Weak positive Poisson stability and Hamiltonian vector fields in mechanical systems

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Abstract: We consider mechanical systems whose configuration manifold is \( Q = G \times S \), where \( G \) is a compact Lie group and \( S \) is a smooth manifold. Under an additional assumption of symmetry, we show that the dynamics of the system over the phase space \( T^*Q \) can be reduced to \( G \times T^*S \). We then show that the component of the dynamics on \( G \) is weakly positively Poisson stable. We apply this result to analyze global attitude controllability of a spacecraft with two rotors.

Keywords: Weak positive Poisson stability, Hamiltonian vector field, Lie group, Controllability, Spacecraft with rotors.

1. INTRODUCTION

In this paper we consider mechanical systems whose configuration manifold is of the form \( Q = G \times S \), where \( G \) is a compact Lie group and \( S \) is a smooth manifold, termed the shape space. We relate the notion of weak positive Poisson stability (WPPS) to Hamiltonian vector fields on the cotangent bundle of \( Q \). The concept of weak positive Poisson stability was introduced by Lian et al. (1994), which is useful in showing global controllability of certain class of control systems. Birtea et al. (2004) showed that if there is a volume form on a compact manifold which is preserved by a vector field on the manifold, then the vector field is WPPS.

When the mechanical system is described by a Hamiltonian which is invariant under the action of \( G \) on \( T^*Q \), the dynamics can be reduced to \( G \times T^*S \). The idea of this paper is to construct a volume form on \( G \) which is preserved under the flow of the reduced dynamics on \( G \), thus enabling us to deduce the WPPS property.

For mechanical control systems such as spacecrafts or underwa- ter vehicles, the actuation is often provided by internal motion that is represented by the shape space \( S \). In such cases, the component of the reduced dynamics on \( G \) corresponds to the drift or uncontrolled vector field. When the drift vector field is WPPS, global controllability is equivalent to the Lie algebra rank condition being satisfied [Lian et al. (1994)], thus enabling us to study global controllability of mechanical systems which are internally actuated.

The paper is organized as follows. After a brief review of preliminaries in section 2, we first analyze the dynamics on \( T^*G \) and derive the form of reduced dynamics in section 3. In section 4 we construct a volume form on \( G \) using the canonical two form on \( T^*G \) and use this form to show WPPS property of the reduced dynamics on \( G \). In section 5 we generalize the result of section 3 to the case when \( Q = G \times S \). Finally in section 6 we use these results to show global controllability of the attitude for a spacecraft with two rotors.

2. PRELIMINARIES

Geometric mechanics

For detailed descriptions of the concepts in this subsection, we refer the reader to Abraham and Marsden (1972) or Marsden and Ratiu (1999).

If \( Q \) is a smooth manifold, we denote the tangent and cotangent bundles of \( Q \) by \( TQ \) and \( T^*Q \) respectively. If \( G \) is Lie group, \( \mathfrak{g} \) denotes the Lie algebra of \( G \) and \( \mathfrak{g}^* \) denotes the dual of \( \mathfrak{g} \).

Let \( Q \), the configuration space of the mechanical system, be a smooth manifold and let a Lie group \( G \) act freely and properly on \( Q \). The phase space of the mechanical system is the cotangent bundle \( T^*Q \) and the action of \( G \) on \( Q \) induces a lifted action on \( T^*Q \). The canonical symplectic structure on \( T^*Q \) induces a momentum map \( J : T^*Q \rightarrow \mathfrak{g}^* \), given by

\[
(J(q), \xi) = (\alpha_q, \xi_Q(q)), \quad \forall \xi \in \mathfrak{g}, \forall \alpha_q \in T^*Q
\]

where \( \xi_Q \) denotes the infinitesimal generator of the \( G \) action on \( Q \). The momentum map satisfies

\[
X_{J(q), \xi}(\alpha_q) = \xi^*\mu(\alpha_q),
\]

where \( X_{J(q), \xi} \) is the Hamiltonian vector field on \( T^*Q \) corresponding to the function \( J(q), \xi \). By E. Noether’s theorem, the value of \( J \) is conserved along the flow of any Hamiltonian vector field corresponding to a Hamiltonian which is invariant under the action of \( G \). Thus, if \( \mu \) is the conserved value of the momentum map, a Hamiltonian system evolves on \( J^{-1}(\mu) \), which is a submanifold of \( T^*Q \).

When \( Q = G \), \( G \) acts on itself by left or right actions. We denote by \( J_L \) and \( J_R \) the momentum maps corresponding to the left and the right actions. Explicitly, \( J_L \) and \( J_R \) are given by

\[
J_L(\alpha_g) = T^*_g R_g(\alpha_g), \quad J_R(\alpha_g) = T^*_g L_g(\alpha_g),
\]

and they are related by

\[
J_L(\alpha_g) = (\text{Ad}^*_g \circ J_R)(\alpha_g).
\]

where \( \text{Ad}^* \) is the coadjoint action of \( G \) on \( \mathfrak{g}^* \).
If \( \phi \) is a diffeomorphism of \( Q \), then we denote by \( \phi_* \) the push-forward by \( \phi \). For example, if \( X \) is a vector field on \( Q \), then \( \phi_*X \) represents the vector field, which is the push-forward of \( X \) by \( \phi \). Similarly if \( f : Q \to P \) is any smooth function between the manifolds \( Q \) and \( P \), then we denote by \( f^* \) the pull-back by \( f \).

A complete vector field on \( Q \) is said to be weakly positively Poisson stable if for every open set \( U \subset Q \) and for every \( t > 0 \), there exists \( T > t \) such that \( \phi_T(U) \cup \phi_T(U) \neq \emptyset \).

Let \( \mathfrak{so}(3) \) denote the space of skew-symmetric \( 3 \times 3 \) real matrices, which is the Lie algebra of \( SO(3) \). The map \( S : \mathbb{R}^3 \to \mathfrak{so}(3) \) is defined as:

\[
S((a_1, a_2, a_3)) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.
\]

3. HAMILTONIAN VECTOR FIELD RESTRICTED TO \( J^1_L(\mu) \): CASE \( Q = G \)

Under the left action of \( G \) on \( T^*G \), let the Hamiltonian \( H : T^*G \to \mathbb{R} \) be invariant under the left action of \( G \) on \( T^*G \). By Noether’s theorem, the value of the left momentum map \( J_L \) is conserved along the flow of the Hamiltonian vector field and the Hamiltonian vector field is tangential to the submanifold \( J^1_L(\mu) \). We shall get an expression for this vector field in terms of the coordinates of \( G \).

Let \( X_H \) be the Hamiltonian vector field corresponding to the Hamiltonian \( H \) on \( T^*G \). Define \( H^\perp : g^* \to \mathbb{R} \) as

\[
H^\perp(\eta) = H(g \cdot \eta).
\]

We have the following result.

**Theorem 1.** Define \( \phi = (\pi_G, J_L) : T^*G \to G \times g^* \), where \( \pi_G : T^*G \to G \) is the cotangent bundle projection. Then,

\[
(\phi_*X_H)(g, \mu) = \left( T_L g \frac{\delta H^\perp}{\delta \eta} \bigg|_{(g, \mu)} , 0 \right)
\]

**Proof.** First of all, note that since \( J^1_L(\alpha_g) = T^*_L R_g(\alpha_g) \), \( \phi G(\alpha_g) = (\pi_G, T^*_L R_g)(\alpha_g) \). Thus, \( \phi \) is nothing but the right trivialization of \( T^*G \) and hence a diffeomorphism. Note further that this diffeomorphism maps \( J^1_L(\mu) \) to \( G \times \{ \mu \} \subset G \times g^* \), which can be identified with \( G \).

For ease of computation, we make use of an existing result stated as proposition 13.4.3 in Marsden and Ratiu (1999), where they compute the push forward of \( X_H \) by the diffeomorphism \( \lambda(\alpha_g) = (\pi, T^*_L \lambda_0)(\alpha_g) = (\pi_G, J^1_L)(\alpha_g) \). It is shown that

\[
X^\perp_H(g, \Pi) = (\lambda, X_H)(g, \Pi) = \left( g, T_L g \frac{\delta H^\perp}{\delta \Pi} , (\Pi, \text{Ad}^\perp_{\text{inv}}) \frac{\delta H^\perp}{\delta \Pi} \right)
\]

Referring to equation (1), we can see that \( \phi = \varphi \circ \lambda \), where \( \varphi : G \times g^* \to G \times g^* \), given by \( \varphi(g, \Pi) = (g, \text{Ad}^\perp_{\text{inv}}) \Pi \). That is, \( \varphi = (\pi_1, \text{Ad}^\perp_{\text{inv}} \pi_2) \), where \( \pi_1 \) and \( \pi_2 \) are projections onto the first and second factors of \( G \times g^* \) and \( \text{inv} \) is the inversion operation on \( G \). Therefore, \( X^\perp_H = \varphi_*X_H = \varphi_*\lambda_*X_H = \varphi_*X^\perp_H \). Since \( X^\perp_H \) is given by (2), our computations reduce to computing \( \varphi_*X_H \).

By definition, for any diffeomorphism,

\[
(\psi_*X)(\psi(x)) = D\psi(x)X(x).
\]

That is,

\[
(\psi_*X)(\psi(x)) = D\psi(\varphi^{-1}(x))X(\varphi^{-1}(x)).
\]

Since \( \varphi^{-1}(g, \mu) = (g, \text{Ad}^\perp_{\text{inv}} \mu) \), we get

\[
X^\perp_H(g, \mu) = D\varphi(\varphi^{-1}(g, \mu))X(\varphi^{-1}(g, \mu)) = (D\varphi(g, \text{Ad}^\perp_{\text{inv}} \mu))X^\perp_H(g, \text{Ad}^\perp_{\text{inv}} \mu).
\]

Denoting \( X^\perp_H = (X^\perp_1, X^\perp_2) \), we see that

\[
(X^\perp_1, X^\perp_2) = (D\varphi(g, \text{Ad}^\perp_{\text{inv}} \mu))X^\perp_H(g, \text{Ad}^\perp_{\text{inv}} \mu).
\]

With this, we get

\[
X^\perp_1(g, \mu) = T_L g \frac{\delta H^\perp}{\delta \eta} \bigg|_{(g, \mu)}.
\]

Let us now compute the second component \( X^\perp_2(g, \mu) \). Let \( c(t) \) be a curve on \( G \) so that \( c(0) = g \) and \( c'(0) = X^\perp_1 \). Then we have

\[
X^\perp_2(g, \mu) = \left. \frac{d}{dt} \right|_{t=0} (\text{Ad}^\perp_{\text{inv}} \mu) + \text{Ad}^\perp_{\text{inv}} \mu (X^\perp_2).
\]

The first term above is evaluated as \([\text{proposition 6.54, Holm et al. (2009)}]\)

\[
\left. \frac{d}{dt} \right|_{t=0} (\text{Ad}^\perp_{\text{inv}} \mu) = \text{Ad}^\perp_{\text{inv}} \mu (X^\perp_2).
\]

Evaluating at \( t = 0 \), we see that the second term is

\[
\text{Ad}^\perp_{\text{inv}} \mu (X^\perp_2) = \text{Ad}^\perp_{\text{inv}} \mu (\delta H^\perp / \delta \Pi \text{Ad}^\perp_{\text{inv}} \mu) (X^\perp_2).
\]

Thus we get

\[
X^\perp_2(g, \mu) = 0.
\]

**Example.** For a free rigid body, the configuration space \( Q = SO(3) \). The reduced Hamiltonian on \( \mathfrak{so}(3) \) is given by \( H^\perp(\Pi) = \Pi^T I^2 \Pi \), where \( I \) is the moment of inertia matrix in the body frame. Thus, the pushed-forward vector field \( X^\perp_H \) on \( SO(3) \times \mathfrak{so}(3)^* \) corresponds to \( X^\perp_H(R, \mu) = (RS (I^T R^T \mu), 0) \).

4. A CONSERVED VOLUME FORM ON \( G \)

In this section, we construct a volume form on \( G \) which is conserved under the flow of the first component of \( X^\perp_H \). Define the two form \( \Omega^\perp \) on \( G \times g^* \) as \( \Omega^\perp = \phi_* \Omega_T \), where \( \Omega_T \) is the canonical two form on \( T^*G \). Given a vector \( \xi \in g^* \), we define a one-form \( \Theta_\xi \) on \( G \) using the two form \( \Omega^\perp \) as follows. Note that \( (0, \xi) \) defines a vector field over \( G \times g^* \). Now,

\[
\Theta_\xi := \iota^T \Omega(0,0) \Omega^\perp,
\]

where \( \iota^T \) is the back of \( T \)
where \( i : G \rightarrow G \times g^* \) is the inclusion and \( i \) is the contraction operator.

\[
\begin{align*}
(\omega_{i(eG)})_{T^*G}^\phi & \rightarrow (\omega_{i(eG)})_{G \times g^*}^\phi \quad \theta_{i(eG)}^\phi \rightarrow i((\omega_{i(eG)})_{G \times g^*})_{G \times g^*}.
\end{align*}
\]

Given a set of \( n \) linearly independent vectors \( \{\xi_1, \xi_2, \ldots, \xi_n\} \subset g^* \), we define the volume form \( \Gamma \) on \( G \) as follows

\[
\Gamma := \Theta_{\xi_1} \land \Theta_{\xi_2} \land \cdots \land \Theta_{\xi_n} = i^* 1_{(0,\xi_1)}(\omega) \land i^* 1_{(0,\xi_2)}(\omega) \land \cdots \land i^* 1_{(0,\xi_n)}(\omega).
\]

In this section we shall denote \( X_H^\phi \) as \((X,0)\), the Hamiltonian vector field on \( G \times g^* \), diffeomorphic to the original Hamiltonian vector field \( X_H \) on \( T^*G \). Then the vector field \( X \) on \( G \) and \((X,0)\) on \( G \times g^* \) are \( i \)-related. That is,

\[
T_i X = (X,0),
\]

where \( T_i \) denotes the tangent map of the inclusion function. Now we have the following result.

**Proposition 2.** \( L_X \Theta_\xi = 0 \).

**Proof.** Since \( X \) and \((X,0)\) are \( i \)-related, we know that \( L_X \Theta_\xi = L_X(0,\xi) \). We shall now show that \( L_X(0,\xi) = 0 \). Let \( (X,1) \) be a vector field on \( G \times g^* \). We use the formula [proposition 2.4.15, Abraham and Marsden (1972)]

\[
(L_X(0,\xi))_{(X,1)} = L_X(0,\xi)(\Omega(X,1,\xi)) - i((\omega_{i(eG)})_{(X,0)}(\Omega)).
\]

Expanding the contraction symbol on the right hand side, we have

\[
(L_X(0,\xi))_{(X,1)} = L_X(0,\xi)(\Omega(X,1,\xi),0)) - \Omega((X,0),0),\xi,1)) = 0.
\]

Now, note that \( \Omega = 0 \), therefore to the \( i \)-related of the above equation we can append the term \( \Omega((X,1,\xi),((X,0)),(X,\xi))) \). Therefore we have

\[
(L_X(0,\xi))_{(X,1)} = L_X(0,\xi)(\Omega((X,1,\xi),0,\xi))) - \Omega((X,0),0),\xi,1)) = 0.
\]

where in the last two steps we have made use of the same formula [proposition 2.4.15, Abraham and Marsden (1972)] for \( (L_X(0,\xi))_{(X,1)} = (X,0)) \). Since \( X \) is a \( i \)-related, \( \Omega = 0 \), it follows that \( L_X(0,\xi) = 0 \).

**Corollary 3.** \( L_X \Gamma = 0 \).

**Proof.** We know that

\[
L_X \Gamma = L_X \Theta_{\xi_1} \land \Theta_{\xi_2} \land \cdots \land \Theta_{\xi_n} + \Theta_{\xi_1} \land L_X \Theta_{\xi_2} \land \cdots \land \Theta_{\xi_n} + \cdots + \Theta_{\xi_1} \land \Theta_{\xi_2} \land \cdots \land L_X \Theta_{\xi_n}.
\]

From the previous proposition the result follows. □

It is known that if there is a volume form that is preserved under the flow of a vector field on a compact manifold, then the vector field is WPPS. [See for example proposition 2.8 of Birtea et al. (2004)]. Hence, it follows that

**Corollary 4.** The vector field \( X \) on \( G \) is WPPS.

5. HAMILTONIAN VECTOR FIELD RESTRICTED TO \( J_L^*(\mu) \): CASE \( Q = G \times S \)

Now we consider the case where the configuration manifold is given by \( Q = G \times S \). The Hamiltonian vector field on \( Q \) is given by the standard action of \( G \) on the first element of \( G \times S \). Let \( H : T^*Q \rightarrow \mathbb{R} \) be the \( G \)-invariant Hamiltonian function corresponding to the given mechanical system. Since we have \( T^*Q = T^*G \times T^*S \), by proposition 8 of appendix A, the Hamiltonian vector field \( X_H \) is given by

\[
X_H(\alpha_g,\alpha_s) = (X_H(\alpha_g),X_H\omega_g(\alpha_s)).
\]

To derive an expression for the Hamiltonian vector field on \( J_L^*(\mu) \), we define the following diffeomorphism

\[
\varphi := (\phi,\text{id}_{T^*S}),
\]

where \( \phi : T^*G \rightarrow G \times g^* \) is given by \( \phi = (\pi_G,\mathbf{J}_L) \). Note that \( \varphi \) maps \( J_L^*(\mu) \) to \( G \times (\mu) \times T^*S \). Define \( H_{\alpha_s}(\eta) := H_{\alpha_s}(g,\eta) \). Then we have the following result.

**Theorem 5.**

\[
(\varphi_*X_H^\mu)(g,\mu,\alpha_s) = \left( T_g L_\mu \frac{\delta H_{\varphi_*}}{\delta \alpha} \right)_{(\eta,Ad^\mu_{s} \mu,0),X_H\omega^\mu_2(\alpha_s)}.
\]

**Proof.** The proof for the component corresponding to \( X_H\omega^\mu_2 \) follows from theorem 1, by treating \( H_{\alpha_s} \) as a function on \( T^*G \). The expression for the second component follows from the definition of the push forward. □

**Remark** Note that the vector field \( T_g L_\mu \frac{\delta H_{\varphi_*}}{\delta \alpha} \) on \( G \) is WPPS by corollary 4. This result thus generalizes the result of Bhat and Tiwari (2009) to a compact Lie group, who showed that vector field \( RS(l^{-1}R^Tl - I) \), where \( l \in \mathbb{R}^3 \) is a parameter, is WPPS on \( SO(3) \).

6. APPLICATION TO A SPACECRAFT WITH TWO ROTORS

In this section, we carry out a global controllability analysis of a spacecraft with two rotors. We had done it earlier [Bayadi et al. (2012)] using the Lagrange-Routh reduction approach which required us to first reduce the dynamics to \( J^{-1}(\mu)G \eta \) and then reconstruct the part on \( G \). Here we show that using the form of dynamics derived in section 5, the controllability analysis can be carried out in a rather straightforward fashion.

In our model of the spacecraft with rotors, we consider the spacecraft as a rigid body. The configuration manifold of the model is then \( Q = SO(3) \times T^k_k \), where \( k = 1, \ldots, 3 \). Here \( SO(3) \) corresponds to the orientation of the spacecraft with respect to an inertial frame and the \( k \)-torus \( T^k_k \) is the \( k \)-fold Cartesian product of \( S^1 \) where \( k \) denotes the number of rotors and each \( S^1 \) represents the orientation of a rotor with respect to the spacecraft. We denote the configuration variables of the spacecraft with rotors by \( (R,\Theta) \) where \( \Theta = (\Theta_k) \), \( k = 1, \ldots, 3 \). We treat \( S^1 \) as a submanifold of \( SO(3) \) and hence \( \Theta_k \) act on vectors in \( \mathbb{R}^3 \). We denote by \( q = (R,\Theta) \) the configuration variables and \( v_q = (R,\Theta) \) the tangent vectors at \( q \).

The kinetic energy of the spacecraft can be modeled through a Riemannian metric over \( SO(3) \times T^k_k \).
The reduced Hamiltonian stands for the rotor angular momenta. Using the reduced moments of inertia about their spin axis.

This Lagrangian is invariant under the standard tangent lifted transformation on the full Lagrangian to compute the Hamiltonian to get

where $\theta$ is the angular momentum and $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the canonical projection. Since the value of $\theta$ is not relevant to spacecraft applications, the equations relevant to control of a spacecraft with two rotors are

where $u$ is the control torque applied to the rotors.

Controllability analysis

The equations (8)-(9) represent a control system over $SO(3) \times \mathbb{R}^2$. Thus, we get a control system of the form

where $f_0 = \langle RS\left(\tilde{I}(R^T \mu - \tilde{l})\right), 0\rangle$, $f_1 = (0, e_1)$, $f_2 = (0, e_2)$, where $e_1$ and $e_2$ are unit vectors in $\mathbb{R}^2$. Lian et al. (1994) showed that if $f_0$ is WPPS and $f_0, f_1, f_2$ verify the Lie algebra rank condition (LARC) [Nijmeijer and van der Schaft (1990)], then the control system is globally controllable.

By corollary 4, it follows that $f_0$ is WPPS on $SO(3) \times \mathbb{R}^2$. Now we show the following result:

Proposition 6. The vector fields $f_0, f_1, f_2$ verify LARC on $SO(3) \times \mathbb{R}^2$.

Proof. Let $g_1 = [f_0, f_1]$. We make use of the following formula for computing the Lie bracket:

Therefore we get

Similarly we get $g_2 := [f_0, f_1](R, l) = (RS(\tilde{I}e_2), 0)$. Making use of the fact that $\tilde{I}$ is diagonal, we see that $g_2(R, l) := [g_1, g_2](R, l) = \tilde{I}_e \tilde{I}_e RS(e_3)$. Thus, $g_1, g_2, g_3$ span $T_RSO(3)$ at every $R \in SO(3)$ and $f_1, f_2$ span $\mathbb{R}^2$ at all $l \in \mathbb{R}^2$. □

Thus we have the following result, which follows from the theorem 3 of Lian et al. (1994):

Theorem 7. The control system (8)-(9) is globally controllable on $SO(3) \times \mathbb{R}^2$.

This result is the same as the proposition 5.4 we presented in Bayadi et al. (2012), however the analysis is simpler. Therefore the same result can be used to characterize the reachable set over $T^* SO(3)$ for the spacecraft.
We have derived an expression for the dynamics of a mechanical system with symmetry, whose configuration manifold is of the form \(Q = G \times S\). Using this expression and a volume form on \(G\) that we construct, we have shown that the component of the dynamics on \(G\) is weakly positively Poisson stable. By applying these results to the case of a spacecraft with two rotors, we have shown that it possible for the spacecraft to achieve any orientation.

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REFERENCES


Appendix A. PRODUCT OF SYMPLECTIC MANIFOLDS

Let \((S_1, \Omega_1)\) and \((S_2, \Omega_2)\) be two symplectic manifolds. It can be seen that the two form \(\Omega\) defined below satisfies all requirements of a symplectic form on \(S := S_1 \times S_2:\)

\[
\Omega = i_1^* \Omega_1 + i_2^* \Omega_2.
\]

Let \(H\) be a differentiable function on \(S\). Note that \(H\) defines parametrized family of functions \(H_{1s_2}\) and \(H_{2s_1}\) on \(S_1\) and \(S_2\) respectively as

\[
H_{1s_2}(s_1) := H(s_1, s_2) \ H_{2s_1}(s_2) := H(s_1, s_2).
\]

**Proposition 8.** The Hamitonian vector field on \(S\) corresponding to \(H\) is given by

\[
X_H(s_1, s_2) = \left( X_{H_{1s_2}}(s_1), X_{H_{2s_1}}(s_2) \right).
\]

**Proof.** By definition,

\[
\Omega(s_1, s_2)(X_H(u, v)) = \left. \frac{d}{dt} \right|_{t=0} H(c_1(t), c_2(t)),
\]

where \(c_1(t)\) and \(c_2(t)\) are curves on \(S_1\) and \(S_2\) such that \(c_1(0) = s_1, c_1'(0) = u\) and \(c_2(0) = s_2, c_2'(0) = v\). Employing the product trick, we see that

\[
\frac{d}{dt} \bigg|_{t=0} H(c_1(t), c_2(t)) = dH(s_1, s_2) \cdot (u, v)
\]

\[
= dH(s_1, s_2)(u, 0) + dH(s_1, s_2)(0, v)
\]

\[
= \left. \frac{d}{dt} \right|_{t=0} H(c_1(t), s_2) + \left. \frac{d}{dt} \right|_{t=0} H(s_1, c_2(t))
\]

\[
= dH_{1s_2}(s_1) \cdot u + dH_{2s_1}(s_2) \cdot v
\]

\[
= \Omega_1(X_{H_{1s_2}}(s_1), s_2) + \Omega_2(X_{H_{2s_1}}(s_2), v)
\]

\[
= \Omega_1(s_1, s_2) \left( X_{H_{1s_2}}(s_1), X_{H_{2s_1}}(s_2), (u, v) \right),
\]

where the last equality follows by definition of \(\Omega\). □

Appendix B. DYNAMICS OF A SPACECRAFT WITH TWO ROTORS

We know that the Hamiltonian \(H^- : \mathfrak{so}(3)^* \times T^*T^2 \to \mathbb{R}\) is given by

\[
H^-((\Pi, \mu), \frac{d}{dt} \bar{\Pi} - \bar{\Pi}^T \bar{\Pi}) + \frac{1}{2} T \bar{I} \frac{d}{dt} \bar{I},
\]

as noted in section 6. We have \(H_{1\mu} : \mathfrak{so}(3)^* \to \mathbb{R}\),

\[
H_{1\mu}(\eta) = \frac{1}{2} (\eta - \eta) \bar{I} (\eta - \bar{I}) + \frac{1}{2} T \bar{I} \frac{d}{dt} \bar{I}.
\]

Therefore \(\delta H_{1\mu}/\delta \eta = \bar{I} (\eta - \bar{I})\), which gives \(\delta H_{1\mu}/\delta \eta|_{\eta=\eta} = \bar{I} (R^T \mu - \bar{I})\). Hence

\[
T \mu \left( \delta H_{1\mu}/\delta \eta |_{\eta=\eta} \right) = R \bar{I} (R^T \mu - \bar{I}).
\]

Similarly, \(H_{2(\mu, l)} : T^*T^2 \to \mathbb{R}\) is given by

\[
H_{2(\mu, l)}(\theta, l) = \frac{1}{2} (R^T \mu - \bar{I}) R (R^T \mu - \bar{I}) + \frac{1}{2} T \bar{I} \frac{d}{dt} \bar{I}.
\]

On \(T^*T^2\), the Hamiltonian vector field corresponding to \(H_{2(\mu, l)}\) is obtained by the standard coordinate formula

\[
\frac{\partial H_{2(\mu, l)}}{\partial \bar{I}} \frac{d}{dt} \bar{I} + \frac{\partial H_{2(\mu, l)}}{\partial \theta} \frac{d}{dt} \theta = 0.
\]

Obviously, \(\partial H_{2(\mu, l)}/\partial \theta = 0\). To compute \(\partial H_{2(\mu, l)}/\partial \bar{I}\), we proceed as follows. We have \(\forall \nu \in \mathbb{R}^2\),

\[
\frac{d}{d\bar{I}} \bigg|_{\bar{I}=0} H_{2(\mu, l)}(\theta, l + s \nu)\]

\[
= -i(v) \bar{I} \bar{R}^T \mu + v^T \bar{I} \frac{d}{dt} \bar{I},
\]

where, as noted in section 6, \(i : \mathbb{R}^2 \to \mathbb{R}^3\) is the inclusion. But \(i(v)^T \bar{I} \bar{R}^T \mu = v^T \pi(\bar{I} \bar{R}^T \mu)\), where \(\pi : \mathbb{R}^3 \to \mathbb{R}^2\) is the projection. Therefore we get

\[
\frac{\partial H_{2(\mu, l)}}{\partial \bar{I}} = -\pi(\bar{I} \bar{R}^T \mu) + \bar{I} \frac{d}{dt} \bar{I}.
\]