Almost global attitude stabilization of a rigid body for both internal and external actuation schemes

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Outline

Introduction

AGAS with internal actuation

The modified trace function and the spinning top potential

Conclusions
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The notion of almost global stability is what one looks for in this situation.
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Almost global stabilization of a rigid body has been so far addressed only for external actuation.
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Equations of motion

We consider the rigid body actuated by three orthogonally mounted rotors along the principal axes.

\[ R \in SO(3) \] is the orientation matrix, \( \Omega \in \mathbb{R}^3 \) is the angular velocity in the body frame, \( I_s \) is the inertia matrix, \( \mu \in \mathbb{R}^3 \) is the conserved value of the net angular momentum, \( u \) is the torque applied at the rotors.
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Using conservation of angular momentum, the equations of motion\(^1\) can be reduced to \(SO(3) \times \mathbb{R}^3\)

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\dot{R} &= R\hat{\Omega}, \\
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We choose the feedback torque of the form $u(R) = dV(R)$, where $V : SO(3) \longrightarrow \mathbb{R}$. 

**Damping**
Add damping so that $u(R) \rightarrow \Omega = C \Omega + dV(R)$, where $C$ is positive definite.
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EQUILIBRIA
If \( dV(R_d) = 0 \), then \( (R_d, 0) \) is an equilibrium of the closed loop system.

\footnote{D. E. Koditschek, Contemporary Mathematics, vol. 97, 1989.}
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The main question

How to choose \(V\) so as to achieve AGAS at \((R_d, 0)\)?

A comparison with external actuation

Equations of motion in case of external actuation

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\begin{align*}
\dot{R} &= R\dot{\Omega}, \\
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AGAS has already been addressed for external actuation. Since the equations in case of external and internal actuation are so similar, why not apply the same analysis to this case?
The main task to show AGAS is to prove stability of $R_d$ and instability of the other undesired equilibria.
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Therefore to analyze stability for internal actuation, we resort to a general linearization procedure on the nonlinear manifold $SO(3) \times \mathbb{R}^3$. 
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Suppose $\psi : \mathbb{R}^n \rightarrow M$ is a local parameterization of $M$ around $x_0$ ($\psi(0) = x_0$). Define

$$Y := \psi_* X,$$

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**Linearization of $Y$**

The linearization of $Y$ at 0 can be obtained as follows:

$$DY(0) \gamma := \frac{d}{dt} \bigg|_{t=0} Y(t\gamma).$$

This is equivalent to linearizing $X$ at $x_0$. 
LINEARIZATION USING EXPONENTIAL COORDINATES

THE EXPONENTIAL COORDINATES FOR $SO(3)$

To linearize around the equilibrium $(R_e, 0)$, define
$$\psi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow SO(3) \times \mathbb{R}^3$$

as $\psi = (R_e \circ \exp, \text{id})$, where $\exp$ is the usual matrix exponential map.

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This refinement is required so as to

- rigorously establish the correspondence between the stability or instability of the linear system and the original system on the manifold,

- analyze the feedback torque of the form $u = C\Omega + dV(R)$ for any general $V : SO(3) \rightarrow \mathbb{R}$.

The linearized dynamics

Linearized dynamics
The linearization around \((R_e, 0)\) is

\[
I_s \ddot{\eta} + \left( C - \overrightarrow{R_e} \mu \right) \dot{\eta} - \delta^2 \tilde{V}(R_e) \eta = 0.
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\(^4\text{P. C. Hughes, Spacecraft Attitude Dynamics, 1986.}\)
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Theorem (Kelvin-Tait-Chetaev\(^4\))

The origin of the system \((\eta, \dot{\eta}) = (0, 0)\) is

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Theorem

The closed loop system is AGAS at $(R_d,0)$.
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Suppose $V$ is chosen appropriately as indicated. Then,
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**Theorem**

- The control law of the form

  $$u_{ext}(R, \Omega) = -C\Omega - dV(R)$$

  almost globally stabilizes the equilibrium point $(R_d, 0)$ for the externally actuated system.

- The control law of the form

  $$u_{int}(R, \Omega) = C\Omega + dV(R)$$

  almost globally stabilizes the equilibrium point $(R_d, 0)$ for the internally actuated system.

However, it does not follow that if any $u(R, \Omega)$ applied internally stabilizes $(R_d, 0)$, then $-u(R, \Omega)$ applied externally stabilizes the same. Counterexamples can be constructed.
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The modified trace function

Modified trace functions\(^5\) (MTF) have been used earlier in connection with rigid body stabilization. It is defined as

\[ \text{trm}_P(R) = \text{trace}(PR), \]

where \(P\) is a symmetric matrix.

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**Lemma**

If \(P\) is a symmetric \(3 \times 3\) matrix with distinct eigenvalues \(\pi_1, \pi_2, \pi_3\) and if

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(\pi_1 + \pi_2)(\pi_2 + \pi_3)(\pi_3 + \pi_1) \neq 0,
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then there are exactly four critical points of \(\text{trm}_P(.)\).

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It further follows that \(\delta^2V\) is positive definite at only one of the critical points; nonsingular at the others. *Thus, MTFs are tailor-made for AGAS.*

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The spinning top potential

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The gravitational potential of a spinning top can be written as:

\[ V(R) = -mglk \cdot Rk, \quad k \in \mathbb{R}^3 \]

under a suitable choice of the body frame.

In fact, potential functions of the form:

\[ V(R) = -k \cdot Rk, \quad k \in \mathbb{R}^3 \]

have been used earlier to stabilize relative equilibria.

However, \( \delta^2 V(e) \), where \( e \in SO(3) \) is the identity, is positive semidefinite.

If we modify so that:

\[ V(R) = -k_1 \cdot Rk_1 - k_2 \cdot Rk_2, \quad k_1, k_2 \in \mathbb{R}^3 \]

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**Proposition**

There exists a \( P \in \text{sym}(3 \times 3) \) such that

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V(R) = -k_1 \cdot Rk_1 - k_2 \cdot Rk_2 = -\text{trace}(PR),
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where \( P \) is diagonalizable to \( \text{diag}(a_1, a_2, 0) \), \( a_1, a_2 > 0 \).
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Thus, $V$ is indeed a MTF and can be used to achieve AGAS. It can be shown,

$$dV(R) = k_1 \times R^Tk_1 - k_2 \times R^Tk_2$$
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These results can be generalized to achieve AGAS at any $(R_d, 0)$. 
**Simulations**

\[ R_d = e. \]

**Control law**

We consider the potential

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**Model and control parameters**
$I_s = \text{diag}(40, 45, 42.5) \ kg \ m^2$, $C = 10I_{3 \times 3}$, $c_1 = 1$, $c_2 = 1.2$, $C = 10I_{3 \times 3}$. 
Simulations

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**Initial conditions**

\( R_0 = R_x(\pi/6)R_y(\pi/8)R_z(5\pi/12) \) and \( \mu = (1, 1.5, -2) \)
Simulation results

Figure: Error norm of $R$ and the rigid body angular velocity $\Omega$
**Simulation Results**

**Figure:** Torque in $Nm$ applied at the three rotors
Simulation results

**Figure:** Angular velocity in RPM of the three rotors
Outline

Introduction

AGAS with internal actuation

The modified trace function and the spinning top potential

Conclusions
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We showed that if an externally applied control torque derived from a potential can stabilize a desired equilibrium, then the negative of the same torque applied internally can stabilize the same equilibrium.

We also showed that the classical spinning top potential can be used as a motivation for deriving such a stabilizing potential, which leads to the MTF.
Thank You