SC 618: Flows, derivatives and brackets

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Outline

1. Linear and nonlinear systems

2. The flow of a vector field

3. Lie groups

4. Push-forward and pull-back

5. The Lie derivative and the Jacobi-Lie bracket

6. Lie algebras

Lectures 3, 4 and 5
Outline

1 Linear and nonlinear systems
2 The flow of a vector field
3 Lie groups
4 Push-forward and pull-back
5 The Lie derivative and the Jacobi-Lie bracket
6 Lie algebras
Linear systems - preliminaries

A linear system with an input

\[ \dot{x} = Ax + Bu \quad x(t) \in \mathbb{R}^n \]

\( x(t) \) lives in \( \mathbb{R}^n \), a vector space.

A linear autonomous system

\[ \dot{x} = Ax, \quad x(t) \in \mathbb{R}^n \]

\( x(t) \) lives in \( \mathbb{R}^n \), a vector space.
Linear systems - preliminaries

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A linear autonomous system

\[ \dot{x} = Ax, \quad x(t) \in \mathbb{R}^n \]

\(x(t)\) lives in \(\mathbb{R}^n\), a vector space.

The right-hand side of the differential equation is termed a vector field. For the linear system, it is a linear vector field.

Linearity

\[ A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 Ax_1 + \alpha_2 Ax_2 \]
Solution and flow

Solution to the set of differential equations

\[ x(t) = e^{At} x_0, \quad x(0) = x_0, \quad e^{At} \triangleq I + A + A^2/2! + \ldots \]

The term \( e^{At}x_0 \) is termed the flow associated with the linear vector field \( Ax \).
Nonlinear systems - preliminaries

A nonlinear system with an input

\[ \dot{x} = f(x) + g(x)u \quad x(t) \in M \]

\( f(\cdot), g(\cdot) \) are smooth functions, \( x(t) \) lives in \( M \), a smooth manifold.

A nonlinear autonomous system

\[ \dot{x} = f(x), \quad x(t) \in \mathbb{R}^n \]

\( x(t) \) lives in \( M \), a smooth manifold.
Nonlinear systems - preliminaries

A nonlinear system with an input

$$\dot{x} = f(x) + g(x)u \quad x(t) \in M$$

$f(\cdot), g(\cdot)$ are smooth functions, $x(t)$ lives in $M$, a smooth manifold.

A nonlinear autonomous system

$$\dot{x} = f(x), \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in $M$, a smooth manifold.
The right-hand side of the differential equation is a nonlinear vector field.
Nonlinear systems - preliminaries

A nonlinear system with an input

\[ \dot{x} = f(x) + g(x)u \quad x(t) \in M \]

\( f(\cdot), g(\cdot) \) are smooth functions, \( x(t) \) lives in \( M \), a smooth manifold.

A nonlinear autonomous system

\[ \dot{x} = f(x), \quad x(t) \in \mathbb{R}^n \]

\( x(t) \) lives in \( M \), a smooth manifold. The right-hand side of the differential equation is a nonlinear vector field. Linearity does not hold.

\[ f(\alpha_1 x_1 + \alpha_2 x_2) \neq \alpha_1 f(x_1) + \alpha_2 f(x_2) \]
Solution and flow

Solution to the set of differential equations

\[ x(t) = \Phi(t, x_0) \quad x(0) = x_0 \]

The term \( \Phi(t, x_0) \) is termed the flow associated with the nonlinear vector field \( f(x) \).
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Lectures 3, 4 and 5
Flow of a vector field

Flow of $X(x)$

The flow of the vector field $X(x)$, denoted by $\Phi(t, x_0)$, is a mapping from $(-a, a) \times U \rightarrow \mathbb{R}^n$ (where $a(>0) \in \mathbb{R}$ and $U$ is an open region in the state-space) and satisfies the differential equation

$$\frac{d\Phi(t, x_0)}{dt} = X(\Phi(t, x_0)) \quad \forall t \in (-a, a), x(0) = x_0 \in U.$$ 

over the interval $(-a, a)$ and with initial conditions starting in the region $U$. 

Lectures 3, 4 and 5
Figure: Flow of a vector field
Properties of flows

The group structure
Denote
\[ \Phi_t(x_0) \triangleq \Phi(t, x_0) \]

The set of transformations \( \{ \Phi_t \} \) : \( U \to \mathbb{R}^n \) satisfies the following properties.

- \( \Phi_{t+s}x_0 = \Phi_t \circ \Phi_s x_0 \quad \forall t, s, t + s \in (-a, a) \) (the group binary operation.)
- \( \Phi_0x_0 = x_0 \) (the group identity.)
- For a fixed \( t \in (-a, a) \) we have \( \Phi_t \Phi_{-t} x_0 = x_0 \Rightarrow [\Phi_t]^{-1} = \Phi_{-t} \) (existence of an inverse.)
The flow of a linear system

The group property

Remark

*The three properties mentioned above impart a group structure to the set \( \{ \Phi_t \} \). This set is called a one-parameter (time) group of diffeomorphisms (\( \Phi_t \) and its inverse are smooth mappings).*
The flow of a linear system

The group property

Remark

The three properties mentioned above impart a group structure to the set \( \{ \Phi_t \} \). This set is called a one-parameter (time) group of diffeomorphisms (\( \Phi_t \) and its inverse are smooth mappings).

Linear flow

Remark

For a linear system described by

\[
\dot{x} = Ax \quad A \in \mathbb{R}^{n \times n}
\]

the flow \( \Phi_t x_0 = e^{At} x_0 \) where \( \{ e^{At} : t \in (-\infty, \infty) \} \) constitutes the one-parameter group of diffeomorphisms.
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Lectures 3, 4 and 5
A group

Definition
A group is a set $G$ with a binary operation $+$ such that

- For any $x, y \in G$, $x + y \in G$ (Closure) and $(x + y) + z = x + (y + z)$ (Associativity)

- There exists a unique $i \in G$ such that $x + i = i + x = x$ for every $x \in G$ (Existence of the identity element)

- For every $x \in G$ there exists a unique $y \in G$ such that $x + y = i$. (Existence of inverse)
A Lie group

Definition
A smooth manifold $M$ together with a group structure is called a **Lie group** $G$ if the group operation $+$ is **smooth**.

$$(g, h) \rightarrow g + h \quad (\forall g, h \in G) \quad \text{is smooth}$$

- The identity element of the Lie group is usually denoted by $e$.
- **Left translation** of a group
  $$L_g : G \rightarrow G \quad h \rightarrow g + h$$
- **Right translation** of a group
  $$R_g : G \rightarrow G \quad h \rightarrow h + g$$
Examples of Lie groups

- $\mathbb{R}$ or multiple copies of $\mathbb{R}$ (as $\mathbb{R}^n$) with the binary operation being the usual component-wise addition $+$.

- The unit circle $S^1$ with elements denoted as $\theta (\in [0, 2\pi])$ and the binary operation being the usual addition. Similarly, multiple copies of $S^1$ (as $S^1 \times \ldots \times S^1$).

- The set of $n \times n$ invertible matrices with real entries with the binary operation being matrix multiplication. This group is called $GL(n, \mathbb{R})$.

- The set of $n \times n$ real-orthogonal matrices $O(n)$, a subset of $GL(n, \mathbb{R})$. The set of $n \times n$ rotation matrices $SO(n)$, a subset of $O(n, \mathbb{R})$. 
Rigid body motion

Definition
Rigid body motion is characterized by two properties

- The distance between any two points remains invariant
- The orientation of the body is preserved. (A right-handed coordinate system remains right-handed)
Two groups which are of particular interest to us in mechanics and control are $SO(3)$ - the special orthogonal group that represents rotations - and $SE(3)$ - the special Euclidean group that represents rigid body motions. These are Lie groups.

Elements of $SO(3)$ are represented as $3 \times 3$ real matrices and satisfy

$$R^T R = I$$

with $\det(R) = 1$.

An element of $SE(3)$ is of the form $(p, R)$ where $p \in \mathbb{R}^3$ and $R \in SO(3)$. 

$SO(3)$ and $SE(3)$
Frames of reference or coordinate frames

- In describing rigid body motions we always fix two frames of reference. One is called the **body frame** that remains fixed to the body and the other is the **inertial frame** that remains fixed in inertial space.

![Frame Diagram](image)

**Figure:** Rigid body motion
Rigid body motions and groups

- Suppose \( q_a \) and \( q_b \) are coordinates of a point \( q \) relative to frames \( A \) and \( B \), respectively.

\[
q_a = p_{ab} + R_{ab} q_b
\]

Here \( p_{ab} \) represents the position of the origin of the frame \( B \) with respect to frame \( A \) in frame \( A \) coordinates and \( R_{ab} \) is the orientation of frame \( B \) with respect to frame \( A \).

- Appending a ”1” to the coordinates of a point (to render the group operation as the usual matrix multiplication)

\[
\bar{q} = \begin{pmatrix} q_a \\ 1 \end{pmatrix} = \begin{pmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_b \\ 1 \end{pmatrix} = \bar{g}_{ab} \bar{q}_b
\]
Two results on rotations

Rotation in a plane

Claim
The rotation group $SO(2)$ can be identified with $S^1$ (the unit circle).

Proof:

$S^1 = \{ x \in \mathbb{R}^2 : \|x\| = 1 \}$

Parametrize the elements of $S^1$ in terms of $\theta \in [0, 2\pi]$. For each $\theta \in [0, 2\pi]$, the counter-clockwise rotation of the vectors $\{(1,0), (0,1)\}$ in $\mathbb{R}^2$ (these form a basis) by the angle $\theta$

$$(1,0) \rightarrow (\cos \theta \quad \sin \theta) \quad (0,1) \rightarrow (-\sin \theta \quad \cos \theta)$$

is given by the matrix

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

which is an element of $SO(2)$. 
Conversely, take an element of $SO(2)$ of the form

$$R = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

Then from the properties of an element of $SO(2)$, we have

$$a_1a_4 - a_2a_3 = 1; \quad a_1^2 + a_3^2 = 1; \quad a_2^2 + a_4^2 = 1; \quad a_1a_2 + a_3a_4 = 0$$

It is possible to find a $\theta \in [0, 2\pi]$ such that that $R$ can be represented in the form $R_\theta$. \( \square \)
Euler’s theorem

Theorem
*(Euler’s theorem)*

*Every* $A \in SO(3)$ *is a rotation through an angle* $\theta \in S^1$ *about an axis* $\omega \in \mathbb{R}^3$.

**Proof:** Since 1 is an eigen value of $A$, we have $Aw = w$ where $w \in \mathbb{R}^3$ is an eigen vector. Choose two vectors $e_1$ and $e_2$ that are orthogonal to each other as well as $w$. So

$$< w, e_1 > = 0, \quad < w, e_2 > = 0, \quad < e_1, e_2 > = 0$$

The two vectors $\{e_1, e_2\}$ lie in the plane perpendicular to $w$ and it follows that $\{w, e_1, e_2\}$ form a basis for $\mathbb{R}^3$. Since $A$ is orthogonal, the matrix of $A$ in this basis is of the form

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & a_1 & a_3 \\
0 & a_2 & a_4
\end{bmatrix}.$$
Proof (contd.)

(why ?) Now

\[
\begin{bmatrix}
a_1 & a_3 \\
a_2 & a_4
\end{bmatrix}
\]

is an element of $SO(2)$ and hence there exists a $\theta \in [0, 2\pi]$ such that

\[
\begin{bmatrix}
a_1 & a_3 \\
a_2 & a_4
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

It follows that $A$ is a rotation about $w$ through the angle $\theta$.
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Lectures 3, 4 and 5
The "star" map \((f_*)\) associated with a smooth function \(f\)

Consider a smooth map \(f : X \to Y\). At each \(p \in X\) we define a linear transformation as follows

\[
f_*p : T_p(X) \to T_{f(p)}(Y)
\]
called the derivative of \(f\) at \(p\), which is intended to serve as a "linear approximation to \(f\) near \(p\).” Visualize this as follows.

- Choose a parametrized curve \(c(\cdot) : (-\epsilon, \epsilon) \to X\) with \(c(0) = p\) and \(\frac{dc}{dt}\big|_{t=0} = v_p\).
- Construct the curve \(f \circ c\). Then define

\[
f_*p(v_p) \triangleq T_p f \cdot v_p = \frac{d}{dt}\big|_{t=0}(f \circ c)(t)
\]

- The rank of \(f\) at \(p\) is the rank of the Jacobian matrix at \(x(p)\) and this is independent of the choice of coordinates \(x\).
Push-forward and pull-back of a function

Push-forward

Pull-back

$\phi_t$ is the flow associated with the vector field $X$. 

Figure: Pull-back and push-forward of functions
Push-forward and pull-back of a function

- Suppose $X$ is a vector field on $M$ and $f : M \to \mathbb{R}$ is a smooth function. Then the *push-forward* of the function $f$ on $M$ by the flow of $X$ is the function $\Phi_t^* f$ defined by
  \[(\Phi_t^* f)(x) \triangleq f \circ \Phi_t^{-1}(x) \hspace{1em} \forall x \in M\]

- Suppose $X$ is a vector field on $M$ and $f : M \to \mathbb{R}$ is a smooth function. Then the *pull-back* of the function $f$ on $M$ by the flow of $X$ is the function $\Phi_t^* f$ defined by
  \[(\Phi_t^* f)(x) \triangleq f \circ \Phi_t(x) \hspace{1em} \forall x \in M\]
Push-forward and pull-back of a vector field

\[ \phi_t^* \] is associated with \( X \).

\[ Y(\phi_t^{-1}x) \]

\[ (\phi_t Y)(x) = D\phi_t(Y(\phi_t^{-1}x)) \]

\[ \text{Push-forward of a vector field } Y \text{ by } X. \]
Push-forward of a vector field by another vector field

Push-forward

• Suppose $\Phi_t : M \to M$ is the flow associated with a vector field $X$, then the *push-forward of a vector field* $Y$ on $M$ by $f$ is the vector field $(\Phi_t*Y)$ on $M$ defined by

$$(\Phi_t*Y)(x) = T(\Phi_t^{-1}x)[Y(\Phi_t^{-1}x)] \quad \forall x \in M$$

• In coordinates

$$(\Phi_t*Y)(x) = (D\Phi_t)(Y(\Phi_t^{-1}x))$$
Push forward of vector fields under a diffeomorphism $f$
Push-forward of vector fields under a diffeomorphism \( f \)

Push-forward

- Suppose \( f : M \to N \) is a diffeomorphism, then the push-forward of a vector field \( X \) on \( M \) by \( f \) is the vector field \( f_*X \) on \( N \) defined by

\[
(f_*X)(f(x)) = T_x f (X(x)) \quad \forall x \in M
\]

- In coordinates

\[
y = f(x) \quad (f_*X)(y) = Df(x).X(x) = \frac{dy}{dx} \cdot X(x)
\]
Pull back of a vector field under a diffeomorphism $f$
Pull-back of vector fields under a diffeomorphism $f$

The pull-back

- Suppose $f : M \to N$ is a diffeomorphism, then the **pull-back** of a vector field $Y$ on $N$ by $f$ is the vector field $f^* Y$ on $M$ defined by

$$f^* Y = (f^{-1})_* Y = T f^{-1} \circ Y \circ f$$

- In coordinates

$$y = f(x) \quad (f_* X)(y) = Df(x).X(x) = \frac{dy}{dx} \cdot X(x)$$
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Operations on vector fields

The Gradient
Consider a smooth function $g(\cdot) : U \rightarrow \mathbb{R}$. The gradient of such a function, denoted by $\nabla g$, is defined as

$$\nabla g(x) = \left[ \frac{\partial g}{\partial x_1} \quad \cdots \quad \frac{\partial g}{\partial x_n} \right]$$

alternate notation: $\text{grad}(g)$. 
The Lie derivative of a function

The Lie derivative
The Lie derivative of a function $f$ along $X$ is

$$(L_X f)(x) = \frac{d}{dt} |_{t=0} (\Phi_t^* f)(x) = \frac{d}{dt} |_{t=0} f \circ \Phi_t(x)$$

In coordinates we have the familiar

$$(L_X f)(x) = \left[ \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right] X(x)$$
The Lie derivative of a function

The Lie derivative
The Lie derivative of a function \( f \) along \( X \) is

\[
(L_X f)(x) = \frac{d}{dt}\bigg|_{t=0} (\Phi_t^* f)(x) = \frac{d}{dt}\bigg|_{t=0} f \circ \Phi_t(x)
\]

In coordinates we have the familiar

\[
(L_X f)(x) = \left[ \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right] X(x)
\]

Alternate notation

\[
(X f)(x) = \frac{d}{dt}\bigg|_{t=0} f \circ \Phi_t(x) = \lim_{t \to 0} \frac{f(\Phi_t(x)) - f(x)}{t}
\]
High school physics

The cross product

- Vector space $\mathbb{R}^3$ and the cross-product operation $\times$.
  - $(\alpha_1 a_1 + \alpha_2 a_2) \times b = \alpha_1 (a_1 \times b_1) + \alpha_2 (a_2 \times b_2) -$ linearity. (*holds in the second argument as well.*)
  - $a \times b = -b \times a -$ skew-commutative.
  - $a \times (b \times c) + c \times (a \times b) + b \times (c \times a) = 0 -$ the Jacobi-Lie identity.

Comment: the cross-product of two linearly independent vectors in $\mathbb{R}^3$ yields a vector in a new direction.
High school physics

The cross product

- Vector space $\mathbb{R}^3$ and the cross-product operation $\times$.
  - $(\alpha_1 a_1 + \alpha_2 a_2) \times b = \alpha_1 (a_1 \times b_1) + \alpha_2 (a_2 \times b_2)$ - linearity. (holds in the second argument as well.)
  - $a \times b = -b \times a$ - skew-commutative.
  - $a \times (b \times c) + c \times (a \times b) + b \times (c \times a) = 0$ - the Jacobi-Lie identity.

Comment: the cross-product of two linearly independent vectors in $\mathbb{R}^3$ yields a vector in a new direction.

An alternate notation

$$a \times b \leftrightarrow \hat{a}b$$

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$
The Lie derivative of a vector field

The pull back of a vector field

The Lie derivative of $Y$ along $X$ is

$$\mathcal{L}_X Y \triangleq \frac{d}{dt} \big|_{t=0} \Phi_t^* Y$$

where $\Phi$ is the flow of $X$. 
The Lie derivative of a vector field

The pull back of a vector field
The Lie derivative of $Y$ along $X$ is

$$\mathcal{L}_X Y \triangleq \frac{d}{dt} \bigg|_{t=0} \Phi_t^* Y$$

where $\Phi$ is the flow of $X$.
Explicitly

$$(\mathcal{L}_X Y)(x) = \frac{d}{dt} \bigg|_{t=0} (D\Phi_t(x))^{-1} \cdot Y(\Phi_t(x))$$

The Lie bracket
In coordinates we have the familiar expression

$$\frac{d}{dt} \bigg|_{t=0} (D\Phi_t(x))^{-1} \cdot Y(\Phi_t(x)) = \frac{\partial Y}{\partial x} X(x) - \frac{\partial X}{\partial x} Y(x) = [X, Y](x)$$
The Jacobi-Lie bracket

The Jacobi-Lie bracket of two vector fields is an operation between two vector fields that yields another vector field. For two vector fields $X$ and $Y$, both defined from $U$ to $\mathbb{R}^n$, it is defined as

$$[X, Y] = (\mathcal{L}_XY) = (DY) \cdot X - (DX) \cdot Y$$

and satisfies the following properties (for any three vector fields $X, Y, Z$)

- $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ - linearity in the first argument (also hold for the second argument)
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [X, Z]] = 0$ - the Jacobi-Lie identity
More properties

The Jacobi-Lie bracket
Let $X$ generate the flow $\{\Phi_t\}$ and $Y$ generate the flow $\{\Psi_t\}$. Then $[X, Y] = 0$ if and only if $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for all $s, t \in \mathbb{R}$. 

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The Lie algebra - $\mathfrak{so}(3)$

$3 \times 3$ skew-symmetric matrices

Recall

$\omega \times x \leftrightarrow \hat{\omega}x \quad \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

The eigen values of $\hat{\omega}$ are $0, \pm \|\omega\| i$ (Hint: The trace of a matrix is the sum of its eigen values.)

Claim

*Exponential of a skew-symmetric matrix is a rotation matrix*

To show $e^{\hat{\omega}} \in SO(3)$

$$(e^{\hat{\omega}})(e^{\hat{\omega}})^T = (e^{(\hat{\omega}-\hat{\omega})}) = I \Rightarrow \det(e^{\hat{\omega}}) = \pm 1$$

Now from $\omega = 0$, $e^{\hat{\omega}} = I$ and $\det(I) = 1$. The determinant is a continuous function of the elements of the matrix

$\Rightarrow \det(e^{\hat{\omega}}) = 1$
Properties of the Lie algebra - $\mathfrak{so}(3)$

It is a vector space of dimension 3.
The tangent space of the identity of $SO(3)$ i.e. $T_e SO(3) = \mathfrak{so}(3)$.
The bracket operation $[\cdot, \cdot] : \mathfrak{so}(3) \times \mathfrak{so}(3) \rightarrow \mathfrak{so}(3)$ satisfies

- $[\alpha \hat{x} + \beta \hat{y}, \hat{z}] = \alpha [\hat{x}, \hat{z}] + \beta [\hat{y}, \hat{z}]$ - linearity in the first argument (also hold for the second argument)
- $[\hat{x}, \hat{z}] = -[\hat{z}, \hat{x}]$ - skew-commutative.
- $[\hat{x}, [\hat{y}, \hat{z}]] + [\hat{z}, [\hat{x}, \hat{y}]] + [\hat{y}, [\hat{z}, \hat{x}]] = 0$ - the Jacobi-Lie identity

Notice that the cross product relates as

$$[\hat{x}, \hat{z}] = \hat{x} \times \hat{z}$$