Characterizing the reachable set for a spacecraft with two rotors

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Abstract—One commonly employed actuation device to change the orientation and angular velocity of a satellite is the reaction wheel (or internal rotor.) With this mode of actuation, to achieve any arbitrary orientation and angular velocity, three internal rotors suffice. A problem of long standing interest however, is the aspect of controllability when there are only two rotors - either by design or due to the failure of one rotor. An early result on this problem is by Crouch [1]. The paper affirms that the system is not accessible when there are two rotors. However, there is no attempt to characterize the reachable set in this case. In this paper, using a geometric framework based on the Lagrange -Routh reduction procedure, we characterize the reachable set for the case of two rotors.

I. INTRODUCTION

Attitude control of spacecrafts using momentum exchange devices such as momentum or reaction wheels and control moment gyroscopes has been studied widely during the past two decades.

One of the earliest papers on the controllability of a spacecraft with rotors is by Crouch [1]. The paper shows that with three orthogonally mounted rotors a spacecraft is globally controllable with respect to the orientation and angular velocity, but it is not possible to achieve this with less than three rotors. There have been numerous investigations of this case in the subsequent years in terms of deriving control laws for a spacecraft actuated by rotors. However, most of the work has been towards controlling angular velocity equations alone [2], [3], [4], [5], or for the case when the inertial angular momentum, which is a conserved quantity for the system, is zero [6], [7], [8].

Krishnan and McClamroch [6] analyze the complete attitude dynamics of a spacecraft actuated by two rotors when the inertial angular momentum is zero and conclude that the system is small time locally controllable. The problem of complete attitude control with two rotors with a nonzero inertial angular momentum has been dealt with only recently [9], [10]. In [9], the authors propose a control law based on feedback linearization for the initial attitude acquisition maneuver. Controllability analysis in this case is carried out by Boyer and Alamir [10] where they conclude that a five-dimensional subspace of feasible states is potentially reachable. They also propose control laws for a few attitude maneuver problems.

In this paper we explicitly characterize the reachable set - in the orientation and angular velocity space - for a spacecraft with two internal rotors. The approach is geometric and employs the theory of reduction. Exploiting the fact that the dynamics of the system evolves on a level set of the angular momentum function, we study the system using the Lagrange-Routh reduction theory [12] which is an appropriate framework for such systems. From a system evolving on the configuration space \( Q = SO(3) \times T^2 \) with the dynamics described on \( TQ \), reduction helps us to study the equations over the manifold \( SO(3) \times \mathbb{R}^2 \). Through a reconstruction procedure we obtain local representations over \( S^1 \times S^2 \times \mathbb{R}^2 \) of the equations over \( SO(3) \times \mathbb{R}^2 \) and obtain a characterization of the reachable set.

II. OBSERVATIONS BY CROUCH [1]

We reproduce the part from [1] relevant to our work.

Consider a spacecraft with two internal rotors under a zero-gravity condition. The configuration space for this system is \( Q = SO(3) \times S^1 \times S^1 \). Let \( R \) denote the orientation-transformation that transforms the spacecraft body coordinates to the inertial coordinates. Further, let \( I_s \) denote the moment of inertia of the spacecraft alone in the body frame and \( (I_r)_i \) denote the inertia of the \( i \)th rotor, \( \mu \) the total angular momentum in the inertial frame which is conserved and \( \Omega_b \) the spacecraft angular velocity expressed in the bodyframe. The inertial angular momentum of each rotor is denoted by \( h_i \). The total angular momentum being conserved is expressed as

\[
\sum_{i=1}^{2} (I_r)_i (\Omega_b + (\Omega_r)_i) + I_s \Omega_b = R^T \mu
\]  

(1)

The angular momentum of each rotor and the dynamics of each rotor due to a torque applied along the principal axis \( b_i \) are expressed as

\[
(I_r)_i (\Omega_b + (\Omega_r)_i) = R^T h_i
\]

\[
\langle b_i, \frac{d}{dt} (R^T h_i) \rangle = \tau_i = \langle b_i, -b_i u_i \rangle
\]

(2)

where \( b_i \) is defined as the control input to the \( i \)th rotor. The equations of motion are

\[
\dot{R} = R \hat{\Omega}_b
\]

\[
I_s \hat{\Omega}_b = \sum_{i=1}^{2} b_i u_i - \hat{\Omega}_b \dot{R}^T \mu
\]

\[
\mu = \text{constant}
\]

(3)

With two rotors, there exists a vector \( c \in \mathbb{R}^3 \) such that

\[
\langle c, b_i \rangle = 0 \quad \text{for} \quad i = 1, 2.
\]
The above constraint translates to the following algebraic constraint in $T^2Q$,

$$\langle c, I_s \dot{\Omega}_b - \Omega_b R^T \mu \rangle = 0,$$

which defines a level set in the manifold $T^2Q$. The existence of constraint (1) precludes the system from reaching every two-tuple $((R, \theta_1, \theta_2), (\Omega_b, \dot{\theta}_1, \dot{\theta}_2))$ in the tangent bundle.

Our notation here differs from Crouch’s and is related to the one in Crouch’s paper as follows:

$$I_s \rightarrow I_L, \sum_{i=1}^{m} (I_i), \rightarrow I_r, \sum_{i=1}^{m} h_i \rightarrow I, \text{ diag}(\theta_i) \rightarrow \Omega_r,$$

where $m$ is the number of rotors.

III. THE LAGRANGE-ROUTH EQUATIONS FOR A SPACECRAFT WITH ROTORS

In this section, we derive the Lagrange-Routh equations for a spacecraft with rotors.

A. Preliminaries

In this section we derive the objects necessary for Lagrange-Routh reduction of the equations of motion for a spacecraft with rotors.

In our model of the spacecraft with rotors, we consider the spacecraft as a rigid body. The configuration manifold of the model is then $Q = SO(3) \times \mathbb{T}^k$, $k = 1, \ldots, 3$. Here $SO(3)$ corresponds to the orientation of the spacecraft with respect to an inertial frame and the $k$-torus $\mathbb{T}^k$ is the $k$-fold Cartesian product of $S^1$ where $k$ denotes the number of rotors and each $S^1$ represents the orientation of a rotor with respect to the spacecraft. We denote the configuration variables of the spacecraft with rotors by $(R, \Theta)$ where $\Theta = (\Theta_k), k = 1, \ldots, 3$. We treat $S^1$ as a submanifold of $SO(3)$ and hence $\Theta_k$ act on vectors in $\mathbb{R}^3$. Often we denote by $q = (R, \Theta)$ the configuration variables and $v_q = (R, \Theta)$ the tangent vectors at $q$.

The kinetic energy of the spacecraft can be modelled through a Riemannian metric over $SO(3) \times \mathbb{T}^k$ defined as

$$\langle \dot{R}_1, \dot{R}_2 \rangle_{(R, \Theta)} = \int_B (\dot{R}_1 X)^T (\dot{R}_2 X) \, dV + \sum_{i=1}^{k} \int_{r_i} (R \dot{\Theta}_{1i} X + \dot{R}_1 \Theta_{1i} X)^T (R \dot{\Theta}_{2i} X + \dot{R}_2 \Theta_{2i} X) \, dV,$$

where the first term is an integral over the spacecraft body excluding the rotors. The second term is the sum of the integral over the rotors. Here $X$ denotes the position vector of the elemental volume expressed in the body coordinates of the spacecraft, as shown in figure 1. The kinetic energy of the spacecraft is given by

$$L(v_q) = \langle v_q, v_q \rangle_{(R, \Theta)}.$$

This Lagrangian is invariant under the standard tangent lifted $SO(3)$ action on the first element given by $\phi_p : SO(3) \times \mathbb{T}^k \rightarrow SO(3) \times \mathbb{T}^k, \phi_p(R, \Theta) = (PR, \Theta)$. Next we compute the locked inertia tensor, momentum map and the mechanical connection associated with this group action.

![Fig. 1. Rigid body model of a spacecraft with rotors.](image)

The locked inertia tensor, the momentum map and the mechanical connection one-form [13] for the model can respectively be worked out as

$$I(q) = Ad^{-1}_R I_L Ad_R^{-1},$$

$$J(v_q) = Ad^{-1}_R (I_L \Omega_b + I_r \Omega_r),$$

$$A(v_q) = Ad_R (\Omega_b + A_r \Omega_r).$$

Here $I_r$ is the combined moment of inertia tensor of the rotors, expressed as a $3 \times 3$ matrix. In writing the last equation above, we have identified $\mathfrak{so}(3)$ with $\mathbb{R}^3$ using the standard identification. The body angular velocity $\Omega_b$ is such that $\Omega_b = R^{-1} \dot{R}$. The rotor angular velocities $\Omega_r \in \mathbb{R}^3$ is the vector representing the spin angular velocities of the rotors with respect to the spacecraft frame, expressed in the spacecraft body frame.

B. Lagrange-Routh equations

In this section, we derive the Lagrange-Routh equations for the spacecraft with rotors. Let us first consider the consequence of the global realization theorem 1.1 (see appendix A) for our system. We have $Q = SO(3) \times \mathbb{T}^k$. Therefore, as we have seen in section A,

$$J^{-1}(\mu)/G_{\mu} \cong (\mathcal{O}_\mu \times \mathbb{T}^k) \times_\mathbb{T}^{k} T(\mathbb{T}^k).$$

We know that $T(\mathbb{T}^k) \cong \mathbb{R}^k \times \mathbb{T}^k$. Hence, $J^{-1}(\mu)/G_{\mu} \cong (\mathcal{O}_\mu \times \mathbb{R}^k) \times \mathbb{T}^{k}$. Thus,

$$J^{-1}(\mu)/G_{\mu} \cong (\mathcal{O}_\mu \times \mathbb{R}^k) \times \mathbb{T}^{k}.$$

The coadjoint orbit $\mathcal{O}_\mu$ for our system is given by

$$\mathcal{O}_\mu = \{ \Pi \in \mathfrak{so}(3)^* \mid \exists R \in SO(3)$$

such that $Ad^*_R \mu = R^T \mu = \Pi \}.$$
Now, the Routhian is defined on $TQ$ as

$$R^\mu(v_q) = L(v_q) - \langle \mu, A(v_q) \rangle.$$ 

This function is $G_\mu$ invariant. Therefore we can restrict this function to $J^{-1}(\mu) \subset TQ$ and quotient it by $G_\mu$. Thus the Routhian induces a function $R^\mu$ on $J^{-1}(\mu)/G_\mu$. The Lagrange-Routh equations on $J^{-1}(\mu)/G_\mu$ are obtained by using variational principle described in equation (30).

When the configuration space has an explicit product structure $G \times S$, the variational principle leads to equations in a simpler form as follows. Let $q$ be the curve that satisfies the Euler-Lagrange equations. Let $\Pi(t) = \pi_{Q,G} q(t)$ be the projection of $q$ onto the first factor of $J^{-1}(\mu)/G_\mu$ and $x(t) = \pi_{Q,G} q(t)$. Then the variational principle (30) can be shown to be globally equivalent to

$$-\frac{\partial V_\mu}{\partial \Pi}(\Pi, x) \cdot \delta \Pi = \beta_\mu(\Pi, \dot{x}, (0,0)), \quad (10)$$

$$\left( \frac{\partial R^\mu}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial R^\mu}{\partial x} \right) \cdot \delta x = \beta_\mu(\Pi, \dot{x}, (0,0)) + u, \quad (11)$$

where $u$ is the control torque and $V_\mu$ is the amended potential, given by

$$V_\mu(\Pi) = \Pi^T \bar{J} \Pi.$$

To compute $\beta_\mu$, we proceed as follows. From equation (8), we get

$$\{ \mu, A(v_q) \} = \{ \mu, \text{Ad}_R(\Omega_b + A_s \Omega_r) \} = \{ \mu, \text{Ad}_R \Omega_b \} + \{ \Pi, A_s \Omega_r \},$$

Motivated by the above equation, we split $A_s$ as $A_s = A'_s + \delta$, where

$$A'_s(q)(v_q) = \{ \mu, \text{Ad}_R \Omega_b \}, \quad \delta(R, x)(v_q) = \{ \mu, \text{Ad}_R A_s \Omega_r \}.$$ 

The one-form $\tilde{\delta}$ can be dropped to $O_\mu \times S$ as $\sigma(\Pi, x)(v_q) = \{ \Pi, A_s \Omega_r \}$. Thus, $\sigma$ satisfies $\sigma = \pi_{Q,G}^{-1} \sigma$. Note that the one-form $A'_s$ does not depend on $S$. It is known [13] that $dA'_s = \pi_{Q,G} \omega$, where $\omega$ is the canonical two form on $O_\mu$. Thus, $\beta_\mu$ satisfies $dA_s = \pi_{Q,G}^{-1} \beta_\mu$. Since $\beta_\mu$ is unique, we have computed the two-form required by theorem 1.1.

In our case $O_\mu \cong S^2$. At $\Pi \in S^2$, for any tangent vector $v_\Pi \in T_\Pi S^2$, we can find a $\Pi_1 \in \mathbb{R}^3$ such that $v_\Pi = \Pi \times \Pi_1$. With this, the canonical two form on $S^2$ can be written as

$$\omega(\Pi)(\Pi \times \Pi_1, \Pi \times \Pi_2) = \frac{1}{\|\Pi\|^2} \Pi \cdot (\Pi_1 \times \Pi_2). \quad (12)$$

For computing $d\sigma$, we proceed as outlined in [11]. The one-form $\sigma$ is actually defined as

$$\sigma(\Pi, \Theta)(\Pi, \Omega_r) = \{ \Pi, A_s \Omega_r \}.$$ 

This is nothing but the pull-back by inclusion $i : O_\mu \times T^k \rightarrow g^* \times T^k$ of the one form $\sigma^*$ on $g^* \times T^k$ given by

$$\sigma^*(\Pi, \Theta)(\Pi, \Omega_r) = \Pi \cdot A_s \Omega_r.$$ 

That is, $\sigma = i^* \sigma^*$. In coordinates, this can be written as

$$\sigma^*(\Pi, \Theta) = \Pi_i (A_s)_{ij} d \Theta_j.$$ 

The exterior derivative of this is the two form on $g^* \times T^k$ given by

$$d\sigma^* = \Theta^* \sigma.$$

Hence we get,

$$d\sigma^* = \gamma^* \sigma, \quad (\Pi_1, \Omega_{r_1}), (\Pi_2, \Omega_{r_2})$$

$$\Pi_1 \cdot A_s \Omega_{r_2} - \Pi_2 \cdot A_s \Omega_{r_1}.$$ 

Now, $d\sigma = d(i^* \sigma^*) = i^* d\sigma = i^* \gamma^*$. Therefore, making use of equations (12) and (13), we have

$$\beta_\mu(\Pi, \Theta) \left( (\Pi_1, \Omega_{r_1}), (\Pi_2, \Omega_{r_2}) \right)$$

$$= \frac{1}{\|\Pi\|^2} \Pi \cdot (\Pi_1 \times \Pi_2) + \Pi_1 \cdot A_s \Omega_{r_2} - \Pi_2 \cdot A_s \Omega_{r_1}.$$ 

With this, equation (10) gives

$$-I_L^{-1} \Pi \cdot \delta \Pi = \frac{1}{\|\Pi\|^2} \Pi \times \delta \Pi = \Omega_r,$$

for all $\delta \Pi$ tangential to $O_\mu$. This essentially implies the following evolution equation for $\Pi$,

$$\dot{\Pi}(t) = \Pi(t) \times I_L^{-1} \Pi(t) - \Pi(t) \times A_s \Omega_r(t).$$

We can show that

$$R^\mu = \frac{1}{2} \Omega_r^T A_s (I_L - I_r) \Omega_r - V_\mu.$$ 

Using equation (11), we get

$$-A_s (I_L - I_r) \Omega_r \cdot \Theta = \Pi \cdot A_s \delta \Theta + u,$$

for all $\delta \Theta$. Since $A_s$ is symmetric, we get

$$A_s (I_L - I_r) \Omega_r(t) = -A_s \Pi(t) - u.$$ 

Therefore the equations of motion over the reduced phase space are given by

$$\dot{\Pi} = \Pi \times I_L^{-1} \Pi - \Pi \times A_s \Omega_r, \quad (14)$$

$$\dot{\Theta} = \Omega_r, \quad (15)$$

$$A_s (I_L - I_r) \Omega_r(t) = -A_s \Pi(t) - u.$$ 

These equations give the evolution of the dynamics over $J^{-1}(\mu)/G_\mu \cong (S^2 \times \mathbb{R}^k) \times T^k$. Now, from the spacecraft control point of view, we are not interested in the configuration variable $\Theta$; we are interested only in $\Omega_r := \Theta$. Therefore we confine our attention to the variables $\Pi$ and $\Omega_r$ governed by equations (14) and (16), described over $S^2 \times \mathbb{R}^k$, which can be considered as a submanifold of $J^{-1}(\mu)/G_\mu$.

C. Equivalence of Lagrange-Routh equations and Lie-Poisson equations

The Lie-Poisson equations on $g^* \times \mathbb{R}^k$ for a spacecraft with three rotors are as follows [14].

$$\dot{\Pi}(t) = \Pi(t) \times (I_L - I_r)^{-1} (\Pi(t) - l(t))$$

$$\dot{l}(t) = u,$$

where $l = I_r(\Omega_r + \Omega_r)$. We now show that these equations are the same (up to a diffeomorphism) as the Routh equations on $O_\mu \times \mathbb{R}^k$, that is equations (14) and (16). For that, we define
the following diffeomorphism \( \phi: g^* \times \mathbb{R}^k \rightarrow g^* \times \mathbb{R}^k \) given by

\[
(\Pi, l) := \phi(\Pi, \Omega_r) = (\Pi, A_s[\Pi + (I_L - I_r)\Omega_r]).
\]  

(17)

It can be shown that indeed \( l = I_r(\Omega + \Omega_r) \). Now, \( \phi \) is nothing but the linear map given by

\[
\begin{pmatrix}
\Pi \\
l
\end{pmatrix} =
\begin{pmatrix}
I_{3 \times 3} & 0 \\
A_s & -A_s I_r + I_r
\end{pmatrix}
\begin{pmatrix}
\Pi \\
\Omega_r
\end{pmatrix}
\]

(18)

Making use of the fact that \( I_r \) is diagonal, we see that this map is an isomorphism and hence a diffeomorphism. Now, note that since the first component is the identity, \( \phi \) maps \( \mathcal{O}_\mu \) to itself. Thus, we can restrict \( \phi \) to \( \mathcal{O}_\mu \times \mathbb{R}^k \) to get a diffeomorphism. Now, differentiating equation (17), we get

\[
(\Pi, l) = (\hat{\Pi}, A_s[I\Pi + (I_L - I_r)\hat{\Omega}_r]).
\]

Substituting for \( \hat{\Omega}_r \) from equation (16) we get the Lie-Poisson equations. Denoting \( \hat{I} := (I_L - I_r)^{-1} \), we can write the governing equations in terms of the new coordinates as

\[
\hat{\Pi} = \Pi \times \hat{I}(\Pi - l),
\]

(19)

\[
l = u.
\]

(20)

We shall use these equations for our analysis. The following identity can be shown which we shall use in the reconstruction equation,

\[
I_l^{-1}I_A - A_s \Omega_r = (I_L - I_r)^{-1}(\Pi - l).
\]

(21)

IV. THE RECONSTRUCTION EQUATION

The Lagrange-Routh equations evolve on \( J^{-1}(\mu)/G_\mu \). Let us denote the vector field given by these equations by \( X_\mu \). These equations in a sense represent the ”essential dynamics”. Let us denote by \( X \) the vector field corresponding to the original dynamics, which evolves over \( J^{-1}(\mu) \). Thus, the vector field \( X_\mu \) is essentially the projection of \( X \) onto \( J^{-1}(\mu)/G_\mu \). The “factored out” dynamics on \( G_\mu \) can be recovered through a reconstruction procedure. If we denote the dynamics on \( G_\mu \) by \( X_{G_\mu} \), then \( (X_{G_\mu}, X_\mu) \) is a local representation of the vector field \( X \). We now use the reconstruction procedure to get local representations for the vector field \( X \).

A. Reconstruction procedure

As outlined in [12], one of the methods we can use to derive the reconstruction equations as is follows. Given a curve \( y \) on \( Q/G_\mu \), we find a curve \( \hat{y} \) in \( Q \) that projects to \( y \). Then we set \( \hat{q}(t) = g(t)\hat{q}(t) \) where \( g(t) \in G_\mu \) and require that \( \hat{q}(t) \in J^{-1}(\mu) \). With this set up, the evolution for \( g(t) \) on \( G_\mu \) turns out to be

\[
\dot{g}(t) = g(t)\left[ I(\hat{q})^{-1}A(\hat{g}(t)) \right],
\]

(22)

where \( I(q) \) is the locked inertia tensor and \( A \) is the mechanical connection.

The procedure outlined above can be also be understood as follows. We know that \( \pi: Q \rightarrow Q/G_\mu \) is a principal \( G_\mu \) bundle. Let \( K: Q/G_\mu \rightarrow Q \) be a local section [16] of the bundle, which is also differentiable. That is, \( K \) is a local function that satisfies the condition \( \pi \circ K = \text{id} \). This local section induces a parameterization of \( Q \) in terms of \( G_\mu \) and \( Q/G_\mu \) through the map

\[
\varphi: Q \rightarrow G_\mu \times Q/G_\mu, \quad \varphi(q) = (g, p),
\]

where \( p = \pi(q) \) and \( g \) is such that \( q = g \cdot K(p) \) [16].

Given a \( G_\mu \)-invariant tangential vector field \( X \) on \( J^{-1}(\mu) \), let \( X_\mu \) be its projection on \( Q/G_\mu \). In terms of the local parameterization induced by \( K \), we can get the local parameterization of the vector field \( X \) as follows. The \( G_\mu \)-component of the vector field \( X \) is given by [12]:

\[
X_G(g, p) = TL_{g}[(I(K(p)))^{-1} - A(T_pKX_\mu(p))].
\]

Thus, \( (X_G(g, p), X_\mu(p)) \) give a local parameterization on \( G_\mu \times Q/G_\mu \) of the vector field \( X \).

For our system, we can proceed with the reconstruction procedure as follows. Given a curve \( y = (\Pi, \Theta) \) in \( Q/G_\mu \), we have to choose the curve \( \bar{q} = (R, \Theta) \) in \( SO(3) \times \mathbb{T}^k \) such that it projects to \( (\Pi, \Theta) \). We know that the projection \( \pi: SO(3) \times \mathbb{T}^k \rightarrow \mathcal{O}_\mu \times \mathbb{T}^k \) is given by \( \pi(R, \Theta) = (RT\mu, \Theta) \). Therefore we have to choose the curve \( R \) such that \( RT\mu = \Pi(t) \) or in other words such that \( R(t) \) rotates \( \Pi(t) \) to \( \mu \).

In the context of local sections, we are thus looking for a map \( K: S^2 \rightarrow SO(3) \) such that \( K(\Pi) \in SO(3) \) as a matrix rotates \( \Pi \) to \( \mu \). In [12], the authors effectively choose a local section \( K_1 \) as follows:

\[
K_1: S^2 \rightarrow SO(3), \quad K_1(\Pi) = \exp(\overrightarrow{w_1}(\Pi)),
\]

where \( w \) defined as

\[
w_1(\Pi) = \varphi \frac{\Pi \times \mu}{\|\Pi \times \mu\|}, \quad \cos \varphi = \frac{(\Pi \cdot \mu)}{\|\mu\|^2}.
\]

Then the required curve \( \bar{R} \) can be obtained as \( \bar{R}(t) = K_1 \circ \Pi(t) \). Note that this section is not defined at two points on \( S^2 \), namely at \( \Pi = \pm \mu \).

It is possible to choose a local section such that the singularities at \( \pm \mu \) have been ‘transferred’ to other points. Let \( \Pi_0 \neq \pm \mu \) be a point on \( S^2 \). Then we define the local section \( K_2 \) as

\[
K_2(\Pi) = \exp(\overrightarrow{w_2}) \exp(\overrightarrow{w_1}(\Pi)),
\]

where

\[
w_1(\Pi) = \varphi_1 \frac{\Pi \times \Pi_0}{\|\Pi \times \Pi_0\|}, \quad \cos \varphi_1 = \frac{(\Pi \cdot \Pi_0)}{\|\Pi_0\|^2},
\]

and

\[
w_2 = \varphi_2 \frac{\Pi_0 \times \mu}{\|\Pi_0 \times \mu\|}, \quad \cos \varphi_2 = \frac{(\Pi_0 \cdot \mu)}{\|\mu\|^2}.
\]

It can be seen that this section is well defined at \( \Pi = \pm \mu \) but has singularities at \( \Pi = \pm \Pi_0 \).

Now, let \( w_\mu = \mu / \|\mu\|^2 \). Then, we know that any \( g \in G_\mu \) can be written as \( g = \exp(\beta w_\mu) \), where \( \beta \in S^1 \). Thus, let \( \alpha \) be the curve in \( G_\mu \cong S^1 \) such that \( g = \exp(\alpha w_\mu) \), then \( \alpha \theta \) satisfies the condition \( \hat{q}(t) \in J^{-1}(\mu) \).
As shown in equation IV.4 of [12], and making use of the expression (8) for the mechanical connection, the evolution of \( \alpha \) such that \( \dot{q}(t) \in J^{-1}(\mu) \) is given by
\[
\dot{\alpha}(t) = \frac{\mu}{[\mu]} \left( (\tilde{R}(t)I^1_3 R^T(t)\mu - \tilde{R}(t)A_s \Omega_r - \left[ \dot{R}(t) R^T(t) \right] ) \right) \tag{23}
\]
This equation gives the \( S^1 \)-component of the local parameterization of the vector field \( X \).

**B. The reconstruction equation**

Suppose the curve \( y : \mathbb{R} \to J^{-1}(\mu)/G_\mu \) be the solution to the controlled equations (25) and (26). Let us denote \( y(t) = (\Pi(t), \Theta(t)) \). Let \( P : \mathbb{R} \to SO(3) \) such that \( P = K \circ y \), where \( K \) is a local section as described in section IV-A. Making use of the equation (25) in the equation (23), it can be shown that \( \alpha \) evolves according to the equation
\[
\dot{\alpha} = (\Pi - p(\Pi)) \cdot \dot{I}(\Pi - I),
\]
where \( p(\Pi) := -\Pi \times (T_{\Pi} w_1^T \Pi) \) and \( T_{\Pi} w_1 \) denotes the tangent map of the map \( w_1 \). Therefore, we get the following equations as a local representation of the vector field \( X \) which is tangential to \( J^{-1}(\mu) \), in the coordinates \( (\alpha, \Pi, l) \).
\[
\dot{\alpha} = (\Pi - p(\Pi)) \cdot \dot{I}(\Pi - I) \tag{24}
\]
\[
\dot{\Pi} = \Pi \times \dot{I}(\Pi - I) \tag{25}
\]
\[
\dot{l} = u. \tag{26}
\]
Along the solutions of the equations above, the spacecraft orientation can now be reconstructed as
\[
R(t) = \exp(\alpha(t) \bar{\omega}_\mu) \tilde{R}(\Pi(t)). \tag{27}
\]

**V. CONTROLLABILITY ANALYSIS**

For a spacecraft with two rotors, the equations (24) – (26) represent a control system over \( S^1 \times S^2 \times \mathbb{R}^2 \). We treat \( S^2 \) as embedded in \( \mathbb{R}^3 \). In this section, we show that the control system (24) – (26) is globally controllable and use this to characterize the reachable set of a spacecraft with rotors.

The control system (24) – (26) can be written in the standard form
\[
\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x),
\]
where \( x = (\alpha, \Pi, l) \in S^1 \times S^2 \times \mathbb{R}^2 \) and
\[
f_0(x) = \begin{pmatrix} \left( \Pi - p(\Pi) \right) \cdot \dot{I}(\Pi - I) \\ \Pi \times \dot{I}(\Pi - I) \end{pmatrix}, \quad f_i = \begin{pmatrix} 0 \\ 0_{3 \times 1} \end{pmatrix},
\]
for \( i = 1, 2 \) where \( e_i \) is the standard basis in \( \mathbb{R}^2 \).

The controllability analysis of a system with drift is difficult in general. However there are specific results when the drift vector field has certain special properties. In [15], it has been established that if the drift vector field is weakly positively Poisson stable (WPPS), then controllability is equivalent to the Lie algebra rank condition (LARC). We proceed along these lines.

We first show that the uncontrolled \( (\dot{l} = 0) \) vector field \( X_t(\Pi) := \Pi \times I(\Pi - I) \) on \( S^2 \) is a WPPS vector field. From this it follows that the vector field \( f_0(\Pi, l) \) is WPPS on \( S^2 \times \mathbb{R} \). We make use of the result that any Hamiltonian vector field on a compact manifold is WPPS [15]. We have the following observation, the proof of which is straightforward.

**Proposition 5.1:** The vector field \( X_t(\Pi) := \Pi \times I(\Pi - I) \) is the Hamiltonian vector field on \( S^2 \) for the function \( H(\Pi) = c(\Pi - I)^T I(\Pi - I) \) (with the canonical symplectic structure), where \( c \) is a constant.

The function \( H(\Pi) = 1/2(\Pi - I)^T I(\Pi - I) \) actually corresponds to the parts of the kinetic energy of the spacecraft due to rotation with respect to the inertial frame.

Next, note that the integral curves of the vector field \( X_t \) lie in the intersection of the ellipse \( (\Pi - I)^T I(\Pi - I) = c_1 \) and the sphere \( \|\Pi\|^2 = c_2 \) where \( c_1 \) and \( c_2 \) are constants. Therefore, almost all integral curves of the uncontrolled vector field \( X \) are periodic on \( S^2 \). Making use of this, we have the following result.

**Proposition 5.2:** Given any \( (\alpha, \Pi) \) the control system can be steered to a state arbitrarily close to it.

**Proof:** We first note that from the above analysis, any \( \Pi \) can be reached on \( S^2 \). With this \( \Pi \) as initial condition, the integral curve of the uncontrolled vector field maybe periodic, an equilibrium point or may be a non-wandering [15]. If it is a non-wandering point, then there exists a value of \( \Pi' \) arbitrarily close to it such that the integral curve is periodic. Taking this \( \Pi' \) as the initial condition, we compute \( \alpha \) as follows, with the uncontrolled vector field.

\[
\alpha(t) = \alpha_0 + \int_0^t (\Pi(\tau) - p(\tau)) \cdot \dot{I}(\Pi(\tau) - I) \, d\tau
\]

Let the period of the integral curve \( \Pi(t) \) be \( T \). At the end of a period, we have \( \Pi(T) = \Pi' \) and \( \alpha(T) = (\alpha_0 + C) \mod{2\pi} \), where
\[
C = \int_0^T (\Pi(\tau) - p(\tau)) \cdot \dot{I}(\Pi(\tau) - I) \, d\tau.
\]

Similarly, \( \alpha(nT) = (\alpha_0 + nC) \mod{2\pi} \). Now, if \( C/2\pi \) is rational, there exists a \( n \) such that \( \alpha_T = (\alpha_0 + nC) \mod{2\pi} \). If \( C/2\pi \) is irrational, a value arbitrarily close to \( \alpha_T \) can be achieved after a finite number of periods.

From this, the following result about the vector field \( X \) holds.

**Proposition 5.3:** The vector field \( X \) on \( SO(3) \) is WPPS.

**Proof:** Given an open set \( U \subseteq SO(3) \), let \( V = \pi(U) \), which is open in \( S^2 \). The projection of the vector field \( X \) on \( S^2 = X_1 \) and this is periodic for almost all initial conditions. Hence, it is possible to choose a local section \( K \) such that it does not have a singularity along the path of the integral curve of \( X_t \) for any initial conditions in \( V \). Let \( q_0 \in U \) and let \( (\alpha_0, \Pi_0) \) be its representation in the local chart induced by \( L \). Now, by proposition 5.2, the flow of the vector field \( X \) originating from \( q_0 \) can be steered arbitrarily close to it. Thus, for any \( T > 0 \), there is a \( t > T \) such that \( \phi_t(U) \cap U \neq \varnothing \), where \( \phi_t \) is the flow of the vector field \( X \) at time \( t \).

Now we will show that the LARC holds for the control system - that is, the Lie algebra generated by the vector fields \( f_0, f_1, f_2 \) span \( T_{q_0}SO(3) \) at every \( R \in SO(3) \). Let \( \tilde{I} = \text{diag}(I_1, I_2, I_3) \). Let \( g_1 := [f_0, f_1] \) and \( g_2 := [f_0, f_2] \).
Then,
\[
g_1 = \begin{pmatrix} -\Pi - p(\Pi) & \tilde{I} e_1 \\ \Pi \times \tilde{I} e_1 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -\Pi - p(\Pi) & \tilde{I} e_2 \\ \Pi \times \tilde{I} e_2 & 0 \end{pmatrix}.\]

Also, let
\[
h_1 := [g_1, g_2] = \begin{pmatrix} q(\Pi, \mu) \\ \tilde{I}_1 \tilde{I}_2 (\Pi_1 e_2 - \Pi_2 e_1) \end{pmatrix},\]
where \(q(\Pi)\) is some function.

When \(\Pi_3 \neq 0\), it can be shown that the vectors \(f_0, g_1, g_2\) are linearly independent. We have omitted these computations due to the lack of space. In fact, the vectors \(-\Pi \times \tilde{I} e_1\) and \(-\Pi \times \tilde{I} e_2\) are linearly dependent when \(\Pi_3 = 0\).

When \(\Pi_3 = 0\), we proceed as follows. Let \(\Pi = (\Pi_1, \Pi_2, 0)\). We get
\[
g_1 = \begin{pmatrix} -\Pi - p(\Pi) & \tilde{I} e_1 \\ \Pi_2 \tilde{I} e_3 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -\Pi - p(\Pi) & \tilde{I} e_2 \\ -\Pi_1 \tilde{I} e_3 & 0 \end{pmatrix}\]
Writing \(\Pi = \Pi_1 e_1 + \Pi_2 e_2\), we see that a vector orthogonal to \(\Pi\) can be written as \(-\Pi_2 e_1 + \Pi_1 e_2 + ae_3\), where \(a\) is arbitrary. Therefore any vector orthogonal to \(\Pi\) can be spanned by \(h_1\) and either of \(g_1\) or \(g_2\). Therefore if we show that the determinant of the matrix formed by
\[
\begin{pmatrix} -\Pi - p(\Pi) & \tilde{I} e_1 \\ \Pi_2 \tilde{I} e_3 & 0 \\ -\Pi_1 \tilde{I} e_3 & 0 \end{pmatrix}
\]
is nonzero, then we have linear independence. It turns out that the determinant of the above matrix is given by \(\tilde{I}_1 \tilde{I}_2\). Hence we have linear independence.

Thus we have shown that the vector fields \(f_0, f_1, f_2\) verify LARC everywhere on \(S^1 \times S^2\). Along with this and proposition 5.3, we have thus shown

\[\text{Proposition 5.4: The control system on } SO(3) \times \mathbb{R}^2 \text{ represented by equations (24) – (26) is globally controllable.}\]

With these results in hand, we now give a characterization of the reachable set of a spacecraft with two rotors.

\[\text{Theorem 5.5: The (left trivialized) reachable set of } TSO(3) \text{ of a spacecraft with two rotors is } (R, I_L^{-1}(R^T \mu - I_r \Omega_r)) \text{ where } R \in SO(3) \text{ and } \Omega_r \in \mathbb{R}^3 \text{ is such that } e^T_3 \Omega_r = 0.\]

\[\text{Proof: By proposition 5.4, any point } (R_f, l_f) \in SO(3) \times \mathbb{R}^2 \text{ is accessible for a spacecraft with two rotors.}\]

Now, there is a diffeomorphism between the variable \(l\) and \(\Omega_r\) which satisfies \(e^T_3 \Omega_r = 0\) through the map \(\Phi\) defined in equation (17). Hence, it follows that any such pair \((R, \Omega_r)\) is also reachable. But \(R^T \mu = I_L \Omega_b + I_r \Omega_r\), which implies
\[
\Omega_b = I_L^{-1}(R^T \mu - I_r \Omega_r).\]

Thus, \((R, I_L^{-1}(R^T \mu - I_r \Omega_r))\) is the reachable set of \(TSO(3)\) for a spacecraft with two rotors.

\[\text{Visualizing the reachable set}\]

Every orientation \(R \in SO(3)\) is reachable by theorem 5.5. The restriction however is on the values of the body angular momentum that the spacecraft can achieve. In equation (29), the term \(I_L^{-1} I_r \Omega_r\) with the restriction specified on \(\Omega_r\), \((e^T_3 \Omega_r = 0)\) corresponds to all vectors with the third component being zero which forms a plane. The set of angular velocities is just the translation of this plane by the vector \(I_L^{-1} R^T \mu\). This is shown in figure 2.

For the purpose of illustration, we take an example of the UoSAT-12, referred to by Horri et. al [7]. The parameters are: \(I_L = \text{diag}(40.45, 42.09, 40.36)\), \(I_r = \text{diag}(8 \times 10^{-3}, 7.7 \times 10^{-3}, 0)\) in kg m². If \(\mu = (0, 0, 10)\) kg ms⁻¹, then at an orientation making an angle 45 degrees with the \(x\) axis, any angular velocity with the third component having the value 0.1752 rad s⁻¹ is possible.

Fig. 2. Reachable set of \(\Omega_b\) at a particular orientation.

\[\text{APPENDIX}\]

\[\text{A. The Lagrange Routh equations}\]

We give here a very brief overview of Lagrange-Routh reduction, which was developed in [11] and [12]. The required mathematical background can be found in [13] and [14].

Consider a simple mechanical system, whose configuration manifold is a smooth manifold \(Q\) and suppose a Lie group \(G\) acts on \(Q\) freely and properly. Let the mechanical system be described by a \(G\)-invariant Lagrangian \(L : TQ \rightarrow \mathbb{R}\). Suppose also that the Lagrangian be hyperregular so that we can equip \(TQ\) with the symplectic structure pulled back from \(T^*Q\) through the fiber derivative \(\overline{FL}\). Suppose an equivariant momentum map \(J : TQ \rightarrow g^*\) exists for this \(G\)-action.

By Noether’s theorem, the value of \(J\) is conserved along the flow of the Lagrangian vector field. If \(\mu \in g^*\) is this value, then it is clear the system evolves on the level set of \(J\) given by \(J^{-1}(\mu)\). However, there is further symmetry in the system so that one can reduce the equations further onto the reduced phase space \(J^{-1}(\mu)/G_\mu\), where \(G_\mu\) is the isotropy subgroup of \(\mu\).

The following is the global realization theorem for \(J^{-1}(\mu)/G_\mu\).

\[\text{Theorem 1.1 ([12]): The bundle } J^{-1}(\mu)/G_\mu \rightarrow Q/G \text{ is bundle isomorphic to } T(Q/G) \times_{Q/G} Q/G_\mu \rightarrow Q/G.\]
Lagrange-Routh equations, which give the equations of motion on \( J^{-1}(\mu) | G_{\mu} \) are derived based on reduced Routhian \( R^h : J^{-1}(\mu) | G_{\mu} \rightarrow \mathbb{R} \).

In the context of the present paper, we look at a special case where \( Q = G \times S \), where \( S \) is a smooth manifold, called the shape space. The Lie group \( G \) acts on \( Q \) by its action on the first component. Under this case, it can be seen that \( J^{-1}(\mu) | G_{\mu} \) is bundle isomorphic to \( TS \times S (O_{\mu} \times S) \rightarrow S \), where \( O_{\mu} \) is the coadjoint orbit of \( \mu \). Now, the Lagrange-Routh equations are given by a reduced variational principle as given in [11].

**Theorem 1.2 ([13]):** If \( q(t), a \leq t \leq b \), is a solution of the Euler-Lagrange equations with momentum value \( \mu \), \( y(t) = \pi_{Q,G_{\mu}} \) and \( x(t) = \pi_{Q,G} \), then \( y \) satisfies the reduced variational principle

\[
\delta \int_a^b R^h(x(t), \dot{x}(t), y(t)) \, dt = \int_a^b I_{y(t)} \beta_{\mu}(y(t)) \cdot \delta y \, dt + \left\langle \dot{x}(t), \delta x(t) \right\rangle|_a^b \, .
\]

Conversely, if \( q \) is a curve such that \( \dot{q}(t) \in J^{-1}(\mu) \) and if its projection to \( y(t) \) satisfies this reduced variational principle, then \( q(t) \) is a solution of the Euler-Lagrange equations. In the above theorem, the two-form \( \beta_{\mu} \) is the unique two-form on \( Q/G_{\mu} \) such that \( dA_{\mu} = \pi_{Q,G_{\mu}}^{*} \beta_{\mu} \), where the one-form \( A_{\mu} \) is the one form on \( Q \) defined as

\[
\left\langle A_{\mu}(q), v_q \right\rangle = \langle \mu, A(v_q) \rangle,
\]

where \( A(q) : TQ \rightarrow \mathfrak{g} \) is the mechanical connection one-form. To get the reduced equations of motion, we take a variation \( \delta q \) that vanishes at the endpoints.

**References**


