{3} \cos^{-1} \left(\frac{K}{2} \sqrt{\frac{|c|^2}{v^2}}\right)\right)$$ \hspace{1cm} (9)

$$R_{\text{min}} = \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} + \frac{v^3}{|c|^2}}\right)^{\frac{1}{3}} + \left(\frac{K}{2} - \sqrt{\frac{K^2}{4} + \frac{v^3}{|c|^2}}\right)^{\frac{1}{3}}$$ \hspace{1cm} (10)

for $K \in [0, 2\sqrt{\frac{v^3}{|c|^2}}]$.

$$R_{\text{max}} = \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} - \frac{v^3}{|c|^2}}\right)^{\frac{1}{3}} + \left(\frac{K}{2} - \sqrt{\frac{K^2}{4} - \frac{v^3}{|c|^2}}\right)^{\frac{1}{3}}$$ \hspace{1cm} (11)

$$R_{\text{min}} = \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} + \frac{v^3}{|c|^2}}\right)^{\frac{1}{3}} + \left(\frac{K}{2} - \sqrt{\frac{K^2}{4} + \frac{v^3}{|c|^2}}\right)^{\frac{1}{3}}$$ \hspace{1cm} (12)

for $K \in (2\sqrt{\frac{v^3}{|c|^2}}, \infty)$, where $K$ is defined in (4).

Proof: The proof of the theorem has been elaborated by Tripathy et al.[11].

The control strategy (2) generates two types of annular patterns: type 1 and type 2 as shown in figures 4a and 4b respectively. In type 1 pattern, the robot goes around itself while covering the area centred around the target. Whereas in type 2 pattern, the robot encircles the target while creating the annulus.

Lemma 2: The annular trajectory of the agent is Type 1 when $K > 0$ and Type 2 when $K \leq 0$.

Proof: For some initial conditions the trajectory of the agent goes around itself as shown in fig. 4a, called the type 1 pattern. In this case, the variation of instantaneous $\theta$ is not monotonous; it increases and decreases as time progresses as is evident from the figure. On the other hand, for some other initial conditions the agent of the trajectory encircles the target as shown in fig. 4b which is referred to as type 2 pattern. In this case, the rate of change of instantaneous $\theta$ is monotonous. Using (4), we get $\dot{\theta} = \frac{\eta(2c^3 + K)}{r^3}$ which implies that for $K > 0$ $\phi$ varies monotonically with time. Since $\dot{\theta} = \frac{v \sin \phi}{r}$, $\theta$ increases and decreases with time which generates a type 1 pattern irrespective of the sign of $\eta$. Again, (4) can be substituted to yield $\dot{\theta} = \frac{\eta(c^2 - K)}{r^3}$. When $K \leq 0$, the variation of $\theta$ with respect to time is monotonous, giving rise to type 2 pattern.

III. GENERATING THE PATTERNS

In Section II, the expressions for $R_{\text{max}}$ and $R_{\text{min}}$ were derived in the situation where the initial conditions $(r_0, \phi_0)$ and the controller gain $\eta$ have been specified. In this section, we address the problem of finding out the appropriate $\eta$ when the bounds of the annulus, $R_{\text{max}}$ and $R_{\text{min}}$, and the initial conditions have been specified.
Lemma 3: Given $R_{\text{max}}$, $r_0$, and $\phi_0$, the agent generates the desired annulus for the controller gains,

\[ \eta_1 = 3v \frac{R_{\text{max}} + r_0 \sin \phi_0}{R_{\text{max}}^3 - r_0^3} \]

\[ \eta_2 = 3v \frac{-R_{\text{max}} + r_0 \sin \phi_0}{R_{\text{max}}^3 - r_0^3} \]

Proof: Lemma 1 gives that $R_{\text{max}}$ occurs at $\phi = \frac{3\pi}{2}$ for $\eta > 0$ or $\phi = \frac{\pi}{2}$ for $\eta < 0$. Substituting this in (3), we get (13) and (14) where it is obvious that $\eta_1 > 0$ and $\eta_2 < 0$.

Remark 1: When $R_{\text{max}}$, $r_0$, and $\phi_0$ have been specified, the annular patterns could be either Type 1 or Type 2. The type of the annulus depends on the sign of $K$ which depends on $r_0$, $\phi_0$ and $R_{\text{max}}$. Replacing (13) in (3) gives $K_1 = R_{\text{max}} r_0^2 \frac{R_{\text{max}}^2 \sin \phi_0 - r_0^2}{R_{\text{max}}^3 + r_0^3}$ and for (14), $K_2 = R_{\text{max}} r_0^2 \frac{R_{\text{max}}^2 \sin \phi_0 - r_0^2}{R_{\text{max}}^3 - r_0^3}$. For $\sin \phi_0 < -\left( \frac{r_0}{R_{\text{max}}} \right)^2$, $K_1 < 0$ and $K_2 > 0$, hence, $\eta_1$ and $\eta_2$ generate type 2 and type 1 pattern respectively. For $-\left( \frac{r_0}{R_{\text{max}}} \right)^2 \leq \sin \phi_0 \leq \left( \frac{r_0}{R_{\text{max}}} \right)^2$, $K_1 > 0$ and $K_2 > 0$, so both $\eta_1$ and $\eta_2$ generate type 1 pattern. Finally for $\sin \phi_0 > \left( \frac{r_0}{R_{\text{max}}} \right)^2$, $K_1 > 0$ and $K_2 < 0$, hence, $\eta_1$ and $\eta_2$ generate type 1 and type 2 patterns respectively.

Lemma 4: Given $R_{\text{min}}$, $r_0$ and $\phi_0$, the agent generates the pattern for,

\[ \eta_1 = 3v \frac{R_{\text{min}} + r_0 \sin \phi_0}{R_{\text{min}}^3 - r_0^3} \]

\[ \eta_2 = 3v \frac{-R_{\text{min}} + r_0 \sin \phi_0}{R_{\text{min}}^3 - r_0^3} \]

Proof: $R_{\text{min}}$ can occur either at $\phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ depending on the sign of $\eta$ and $K$ as was shown in Lemma 1. Substituting this in (3) and combining with (4), we get (15) and (16).

Remark 2: The situation when $R_{\text{min}}$, $r_0$, and $\phi_0$ have been specified, the type of the annulus purely depends on the specifications. The values of $K$ obtained by replacing $\eta_1$ from (13) and $\eta_2$ from from (14) in (3) are $K_1 = R_{\text{min}} r_0^2 \left( \frac{R_{\text{max}}^2 \sin \phi_0 + r_0^2}{R_{\text{min}}^3 + r_0^3} \right)$ and $K_2 = R_{\text{min}} r_0^2 \left( \frac{R_{\text{max}}^2 \sin \phi_0 - r_0^2}{R_{\text{min}}^3 - r_0^3} \right)$ respectively. Now, for $\sin \phi_0 < -\left( \frac{r_0}{R_{\text{max}}} \right)^2$, $K_1 < 0$ and $K_2 > 0$, hence, both $\eta_1$ and $\eta_2$ generate type 2 and type 1 patterns respectively. For $-\left( \frac{r_0}{R_{\text{max}}} \right)^2 \leq \sin \phi_0 \leq \left( \frac{r_0}{R_{\text{max}}} \right)^2$, $K_1 > 0$ and $K_2 > 0$, so both $\eta_1$ and $\eta_2$ generate type 1 patterns. Finally for $\sin \phi_0 > \left( \frac{r_0}{R_{\text{max}}} \right)^2$, $K_1 > 0$ and $K_2 < 0$, hence, $\eta_1$ and $\eta_2$ generate type 1 and type 2 patterns respectively.

Theorem 2: Given $R_{\text{max}}$ and $R_{\text{min}}$, a robot with kinematics (1) and control (2) generates a pattern in the region $[R_{\text{min}}, R_{\text{max}}]$ centred around the target point $T$ if it has a controller gain

\[ \eta = \pm \frac{3v(R_{\text{max}} \pm R_{\text{min}})}{R_{\text{max}}^3 - R_{\text{min}}^3}. \]

Proof: With reference to Lemma 1, when $K > 0$, $f(R_{\text{max}}) = K = g(R_{\text{min}})$. This gives $|\eta| = 3v(R_{\text{max}} + R_{\text{min}})/(R_{\text{max}}^3 - R_{\text{min}}^3)$ when $K \leq 0$, $f(R_{\text{min}}) = K$ which gives $|\eta| = 3v(R_{\text{max}} - R_{\text{min}})/(R_{\text{max}}^3 - R_{\text{min}}^3)$. Since $\eta$ can be positive or negative, we get (17).

Ecn. (17) suggests four different values of $\eta$ - two positive and two negative. We know that $\eta > 0$ makes the robot turn counterclockwise while $\eta < 0$ makes it turn clockwise. $|\eta| = 3v(R_{\text{max}} + R_{\text{min}})/(R_{\text{max}}^3 - R_{\text{min}}^3)$ (that is, $K > 0$) produces a type 1 annulus. For $|\eta| = 3v(R_{\text{max}} - R_{\text{min}})/(R_{\text{max}}^3 - R_{\text{min}}^3)$ (that is, $K < 0$), the loops of the pattern encircle the target generating a type 2 annulus.

![Type 1 pattern](image1)

![Type 2 pattern](image2)

Fig. 4: Two types of pattern with the same $R_{\text{max}}$ and $R_{\text{min}}$. 

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IV. SIMULATION RESULTS
In this section, the theoretical results obtained in the paper so far have been validated with the help of simulations. In all the simulations, the linear speed has been taken as $v = 5$.

Case 1: Here the bounds of an annulus have been evaluated for initial conditions $r_0 = 5$, $\phi_0 = 45^\circ$, $\eta = 0.5$ using Theorem 1. Theoretically, we get $R_{\text{max}} = 7.737$ and $R_{\text{min}} = 4.554$. In simulations, the trajectory is an annulus as shown in Fig. 5a. The values of the bounds of the annulus can be validated using the Fig. 5b which shows the instantaneous variation of $r$.

Case 2: Let the following conditions have been specified where $R_{\text{max}} = 30$, $r_0 = 7$ and $\phi_0 = 205^\circ$. Using Lemma 3, the controller gains have been calculated to be $\eta_1 = 0.0152$ and $\eta_2 = -0.01854$. Using $\eta_2$ along with the initial conditions, the theoretical results have been validated through simulation as shown in Fig. 6a and 6b.

Case 3: Let the following conditions have been specified where $R_{\text{min}} = 10$, $r_0 = 20$ and $\phi_0 = 55^\circ$. Using Lemma 4, the controller gains have been calculated to be $\eta_1 = -0.05653$ and $\eta_2 = -0.01368$. Using these values of the controller gains along with the initial conditions, the theoretical results have been validated through simulation as shown in Fig. 7a and 7b for $\eta_1$.

Case 4: Let $R_{\text{max}} = 38$ and $R_{\text{min}} = 5$ be the desired specifications of the pattern. The theoretical value of controller gain is $\eta = 0.01178$ which is calculated using the results obtained in Theorem 2. The initial conditions are chosen as $r_0 = 20$ and $\phi_0 = -3.3976$ such that they generate. When the values of $r_0$, $\phi_0$ and $\eta$ are used in simulations, the trajectory obtained is shown in Fig. 8a. The instantaneous variation of $r$ is shown in Fig 8b which validates the theoretical results.

V. CONCLUSION
The control strategy presented in this paper achieves coverage of an area by using a unicycle-based mobile robot. The robot autonomously generates annular patterns around the stationary target such that any shape can be covered within the annulus. We have analysed the control strategy to show that feasible controller gains always exist such that the robot can start from an initial condition to finally form a desired annulus when either $R_{\text{max}}$ or $R_{\text{min}}$ or both have been specified. Further, the work can be extended to make change coverage pattern to a rectangle, pentagon and so on. Also, the controller can be modified to become robust.
Fig. 7: $R_{\text{min}}$, $r_0$ and $\phi_0$ specified

Fig. 8: $R_{\text{max}}$ and $R_{\text{min}}$ specified

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