The Euler-Poincaré equations for a spherical robot actuated by a pendulum

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Abstract: Mechanical systems with rolling constraints form a class of nonholonomic systems. In this paper we derive the dynamic model of a spherical robot, which has been designed and realized in our laboratory, using Lagrangian reduction theory defined on symmetry groups. The reduction is achieved by applying Hamilton’s variation principle on a reduced Lagrangian and then imposing the nonholonomic constraints. The equations of motion are in the Euler-Poincaré form and are equivalent to those obtained using Lagrange-d’Alembert’s principle.

Keywords: Nonholonomic systems, Lagrangian mechanics, reduction with symmetry, Spherical robot.

1. INTRODUCTION

Often the motion of various mechanical systems which we wish to model and control, has to satisfy certain restrictions imposed by the natural environment or the structure of the systems. In mechanics, such restrictions are called constraints. Constraints which restrict the possible configuration space are called holonomic. The constraints on velocities which can not be reduced to a holonomic form are termed as nonholonomic constraints. Examples are rolling systems such as a ball that rolls without slipping on a plane. In mechanical systems with nonholonomic constraints, the configuration space \( Q \) is a finite dimensional smooth manifold, the tangent bundle \( TQ \) is the velocity phase space, the Lagrangian is a map \( L: TQ \rightarrow \mathbb{R} \) and a smooth distribution \( D \subset TQ \) determines the constraints. Typically \( L \) is the kinetic energy minus the potential energy. So at a given point of the configuration space, the distribution \( D \) characterizes the allowable velocity directions of the system. The Lagrange-d’Alembert principle that yields the equations of motion states that the motion of the system occurs along trajectories that satisfy Hamilton’s variational principle where the variations of \( L \) are taken along curves which satisfy \( D \), and are assumed to vanish at the endpoints.

Many systems which are nonholonomic admit some symmetry which usually makes analysis simpler. By symmetry we can study the dynamics of a mechanical system on a reduced manifold or shape space. Hence, reduction theory concerns the removal of symmetries and utilizing the conservation laws associated with the symmetry group, for detail see Bloch et al. (1996). The use of group symmetry to simplify the formulation of Euler-Lagrange equations on the tangent bundle of a Lie group \( G \) is well explored in the literature on geometric mechanics, see Holm (2009) and nonholonomic theory in Bloch (2003). The reduction of nonholonomic systems is based on factoring the dependence of symmetry that are defined on a semidirect product (usually the Euclidean group) to obtain Euler-Poincaré equations on reduced space. So, instead of the Lagrange-d’Alembert equation on \( TQ \), we obtain the Euler-Poincaré equations on a reduced space. The procedure of reduction given in Marsden & Ratiu (1994) is for reducing unconstrained systems on a Lie group. If \( G \) is Lie group and \( \mathfrak{g} \) its Lie algebra, then the formulation on Hamilton’s variational principle for determining the Euler-Lagrange equations on \( TG \). This is equivalent to a reduced principle on \( \mathfrak{g} \) which leads to reduced equations called the pure Euler-Poincaré equations. The Lagrange reduction (unconstrained system) process developed in Cendra et al. (1998) is applied to the class of systems where the configuration space is a Lie group \( G \) but for which the \( G \)-invariance has been ‘broken’ by an advected parameter. An advection parameter is a vector which is constant in the inertial frame. Full symmetry can be broken by gravity. Leonard (1997) uses this geometric framework to investigate the effect of gravity on the motion of a bottom-heavy underwater vehicle. The generalization of Cendra et al. (1998) for the constrained case which are defined on a semidirect product of Lie groups is framed in Schneider (2002), Shen et al. (2008). This paper discusses the framework of a particular example of the Chaplygin’s sphere with an internal rotor. In our modeling of the spherical robot, we utilize a similar kind of framework but it must be noted that the center of mass in our case does not coincide with the geometric center of the sphere.

A spherical mobile robot is essentially a spherical shell with some driving mechanism mounted inside the shell to make the robot roll. Different constructions have been suggested by different researchers, which can be classified mainly based on the type of driving mechanism, for example Halme et al. (1996), Bicchi et al. (1996), SpheroBot, by Mukherjee et al. (2002), has radial spokes along which masses are placed. Radial movement of these masses creates a moment about the center causing the motion of
the robot. Another robot developed on a similar concept is the August by Javadi & Mojabi (2002). The robot by Jia et al. (2008) uses pendulum mechanism for driving the sphere. Our spherical robot uses the yoke and pendulum for motion. As the pendulum rotates, the center of mass moves and torque is generated due to gravity which makes the sphere roll. The yoke mechanism steers the robot.

In this paper we initially briefly present the construction of our spherical robot and use a geometric framework for deriving the equations of motion of this robot. The organization of the paper is as follows. In section II we describe our system in detail and introduce the prototype of spherical robot. In section III we derive equations of motion for the system using the Lagrangian and reduction theory to derive the nonholonomic Euler-poincaré structure. Section IV comprises of equilibria calculations and we define the equilibrium manifold. In section V we summarize the result and state some future goals.

2. SPHERICAL ROBOT-CONCEPT AND HARDWARE

Consider a spherical robot rolling on horizontal plane as shown in Fig.(1). The assembly consists of three interconnected elements - a sphere, a yoke and a pendulum (unbalanced mass). The prototype model is shown in Fig.(2). The yoke rotates a complete circle about a fixed diameter of the sphere, the pendulum rotates a complete circle about an axis fixed to the yoke. The motion of the sphere occurs due to the displacement of the center of mass when the pendulum moves. As the pendulum moves in the forward direction, a torque is generated due to gravity which causes the sphere to rotate. The first DC motor actuates the pendulum and the angular offset of the pendulum result in back or forth motion. The torque available to rotate the sphere depends on the pendulum, its arm length and the forces generated due to the displacement of the center of mass when the pendulum moves. As the pendulum moves in the forward direction, a torque is generated due to gravity which causes the sphere to rotate. The first DC motor actuates the pendulum and the angular offset of the pendulum result in back or forth motion. The torque available to rotate the sphere depends on the pendulum, its arm length and the forces generated due to the displacement of the center of mass when the pendulum moves. As the pendulum moves in the forward direction, a torque is generated due to gravity which causes the sphere to rotate. The first DC motor actuates the pendulum and the angular offset of the pendulum result in back or forth motion. The torque available to rotate the sphere depends on the pendulum, its arm length and the forces generated due to the displacement of the center of mass when the pendulum moves. As the pendulum moves in the forward direction, a torque is generated due to gravity which causes the sphere to rotate. The first DC motor actuates the pendulum and the angular offset of the pendulum result in back or forth motion. The torque available to rotate the sphere depends on the pendulum, its arm length and the forces generated due to the displacement of the center of mass when the pendulum moves.

3. MODELING

An inertial coordinate frame is attached to the origin on the plane on which the sphere is rolling and denoted by $xyz$.

![Fig. 1. Schematic of the spherical robot](image)

The sphere-body frame $x_s y_s z_s$ is attached to the center of the sphere. The yoke-body frame $x_y y_y z_y$, attached to the center of the sphere, gives the orientation of the yoke with respect to the sphere-body frame. Similarly, we have a pendulum-body frame $x_p y_p z_p$ that describes the pendulum orientation relative to the yoke-body frame as shown in Fig.(3). The set of generalized coordinates describing the system consists of:

1. Coordinates of the point of contact of the sphere on the plane $(x_c, y_c) \in \mathbb{R}^2$
2. Orientation of the sphere with respect to the inertial frame $(R_s \in SO(3))$
3. A set of variables describing the orientation of the yoke $(\alpha \in S^1)$ and the pendulum $(\varphi \in S^1)$

The configuration space of this system is thus $\mathbb{R}^2 \times SO(3) \times S^1 \times S^1$.

The orientation $(R_s)$ of the sphere can be represented in different local coordinate systems. Let $P$ be a point on the sphere. Here, we have adopted the following notation. $(*)^I$- inertial frame, $(*)^S$- sphere body frame, $(*)^Y$- yoke body frame, $(*)^P$- pendulum body frame.

The triples $(\hat{e}_1, \hat{e}_2, \hat{e}_3), (\hat{i}_s, \hat{j}_s, \hat{k}_s)$ and $(\hat{i}_y, \hat{j}_y, \hat{k}_y)$ denote unit vectors of the inertial, sphere and pendulum coordinate system respectively. $\alpha$ is the yoke angle about $x_Y$ axis and $\varphi$ the Pendulum angle about the $y_P$ axis.

Then the relationship between the coordinates of point $P$ in different frames is given by

Differentiating with respect to time, we get the velocity \( \dot{r} \) given as
\[
\begin{bmatrix}
C_\varphi & 0 & S_\varphi \\
0 & C_\alpha & -S_\alpha \\
-S_\varphi & 0 & C_\varphi
\end{bmatrix}.
\]
The angular velocity vector of the pendulum as expressed in equation (2) can be expressed in different coordinates as
\[
\omega_{\text{pendulum}} = \dot{\varphi} \hat{p} + \dot{\alpha} \hat{y} + \omega_s^*.
\]
The angular velocity vector of the pendulum in equation (2) can be expressed in different coordinates as follows:

1. The pendulum-body coordinate system:
\[
\omega_p = R^T(y_p, \varphi)R^T(x_p, \alpha) \begin{bmatrix}
(\omega_s)_x \\
(\omega_s)_y \\
(\omega_s)_z
\end{bmatrix} + R^T(y_p, \varphi) \begin{bmatrix}
\dot{\alpha} \\
0 \\
0
\end{bmatrix} + R(x_p, \alpha) \begin{bmatrix}
0 \\
\dot{\varphi} \\
0
\end{bmatrix}.
\]

2. The sphere-body coordinate system:
\[
(\omega_p)^s = \begin{bmatrix}
(\omega_s)_x \\
(\omega_s)_y \\
(\omega_s)_z
\end{bmatrix} + \begin{bmatrix}
\dot{\alpha} \\
0 \\
0
\end{bmatrix} + R(x_p, \alpha) \begin{bmatrix}
0 \\
\dot{\varphi} \\
0
\end{bmatrix}.
\]

3. The inertial coordinate system:
\[
(\omega_p)^i = \begin{bmatrix}
(\omega_s)_x \\
(\omega_s)_y \\
(\omega_s)_z
\end{bmatrix} + R_s \begin{bmatrix}
\dot{\alpha} \\
0 \\
0
\end{bmatrix} + R_s R(x_p, \alpha) \begin{bmatrix}
0 \\
\dot{\varphi} \\
0
\end{bmatrix}.
\]

Let \( r_s \) be the position vector of the center of the sphere as
\[
r_s = x_s \hat{e}_1 + y_s \hat{e}_2 + r \hat{e}_3,
\]
where \( r \) is the radius of the sphere. Differentiating the above equation we get the velocity vector for the center of the sphere as
\[
\dot{r}_s = \dot{x}_s \hat{e}_1 + \dot{y}_s \hat{e}_2 + \hat{r} \hat{e}_3.
\]

Now \( r_p \), the position vector of the pendulum is
\[
r_p = x_p \hat{e}_1 + y_p \hat{e}_2 + r \hat{e}_3 + \hat{k} \hat{e}_3.
\]

Differentiating with respect to time, we get the velocity vector of the pendulum as
\[
\dot{r}_p = \dot{r}_s + l \frac{d\hat{k}_p}{dt}.
\]

To express \( \frac{d\hat{k}_p}{dt} \) with the cross product as
\[
\frac{d\hat{k}_p}{dt} = \omega_p \times \hat{k}_p.
\]

Substituting \( \omega_p \) from equation (3) we have in the pendulum-body frame
\[
\frac{d\hat{k}_p}{dt} = [R^T(y_p, \varphi)R^T(x_p, \alpha)] \begin{bmatrix}
(\omega_s)_x \\
(\omega_s)_y \\
(\omega_s)_z
\end{bmatrix} + R^T(y_p, \varphi) \begin{bmatrix}
\dot{\alpha} \\
0 \\
0
\end{bmatrix} + R(x_p, \alpha) \begin{bmatrix}
0 \\
\dot{\varphi} \\
0
\end{bmatrix} \times \hat{k}_p.
\]

4. VELOCITIES, LAGRANGIAN AND CONSTRAINT

Angular velocities
The angular velocity of the sphere is denoted by
\[
\omega_s^* = \begin{bmatrix}
(\omega_s)_x \\
(\omega_s)_y \\
(\omega_s)_z
\end{bmatrix}^T
\]
and the angular velocity of the pendulum is given by
\[
\omega_{\text{pendulum}} = \dot{\varphi} \hat{p} + \dot{\alpha} \hat{y} + \omega_s^*.
\]

The Lagrangian is
\[
L = \frac{1}{2} m_s \|\dot{r}_s\|^2 + \frac{1}{2} \|\dot{\omega}_s\|^2 + \frac{1}{2} m_p \|\dot{r}_p\|^2 - m_p g \ell_b \dot{\varphi},
\]
where \( m_s \) and \( m_p \) are the mass of the sphere and the pendulum respectively, \( \ell_b = \text{diag}(\ell_3, \ell_3, \ell_3) \) is the moment of inertia of the sphere about the sphere-body axis.

Kinetic energy
The kinetic energy of the system is the sum of the kinetic energy of the sphere and the pendulum.

Potential Energy
The PE ‘V’ due to the pendulum is
\[
V = -m_p g \ell_b \dot{\varphi}.
\]

Lagrangian
The Lagrangian is
\[
L = T - V
\]
where
\[
T = \frac{1}{2} m_s \|\dot{r}_s\|^2 + \frac{1}{2} \|\dot{\omega}_s\|^2 + \frac{1}{2} m_p \|\dot{r}_p\|^2 - m_p g \ell_b \dot{\varphi},
\]
which gives
\[
\frac{d\hat{k}_p}{dt} = [L^T R^T(y_p, \varphi)R^T(x_p, \alpha)] \begin{bmatrix}
(\omega_s)_x \\
(\omega_s)_y \\
(\omega_s)_z
\end{bmatrix} + R^T(y_p, \varphi) \begin{bmatrix}
\dot{\alpha} \\
0 \\
0
\end{bmatrix} + R(x_p, \alpha) \begin{bmatrix}
0 \\
\dot{\varphi} \\
0
\end{bmatrix} \times \hat{k}_p.
\]

Constraint
Now, pure rolling implies that the velocity of the center of mass of the sphere is solely due to the rotation of the sphere around an axis passing through the point of contact. This rolling constraint is given as
\[
\dot{r}_s = (\omega_s)^i \times r \hat{e}_3 \implies \dot{r}_s = (\omega_s)^i \hat{e}_3.
\]

Here we use the vector product notation
\[
a \times b = [a \times b] \hat{e}_3
\]
where \( \hat{a} \) is skew-symmetric.

GROUP ACTION AND INVARIANCE OF THE LAGRANGIAN AND DISTRIBUTION
The action (left or right) of a Lie group on smooth manifold \( M \) is denoted by a smooth mapping \( \Phi : G \times M \rightarrow M \). Assuming \( G \) is free and proper, the Lagrangian \( L \) is said to be invariant under the group action if \( L \) is
invariant or symmetric under the induced action of $G$ on $TM$ Marsden & Ratiu (1994). In this paper we will consider left group action.

Let us consider the configuration space as a submanifold of the space $\hat{Q} = (SO(3) \times \mathbb{R}^3) \times S^1 \times S^1$. On this space let the Lagrangian be $L : T\hat{Q} \rightarrow \mathbb{R}$ and let $\hat{D} \subset T\hat{Q}$ be its distribution. The Lagrangian $L$ for the system given by $L = L_{\hat{T}\hat{Q}}$ and the distribution $\hat{D}$ is given by $\hat{D} = \hat{D}_{\hat{T}\hat{Q}} \subset T\hat{Q}$. The constraint equation $\hat{r}_s = \omega_1 \times r\hat{e}_3$ determines a distribution of an appropriate form. The configuration space is determined by the range space of this distribution. And the range is the horizontal plane with the normal vector $\hat{e}_3$, so $\hat{Q} = SO(3) \times \mathbb{R}^2 \times S^1 \times S^1$. In coordinates, $q^i, i = 1, \ldots, 5$, on $\hat{Q}$ with induced coordinates $(\eta^i, \tilde{\eta}^i)$ i.e. $q = (r_s, r, \alpha, \varphi)$ and $\dot{q} = (\dot{r}_s, \dot{r}, \dot{\alpha}, \dot{\varphi})$, where $r_s \in \mathbb{R}^3; R_s \in SO(3)$; $\alpha, \varphi \in S^1$; $\omega_3$ and $\omega_2$ denote the angular velocity of the yoke and pendulum body in their own coordinate frames respectively. The left action of the Lie group $G = SO(3) \times \mathbb{R}^3$ on the manifold $\hat{Q}$ is given by $\Phi_{\hat{(R_s, \alpha)}} : (r_s, R_s, \alpha, \varphi) \rightarrow (R_sR_s, R_s, \alpha, \varphi)$. The left action of $G$ on the tangent-pendulum co-ordinates of the manifold $\hat{Q}$ is

$$T\Phi_{\hat{(R_s, \alpha)}} : (\dot{r}_s, \dot{\alpha}, (\omega_3)^Y, (\omega_2)^P) \rightarrow (R_s\dot{R}_s, R_s\dot{\alpha}, (\omega_3)^Y, (\omega_2)^P).$$

Under the left action of $G$, the Lagrangian is given by

$$L(T\Phi_{\hat{(R_s, \alpha)}}(r_s, R_s, \alpha, \varphi)) = L(R_sR_s, R_s, \alpha, \varphi) = \frac{1}{2} m_t ||\dot{\alpha}||^2 + \frac{1}{2} \left(\omega^Y_2, \omega^P_2\right) + \frac{1}{2} m_p ||\dot{r}_s||^2 + m_p g l (\dot{R}_s^T \dot{\alpha}, \dot{R}_s, R_s, R_s, \dot{\alpha}).$$

The distribution $\hat{D}$ is determined by the constraint equation (7) and $\hat{D}_q \subset T\hat{Q}$ is mapped by the tangent of the group action as follows

$$T\Phi : G \times \hat{D}_q \rightarrow \hat{D}_q \subset T\hat{Q}.$$

So, under the left group action $\hat{D}$ becomes

$$\hat{r}_s = R_s\dot{R}_s^T \dot{R}_s \dot{\alpha} = r(\omega_3)^Y R_s^T \dot{\alpha}.$$

Here, we have used $(\omega_3)^Y = R_s^T \dot{\alpha}$, that is, the left group action on $\hat{R}_s$ gives left-invariant inertial angular velocity.

**Claim:** The Lagrangian $L$ and distribution $\hat{D}$ are invariant under the action of the group $G_{\hat{e}_3} = \{(r_s, b) \in SO(3) \times \mathbb{R}^3 | R_s^T \dot{\alpha} = \dot{\alpha}\} = SO(2) \times \mathbb{R}^2$.

**Proof:** We see this from equations (6), (7), (8) and (9).

We note the gravity term has broken the full group symmetry, leaving the left symmetry group $(SO(3) \times \mathbb{R}^3)$ of the system to be $SO(2) \times \mathbb{R}^2$ or $S^1 \times \mathbb{R}^2$. This means that the Lagrangian and distribution remain unchanged if we translate the inertial frame anywhere on the XY-plane and rotate it about $\hat{e}_3$, the direction of the gravity.

From the above arguments we see that the triple $(Q, L, \hat{D})$ has been constructed from $\tilde{Q}$, $\tilde{e}_3$ and $\tilde{L}$. The rolling constraint is now expressed in the sphere body coordinate frame as

$$\dot{Y} = R_s^T \dot{\alpha},$$

(10) where $\dot{Y} = R_s^T \dot{\alpha}$, $\dot{\Gamma} = R_s^T \dot{\alpha}$, $(\tilde{\omega}_3)^i = \dot{R}_s R_s^T$ (11) where $\dot{Y}, \dot{\Gamma} \in \mathbb{R}^3$ and where $\tilde{\omega}_s^i = R_s^T \dot{R}_s$ is the (left-invariant) sphere-body angular velocity.

**Comment:**

There is a significance for using $\Gamma$. Given a curve $R_s(t)$ the gravity vector in the sphere-body frame is represented by the unit vector as $\Gamma(t) = R_s^T(t) \dot{\alpha}$ which is seen as a constant with respect to the inertial and moving with respect to the sphere-body. Such quantities are called *adected vectors* which arises naturally in such systems, see [Holm (2009)].

**NONHOLONOMIC EULER-POINCARÉ EQUATION**

When the Lagrangian $L$ and the distribution $\hat{D}$ are invariant under the action of a group $G_{\hat{e}_3}$, the system is reduced to the quotient space $\hat{D}/G_{\hat{e}_3}$. With the action $\Phi_1 : G_{\hat{e}_3} \times \hat{D} \rightarrow \hat{D}$ and the projection map $\hat{p}_\Gamma : \hat{D} \rightarrow \hat{D}/G_{\hat{e}_3}$, we get the quotient space $\hat{D}/G_{\hat{e}_3}$. Assuming the action $\Phi_1$ is free and proper, the manifold $\hat{D}/G_{\hat{e}_3}$ is a smooth manifold diffeomorphic to $\mathbb{R} \times \mathcal{O}$ where $\mathcal{O}$ is the Lie algebra of $SO(3)$ and $\mathcal{O}$ is identified as a orbit space $G/H_{\varphi}$ of $\varphi \in \mathbb{R}^3$.

The configuration space is $\hat{Q} = SO(3) \times \mathbb{R}^3 \times S^1 \times S^1$ with action $G = SO(3) \times \mathbb{R}^3$ and its Lie algebra $\mathfrak{g}$. Here $G$ is the semidirect product, that is, $SO(3)$ acts from the left on $\mathbb{R}^3$. We have reduced the system from $T\hat{Q}$ to $\mathbb{R} \times \mathcal{O} \times TS^1 \times TS^1$ by first developing a reduced system on $\mathbb{R} \times \mathcal{O} \times TS^1 \times TS^1$. On $\mathcal{O} \times \mathcal{O} \times TS^1 \times TS^1$, we define a reduced Lagrangian and a reduced constrained principle. From this, the reduced equations of motion are obtained on $\mathbb{R} \times \mathcal{O}$ primarily in terms of the reduced constrained Lagrange-d’Alembert principle with the constraint $(\hat{R}_s^v, \tilde{\alpha}) \in \hat{Q}$ satisfies the Lagrange-d’Alembert principle with the constraint $(\hat{R}_s^v, \tilde{\alpha}) \in \hat{Q}$.

The curve $(R_s^v(t), \tilde{\alpha}(t), R_s^v(t), \tilde{\alpha}(t)) \in \hat{Q}$ satisfies the dynamical equation is given by the nonholonomic Euler-Poincaré equations [Holm (2008), Schneider (2002)] as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \omega_3^Y} \right) - ad_{\omega_3^Y} \omega_3^Y = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}} \right),$$

and the advection equation as

$$\dot{\Gamma} = \omega_3^s \times \Gamma + \omega_3^Y \times \Gamma = 0,$$

where $\Pi = \frac{\partial}{\partial \omega_3^Y}, s = s(\Gamma) = \Gamma$. The operator $\dot{\Gamma}$ is given by $\langle \beta \circ \gamma, \xi \rangle = -\langle \beta, \gamma \xi \rangle = \langle (\beta \times \gamma)^\flat, \xi \rangle$ and $ad_{\omega_3^Y} \theta = -\Omega \times \theta$. 

The reduced Lagrangian is defined in the body coordinates by using the definitions in (10) as
\[
\begin{align*}
l &= \frac{1}{2} m_s \|\dot{Y}\|^2 + \frac{1}{2} (\dot{\omega}_s^g, \dot{\omega}_s^g) + \frac{1}{2} m_p \|\dot{\hat{Y}}\|^2 \\
&\quad + [\omega_s^g + (\omega_s^g)^T + R_\alpha(\omega_s^g)^T] \times R_\alpha R_\beta \hat{\beta}_p \|^2 \\
&\quad + m_p g l(\Gamma, R_\alpha R_\beta \hat{\beta}_p).
\end{align*}
\]
\]
Let \( R_\alpha R_\beta \hat{\beta}_p = \chi \). Then the above equation becomes
\[
l = \frac{1}{2} m_s \|\dot{Y}\|^2 + \frac{1}{2} (\dot{\omega}_s^g, \dot{\omega}_s^g) + \frac{1}{2} m_p \|\dot{\hat{Y}}\|^2 + \frac{1}{2} m_p \|\dot{\chi}\|^2 + m_p g l(\Gamma, \chi).
\]
Here, \( \omega_s^g \) and \( \dot{\chi} \) denote the angular velocity of the yoke and pendulum body in their own coordinate frames respectively. Then, \( C_1(\alpha, \varphi) \dot{\alpha} = R_\alpha \omega_s^g \) and \( C_2(\alpha, \varphi) \dot{\varphi} = R_\alpha \dot{\chi} \). Therefore, the constrained reduced Lagrangian \( l_c \) is defined by evaluating \( l \) on the constraints as
\[
l_c = \frac{1}{2} m_s r^2 \|\dot{\omega}_s^g\|^2 + \frac{1}{2} (\dot{\omega}_s^g, \dot{\omega}_s^g) + \frac{1}{2} m_p \|\dot{\chi}\|^2 + \frac{1}{2} m_p \|\dot{\hat{Y}}\|^2 + m_p g l(\Gamma, \chi).
\]
So, the reduced and constrained-reduced Lagrangian are given by
\[
l = \frac{1}{2} \begin{bmatrix} \dot{Y} \\ \omega_s^g \end{bmatrix} \begin{bmatrix} \dot{Y} \\ \omega_s^g \end{bmatrix} + m_p g l(\Gamma, \chi)
\]
where
\[
M(\Gamma, \alpha, \varphi) = \begin{bmatrix} m_T I_3 & K(r) B_1(r) B_2(r) \\ KT(r) & B_1(r) B_2(r) \\ B_1^T(r) & B_2^T(r) \\ B_2^T(r) & B_3^T(r) m_\alpha \end{bmatrix}
\]
and
\[
m_T = m_s + m_p; \quad K(r) = -m_p l \mathcal{H}; \quad B(r) = -m_p l \mathcal{H} C_2(\alpha, \varphi); \quad J = \mathcal{H} m_p l^2 \mathcal{H} C_1(r); \quad B(r) = \mathcal{H} m_p l^2 \mathcal{H} C_2(\alpha, \varphi); \quad m_\alpha = C_1(r) \mathcal{H} m_p l^2 \mathcal{H} C_2(r).\]
Here \( r = (\alpha, \varphi) \) and \( I_3 \) be the identity matrix.

We obtain the terms in (13) as
\[
\begin{align*}
\frac{\partial l}{\partial \omega_s^g} &= M(\Gamma, \alpha, \varphi) \begin{bmatrix} \dot{Y} \\ \omega_s^g \end{bmatrix}, \\
\frac{\partial l}{\partial \dot{\omega}_s^g} &= M(\Gamma, \alpha, \varphi) \begin{bmatrix} \dot{Y} \\ \omega_s^g \end{bmatrix} \frac{\partial l}{\partial \dot{\omega}_s^g} = - \begin{bmatrix} \dot{Y} \\ \omega_s^g \end{bmatrix} \Gamma + \frac{\partial l}{\partial \Gamma} \Gamma, \\
\frac{\partial l}{\partial \chi} &= u_1, \\
\frac{\partial l}{\partial \dot{\chi}} &= u_2.
\end{align*}
\]

5. CHARACTERIZING EQUILIBRIA OF THE SPHERICAL ROBOT
We now characterize the equilibria of the system. By making \( (\omega_s^g, \dot{\alpha}, \dot{\varphi}) = 0 \) and assuming constant holding...
torques $\tau_{\alpha}$ and $\tau_{\varphi}$, we obtain conditions for the equilibria for the reduced system as

$$m_p g l \mathcal{X} \times \Gamma = 0; \quad \frac{\partial V(\Gamma, \alpha, \varphi)}{\partial \alpha} = \tau_{\alpha}; \quad \frac{\partial V(\Gamma, \alpha, \varphi)}{\partial \varphi} = \tau_{\varphi}.$$

Now

$$m_p g l \mathcal{X} \times \Gamma = 0 \implies \mathcal{X} \times R_s^T \hat{e}_3 = 0, \quad (21)$$

where $\mathcal{X} = R_s \hat{e}_3' k = [\sin \varphi - \sin \alpha \cos \varphi \cos \alpha \cos \varphi]$. Since $\Gamma$ and $\mathcal{X}$ have unit magnitudes, condition (21) can be satisfied only when they are aligned with each other as

$$\mathcal{X} = c \Gamma \quad (22)$$

where $c = \pm 1$. Then

$$\mathcal{X} - c \Gamma = 0 \implies [R_s \hat{e}_3' k - c (R_s^T \hat{e}_3)] = 0 \implies [R_s \hat{e}_3 - R_s^T \hat{e}_3] = 0.$$

For a given $R_s$, the equilibrium points consist of all such $\alpha$ and $\varphi$ such that $R_s \hat{e}_3 = R_s^T \hat{e}_3$. Therefore, the equilibrium configuration of the system is characterized by

$$\{(R_s, \alpha, \varphi) | \Gamma \times \mathcal{X} = 0 \} \Rightarrow \{(R_s, \alpha, \varphi) | R_s \hat{e}_3 = R_s^T \hat{e}_3 \}. \quad (23)$$

So, there are two distinct equilibrium manifolds, one when $\Gamma = 1, \omega_s^* = 0$, which is, the pendulum bob is downward and second corresponding to $\Gamma = -1, \omega_s^* = 0$ where the bob is in upright position.

The condition implies that the system is in equilibrium if and only if the vector $\mathcal{X}$ is collinear with the gravity vector $R_s^T \hat{e}_3$ in the sphere-body coordinate frame. That is, the vector originating from the geometric center of the sphere to the center of mass should be in the direction of gravity, either upward or downward. For a fixed $\alpha$ and $\varphi$, with the pendulum in upward (or downward) position, all configuration obtained by a rotation around the vertical axis passing through the point of contact, constitute the equilibrium manifold, see Fig.(4). If $R_s$ be an arbitrary orientation of the spherical robot, then any $\alpha$ and $\varphi$ such that $\mathcal{X}$ is in the downright or upright position constitute an equilibrium (Shen et al. (2008)).

6. CONCLUSION

In this paper we have presented the prototype model of a spherical robot fabricated in our laboratory and followed this up with a dynamic model of the system using the geometric framework of the Euler-Poincaré equations. Our future goals are to study stabilizability of the system around an equilibrium as well as work on path-planning algorithms. We also hope to develop a good simulation algorithm based on the equations developed here.

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