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CONTROL OF THE NONHOLONOMIC INTEGRATOR

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• Nonholonomic systems - a brief introduction
• Nonholonomic integrator (NI) (Kinematic model of a wheeled mobile robot)
• Four stabilizing control laws
• The Extended Nonholonomic Double Integrator (ENDI) - Dynamic model of a wheeled mobile robot
• A Sliding mode controller
• Other problems
Classification of constraints in mechanical systems

- Holonomic constraints - restrict the allowable configurations of the system

- Nonholonomic constraints - do not restrict the allowable configurations of the system but restrict instantaneous velocities/accelerations
  - Velocity level constraints - parking of a car, wheeled mobile robots, rolling contacts in robotic applications
  - Acceleration level constraints - fuel slosh in spacecrafts/launch vehicles, underwater vehicles, underactuated mechanisms (on purpose or loss of actuator) systems - serial link manipulators
A coin rolling on a horizontal plane

Figure 1: Vertical coin on a plane
The Differential geometric view

- Rewriting the constraints in terms of annihilator codistributions

\[
\begin{bmatrix}
\sin(\theta) & -\cos(\theta) & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} = 0
\]

- Permissible motions of the coin are such that the vector field is annihilated by the codistribution

\[
\Omega = \begin{bmatrix}
\sin(\theta) & -\cos(\theta) & 0
\end{bmatrix}
\]

- Can \( \Omega \) be expressed as the gradient of a function

\[
\lambda : (\mathbb{R}^1 \times \mathbb{R}^1 \times S^1) \to \mathbb{R}^1 \text{ as }
\]

\[
\Omega = \begin{bmatrix}
d\lambda
\end{bmatrix}
\]
The NI or Brockett Integrator

- Nonholonomic integrator

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1 u_2 - x_2 u_1
\end{align*}
\]

- Third order driftless system
  - Equilibria - the whole of \( IR^3 \)
  - Not linearly controllable at any equilibrium point
  - No continuous feedback control law that can globally stabilize the system

- **Solution:** Time varying feedback OR discontinuous feedback
  - Mimics the kinematic model of a mobile robot with a nonholonomic constraint
Past work on the NI

- **Motion planning** - Given $x(0) = x_0$ and $x_f$ and a time interval $[0, t_f]$, find a control history $u(\cdot)$ (from an admissible class of functions) such that $x(t_f) = x_f$.

- **Stabilization** - Compute a feedback control law that stabilizes the system in a region (set) near the origin

A wheeled mobile robot on a horizontal plane

Figure 2: Schematic of a mobile robot
Wheeled mobile robot

- Notation

\[ M = \text{Mass of the vehicle} \]
\[ I = \text{Inertia of the vehicle} \]
\[ F = \frac{1}{R}(\tau_1 + \tau_2) \]
\[ \tau = \frac{L}{R}(\tau_1 - \tau_2) \]
\[ L = \text{Distance between the center of mass and the wheel} \]
\[ R = \text{Radius of the rear wheel} \]
\[ \tau_1 = \text{Left wheel motor torque} \]
\[ \tau_2 = \text{Right wheel motor torque} \]

- Nonholonomic constraint - no lateral (sideways) motion

\[ x \sin \theta - y \cos \theta = 0 \]
Kinematics and Dynamics

- Generalized coordinates - position of the center of mass $(x, y)$ and orientation $\theta$
- Constraints of motion
  $$\dot{x} \sin \theta - \dot{y} \cos \theta = 0$$  No lateral motion
- Equations of motion
  $$\dot{\theta} = \omega \quad \dot{x} = v \cos \theta \quad \dot{y} = v \sin \theta$$
- Control inputs (at the kinematic level) - drive $v$ and steer $\omega$
- Dynamics - (control inputs are forces and torques)
  $$M \ddot{v} = F$$
  $$I \ddot{\omega} = \tau$$
Transforming the variables

• Step 1 - State transformation from \((\theta, x, y)\) to \((z_1, z_2, z_3)\)

\[
\begin{align*}
z_1 & \triangleq \theta \\
z_2 & \triangleq x \cos \theta + y \sin \theta \\
z_3 & \triangleq x \sin \theta - y \cos \theta
\end{align*}
\]

• Step 2 - State transformation from \((z_1, z_2, z_3)\) to \((\eta_1, \eta_2, \eta_3)\)

\[
\begin{align*}
\eta_1 & \triangleq z_1 \\
\eta_2 & \triangleq z_2 \\
\eta_3 & \triangleq -2z_3 + z_1 z_2
\end{align*}
\]
The new representation

- Define modified control inputs as
  \[ u_1 \triangleq \omega \quad u_2 \triangleq z_3 \omega + v \]

- The system in the new states and inputs is
  \[
  \begin{bmatrix}
  \dot{\eta}_1 \\
  \dot{\eta}_2 \\
  \dot{\eta}_3
  \end{bmatrix}
  =
  \begin{bmatrix}
  u_1 \\
  u_2 \\
  \eta_1 u_2 - \eta_2 u_1
  \end{bmatrix}
  \]
Technique 1 - Exponential Stability in a Dense Set

Definition 1.1 If the system is exponentially stable in an open dense set $\mathcal{O}$, and the control law is well defined and bounded along the closed loop trajectories, then the closed system is said to be almost exponentially stable (AES).

- The open dense set in the system under consideration is
  $\mathcal{O} = \{x \in \mathbb{R}^3 | x_1 \neq 0\}$.

- Proposition 1.1 The NI is AES with the control law
  $$u = \begin{pmatrix} -k_1 x_1 \\ -k_3 \frac{x_3}{x_1} - k_1 x_2 \end{pmatrix} \quad x_1 \neq 0$$
  where $0 < k_1 < k_3$.  

Proof

• The closed loop dynamics can be written in the form

\[ \dot{x} = Ax + g(t) \]

where

\[
A = \begin{bmatrix}
-k_1 & 0 & 0 \\
0 & -k_1 & 0 \\
0 & 0 & -k_3
\end{bmatrix};
\]

\[
g(t) = \begin{bmatrix}
0 \\
-k_3 \frac{x_3(t)}{x_1(t)} \\
0
\end{bmatrix}
\]

Since \( g(t) \leq |k_3 \frac{x_3(0)}{x_1(0)}| e^{(k_1-k_3)t} \) and the eigen values of \( A \) are negative-real, the system is exponentially stable.

• If \( x_1(0) = 0 \), then an open loop control can be used to steer the system to any non zero value of \( x_1 \) and then the control law can be applied.
**Technique 2 - Bounded Control Law**

- In **Technique 1**, if the initial condition is close to the surface $x_1 = 0$, the magnitude of $u_2$ is very large.

- Define $\alpha \triangleq \left| \frac{x_3}{x_1} \right|$ and a set

$$\mathcal{M}_1 \triangleq \{ x \in \mathbb{R}^3 : \alpha \leq c \}$$

and three other sets as

$$\mathcal{M}_2 \triangleq \{ x \in \mathbb{R}^3 : x \notin \mathcal{M}_1 \text{ and } |x_2 + c| \geq 1 \}$$

$$\mathcal{M}_3 \triangleq \{ x \in \mathbb{R}^3 : x \notin \mathcal{M}_1 \text{ and } |x_2 + c| < 1 \}$$

$$\mathcal{M}_4 \triangleq \{ x \in \mathbb{R}^3 : x_1 = x_3 = 0 \}$$
• **Proposition 1.2** The NI is exponentially stable in the set $\mathcal{M}_1$ containing the origin under the following control law

$$u = \begin{cases} 
-k_1 x_1 & \text{if } x \in \mathcal{M}_1 \\
-k_3 \frac{x_3}{x_1} - k_1 x_2 & \text{if } x \in \mathcal{M}_2 \\
k_s \frac{S^{1/3}}{x_2 + c} & \text{if } x \in \mathcal{M}_2 \\
0 & \text{if } x \in \mathcal{M}_3 \\
0 & \text{if } x \in \mathcal{M}_3 \\
d & \text{if } x \in \mathcal{M}_4 \\
-k_1 x_1 & \text{if } x \in \mathcal{M}_4 \\
-k_1 x_2 & \text{if } x \in \mathcal{M}_4 
\end{cases}$$

where $S(x) \triangleq x_3 - cx_1$ and $0 < k_1 < k_3$, $0 < k_s$ and $d$ is any non zero constant.
Proof of Technique 2

• Proof: Case 2

If $x \in \mathcal{M}_2$, then the closed loop system becomes

$$\begin{align*}
\dot{x}_1 &= k_s \frac{S^{1/3}}{x_2 + c} \\
\dot{x}_2 &= 0 \\
\dot{x}_3 &= -x_2 k_s \frac{S^{1/3}}{x_2 + c}
\end{align*}$$

The dynamics of $S$ becomes

$$\dot{S} = -k_s S^{1/3}$$

This is called a power rate reaching law and it is finite-time stable. So there exists a finite-time $T_1$ such that $S(t) = 0 \ \forall \ t \in [T_1, \infty)$. So all the trajectories will reach the boundary of the set $\mathcal{M}_1$ in some finite-time after which case 1 follows.
Finite time stabilization to the surface $S(x) = 0$

- The dynamics of the surface is given by
  \[ \dot{S} = -k_S S^{1/3} \]

- Consider a Lyapunov candidate function and its rate of change
  \[ V = S^2/2 \]
  \[ \dot{V} = -k_S S^{4/3} < 0 \]

  Hence $S = 0$ is attractive globally.

- For finite time convergence we show
  \[ \dot{V} + k V^\alpha \leq 0 \quad ([\text{Haimo}] \text{ condition for finite-time stability}) \]
  where $\alpha \in (0, 1)$ and $k > 0$
• For $\alpha = 2/3$ and $k < 2^{2/3}k_s$ we have

$$\dot{V} + kV^{2/3} = -S^{4/3}(k_s - \frac{k}{2^{2/3}}) < 0$$

• Further

$$\dot{S} = 0$$

which implies that $S = 0$ is positively invariant
Proof

Case 3
If $x \in \mathcal{M}_3$, then the dynamics of $x_2$ is

$$\dot{x}_2 = d$$

So there exists a finite-time $T_2$ such that $|x_2(T_2) + c| = 1$. Then Case 2 follows.

Case 4
If $x \in \mathcal{M}_4$, then closed loop system becomes

$$\dot{x}_1 = 0$$
$$\dot{x}_2 = -k_1 x_2$$
$$\dot{x}_3 = 0$$

and we have exponential stability
Techniques 1 and 2

- Methods 1 and 2 are related though the AES property of 1 does not hold in 2. Method 2 however guarantees boundedness of the control law in a local domain around the origin.
Technique 3 - Sliding Mode Control

The next two techniques rest on the variable structure (sliding mode) philosophy. The methodologies rest on bringing the system to a surface (defined by the dependent state in the dynamics) in finite time and then exponentially making the other two states reach the origin.

• **Proposition 1.3** The NI is exponentially stable in the set 
\( \mathcal{M} \triangleq \{ x \in \mathbb{R}^3 : x_3 = 0 \} \) with the following control law

\[
  u = \begin{pmatrix}
    -k_1 x_1 \\
    -k_3 x_3^{1/3} - k_1 x_2 \\
  \end{pmatrix} \quad \text{if } x_1 \neq 0
\]

where \( k_1, k_3 > 0 \).
Proof of Technique 3

- If $x_1(0) \neq 0$ then the closed loop system is

$$
\begin{align*}
\dot{x}_1 &= -k_1 x_1 \\
\dot{x}_2 &= \frac{-k_3 x_3^{1/3}}{x_1} - k_1 x_2 \\
\dot{x}_3 &= -k_3 x_3^{1/3}
\end{align*}
$$

Since $k_1, k_3 > 0$, the dynamics of $x_3$ implies finite time stability while that of $x_1$ implies exponential stability. So there exists a finite time $T_1 > 0$ such that $x_3(t) \equiv 0 \ \forall t \geq T_1$.

- The system on $\mathcal{M}$ is then governed by the set of equations

$$
\begin{align*}
\dot{x}_1 &= -k_1 x_1 \\
\dot{x}_2 &= -k_1 x_2 \\
\dot{x}_3 &\equiv 0
\end{align*}
$$
Technique 4

• Define $S(x) \triangleq x_1^2 - 3k_1 x_3^{2/3}$ where $k_1 > 0$ is a predefined constant,

• Proposition 1.4 The NI is exponentially stable in the set $\mathcal{M} \triangleq \{ x \in \mathbb{R}^3 : x_3 = 0 \}$ with the following time-varying control law

$$u = \begin{cases} 
\begin{pmatrix} 
-k_1 x_1 \\
-x_1 x_3^{1/3} - k_1 x_2 
\end{pmatrix} & \forall \ s \in [t, \infty) \text{ if } S(x(t)) > 0 \\
\begin{pmatrix} 
x_1 \\
x_2 
\end{pmatrix} & \text{if } S(x(t)) \leq 0 \\
\begin{pmatrix} 
-k_1 x_1 \\
-k_1 x_2 
\end{pmatrix} & \text{if } x_3 = 0
\end{cases}$$
Proof of Technique 4

Case 1 Assume $S(x(0)) > 0$

- We have

\[
\begin{align*}
\dot{x}_1 &= -k_1 x_1 \\
\dot{x}_2 &= -x_1 x_3^{1/3} - k_1 x_2 \\
\dot{x}_3 &= -x_1^2 x_3^{1/3}
\end{align*}
\]

- The evolution of $x_1(t)$ is given by

\[x_1(t) = x_1(0)e^{-k_1 t}\]

- The evolution of $x_3$ is

\[
x_3(t) = \pm \left[\frac{x_1(0)^2e^{-2k_1 t} - S(x(0))}{3k_1}\right]^{3/2} \quad \forall t \in [0, T_1)
\]

\[x_3(t) = 0 \quad \forall t \geq T_1\]
Proof of Technique 4

- The positive (negative) sign corresponds to the solution for an initial condition $x_3(0) > 0$ ($x_3(0) < 0$).

- The closed loop system after time $T_1$ is

$$\begin{align*}
\dot{x}_1 &= -k_1 x_1 \\
\dot{x}_2 &= -k_1 x_2 \\
\dot{x}_3 &\equiv 0
\end{align*}$$

and is exponentially stable.
Proof of Technique 4

**Case 2** If $x_1(0) \neq 0$ and $S(x(0)) \leq 0$, then we have

\[
\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= x_2 \\
\dot{x}_3 &= 0
\end{align*}
\]

The above dynamics leads to an increase in the value of $S(x(t))$ as time increases. Note that $x_3$ remains a constant and $x_1$ has an exponential growth. So there exists a finite-time such that $S(x(t)) > 0$. Then Case 1 follows.
• The third case is straightforward. If $x_1(0) = 0$ then any open loop control can push $x_1$ to some non-zero value, then either case 1 or case 2 follows.
Techniques 3 and 4

- Methods 3 and 4 bring in a variable structure (or sliding mode) feature; the latter is also time-varying.

- Both techniques use a power rate reaching law to enter a set in which the system is exponentially stable.

- The idea in Method 4 is to utilize the state $x_1$ as a gain for the convergence of $x_3$ to zero in finite-time. So we have the term $x_1x_3^{1/3}$ instead of $\frac{x_3^{1/3}}{x_1}$. But since this control law valid in a local domain, we have introduced a switching strategy to move from anywhere to this domain.

- All the four control laws are not defined for $x_1(0) = 0$. 

• Closest parallels - notions of sigma process, paper by Bloch and Drakunov
Dynamic model of the mobile robot

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{x} \\
\dot{y} \\
\dot{v} \\
\dot{\omega}
\end{bmatrix} =
\begin{bmatrix}
\omega \\
v \cos \theta \\
v \sin \theta \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
\frac{1}{M} \\
0
\end{bmatrix} F +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\frac{1}{I}
\end{bmatrix} \tau
\]

- Fifth order system with drift
- Discontinuous controllers (velocity level) designed from the kinematic model cannot be applied to obtain controllers at the acceleration level for the dynamic model.
Transforming the variables

- **Step 1 - State transformation**

\[
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5 \\
\end{bmatrix} \triangleq \begin{bmatrix}
\theta \\
x \cos \theta + y \sin \theta \\
x \sin \theta - y \cos \theta \\
\omega \\
v - (x \sin \theta - y \cos \theta)\omega \\
\end{bmatrix}
\]

- **The dynamic model in the \( z \) variables**

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4 \\
\dot{z}_5 \\
\end{bmatrix} = \begin{bmatrix}
z_4 \\
z_5 \\
z_2z_4 \\
z_1 \\
\frac{F}{M} - \tau \bar{z}_3 - z_2z_4^2 \\
\end{bmatrix}
\]
Transforming the variables

- Alternate inputs
  \[ u_1 \triangleq \frac{\tau}{I} \quad u_2 \triangleq -\frac{z_4 z_2}{I} - \frac{\tau}{I} z_3 + \frac{F}{M} \]

- Step 2 - State transformation from \((z_1, z_2, z_3, z_4, z_5)\) to \((x_1, x_2, x_3, y_1, y_2)\)
  \[ x_1 \triangleq z_1 \quad x_2 \triangleq z_2 \quad x_3 \triangleq -2z_3 + z_1 z_2 \quad y_1 \triangleq z_4 \quad y_2 \triangleq z_5 \]

- The system equations are now in the form
  \[ \ddot{x}_1 = u_1 \]
  \[ \ddot{x}_2 = u_2 \]
  \[ \dot{x}_3 = x_1 \dot{x}_2 - x_2 \dot{x}_1 \]
The Extended Nonholonomic Double Integrator (ENDI)

\[ \dot{x} = f(x) + \sum_{i=1}^{2} g_i(x)u_i \]

where \( x \triangleq [x_1, x_2, x_3, y_1, y_2]^T \in \mathbb{R}^5 \) is the state vector and \( u \triangleq [u_1, u_2] \in \mathbb{R}^2 \) is the control.

\[ f(x) \triangleq \begin{pmatrix} y_1 \\ y_2 \\ x_1y_2 - x_2y_1 \\ 0 \\ 0 \end{pmatrix}; \quad g_1(x) \triangleq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad g_2(x) \triangleq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
Features of the ENDI

- The equilibria of the ENDI system are of the form
  \[ x_e = \{ x \in \mathbb{R}^5 : y_1 = y_2 = 0 \} \] and satisfy the following properties
  1. \( x_e \) is not stabilized by any smooth feedback control laws.
  2. The ENDI is locally strongly accessible from any \( x \in \mathbb{R}^5 \).
  3. The ENDI is small time locally controllable (STLC) from any \( x_e \).
A Sliding Mode Approach

• Define $S_1(x) \triangleq kx_1 + y_1$ and $S_2(x) \triangleq x_1y_2 - x_2y_1 + k_3x_3$ where $k_3 > k > 0$

• Define the sets

$$\mathcal{M}_1 \triangleq \{ x \in \mathbb{R}^5 : S_1(x) = 0 \} \quad \mathcal{M}_2 \triangleq \{ x \in \mathcal{M}_1 : S_2(x) = 0 \}$$

• Proposition 1.5 The origin of the ENDI is exponentially stable in the set $\mathcal{M}_2$ with the following control law

$$u_1 = -ky_1 - S_1^{1/3} \text{ if } (x_1, y_1) \neq 0$$

$$u_2 = \begin{cases} 
-S_2^{1/3}x_1 - k_3y_2 - (k_3 - k)kx_2 & x \in \mathcal{M}_1 \ (x \neq 0) \\
-x_2 - y_2 & \text{otherwise}
\end{cases}$$
Proof

- The closed loop dynamics for \((x_1, y_1) \neq 0\) and \(x \not\in M_1\) becomes

\[
\begin{align*}
\dot{x}_1 &= y_1 \\
\dot{x}_2 &= y_2 \\
\dot{x}_3 &= x_1y_2 - x_2y_1 \\
\dot{y}_1 &= -ky_1 - S_1(x)^{1/3} \\
\dot{y}_2 &= -x_2 - y_2
\end{align*}
\]

- The dynamics of \(S_1(x)\) is

\[
\dot{S}_1 = -S_1^{1/3}
\]

and the surface \(S_1(x) = 0\) is finite-time stable
• So there exists a time $T_1 \geq 0$ such that the closed loop trajectory for any permissible initial condition reaches the set $\mathcal{M}_1$ and stays there for all future time. The control law $u_1$ allows both $x_1$ and $y_1$ to converge to zero as time gets large.

• At the same time as the system is being driven towards $\mathcal{M}_1$, the PD control law $u_2$ with unity gain makes states $x_2$ and $y_2$ converge to zero.
The dynamics on the set $\mathcal{M}_1$

- The closed loop system on $\mathcal{M}_1$ is

$$\dot{x}_1 = -kx_1$$
$$\dot{x}_2 = y_2$$
$$\dot{x}_3 = x_1y_2 - x_2y_1$$
$$\dot{y}_1 = -ky_1$$
$$\dot{y}_2 = \frac{-S_2^{1/3}}{x_1} - k_3y_2 - (k_3 - k)kx_2$$

- The dynamics of $S_2$ is

$$\dot{S}_2 = -S_2^{1/3}$$

So there exists a time $T_2 \geq T_1 \geq 0$ such that the closed loop trajectories reach the set $\mathcal{M}_2$ and stay there for all future time.
The dynamics on the set $\mathcal{M}_2$

- The closed loop system on $\mathcal{M}_2$

\[
\begin{align*}
\dot{x}_1 &= -kx_1 \\
\dot{x}_2 &= y_2 \\
\dot{x}_3 &= -x_3 \\
\dot{y}_1 &= -ky_1 \\
\dot{y}_2 &= -k_3y_2 - (k_3 - k)kx_2
\end{align*}
\]

- All trajectories converge to the origin
Simulation

- Initial conditions are $x(0) = -1.5m$, $y(0) = 4m$, $\theta(0) = -2.3 \text{rad}$, $\omega(0) = 1 \text{rad/sec}$, $v(0) = -1 \text{m/sec}$.

- Controller parameters are $k = 0.5$, $k_3 = 1$.

- Vehicle parameters are $M = 10 \text{Kg}$, $I = 2 \text{Kgm}^2$, $L = 6 \text{cm}$, $R = 3 \text{cm}$. 
Control of the Nonholonomic Integrator

Figure 3: Stabilization to the origin
Figure 4: Position and orientation of the vehicle
Figure 5: Velocities of the vehicle
Control of the Nonholonomic Integrator

Figure 6: Input to the motors
Thank You