THE EULER–POINCARÉ EQUATION FOR A SPHERICAL ROBOT ACTUATED BY A PENDULUM

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August 29, 2012
Outline

Introduction

The Setting

Modeling of spherical robot

Dynamic equation

Equilibrium Configuration

Controllability
**Outline**

**Introduction**

**The Setting**

**Modeling of spherical robot**

**Dynamic equation**

**Equilibrium Configuration**

**Controllability**
**Spherical robot**

**Construction**
A spherical shell with a driving mechanism mounted inside to make the sphere roll.

**Figure:** Prototype of the spherical robot
Mechanism

Figure: Schematic of the spherical robot

- Sphere rolling on a plane
- Internal driving mechanism consists of a yoke and a pendulum
- Movement of the pendulum causes a change in the CG and the sphere to roll
- Yoke movement may be interpreted as a steering input
BROAD OBJECTIVES AND METHODOLOGY

CONTROL OBJECTIVE

- To move the sphere from one point and orientation to another specified point and orientation.
- Devise motion planning algorithm for the robot to achieve the desired orientation and point.
Broad objectives and methodology

Control objective

- To move the sphere from one point and orientation to another specified point and orientation.
- Devise motion planning algorithm for the robot to achieve the desired orientation and point.

Steps to achieve the objective

- Dynamic model of the robot.
- Study equilibrium configurations.
- Study controllability and devise motion planning algorithms.
Outline

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The Setting

Modeling of spherical robot

Dynamic equation

Equilibrium configuration

Controllability
Lagrangian Mechanics

- The set of all possible configurations of a mechanical system is a smooth manifold $Q$.
- The set of configurations and velocities is the tangent bundle $TQ$.
- The Lagrangian is a map $L : TQ \rightarrow \mathbb{R}$.
- A distribution of velocities is a linear subspace $\mathcal{D} \subset TQ$ (appears in the context of nonholonomic systems.)
- The equations of motion on $TQ$ are given by the principle of least action applied to a Lagrangian function $L$. 
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Symmetry

The Lagrangian function is invariant under a Lie group action

$$L(g \cdot \dot{q}) = L(\dot{q}) \quad \forall \dot{q} \in TQ, \forall g \in G,$$

where $G$ is a Lie group.
**Lagrangian reduction**

- By *identifying* the group symmetry and utilizing the associated conservation law, the dynamics are expressed on a reduced space.

- Start with $Q$, define a Lie group $G$ action. If the Lagrangian and distribution are invariant with respect to this group action, express the reduced Lagrangian on $TQ/G$.  

- Factor the *symmetry* on the semidirect product (Euclidean space) to obtain the *Euler-Poincaré equation* on reduced space.

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Symmetry Breaking

The full Lie group symmetry is sometimes broken - results in an isotropy subgroup (eg. with a gravity term).

\[^{3}\text{D. D. Holm al: Geometric Mechanics and Symmetry, Oxford Texts, 2009.}\]
\[^{4}\text{Schneider D: Dynamical Systems, pp 87-130, 2002.}\]
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- The full Lie group symmetry is sometimes broken - results in an isotropy subgroup (eg. with a gravity term).
- The Lagrangian function’s $G$-invariance is now expressed with an advected parameter. (the terminology "advected" finds its source in fluid modeling as invariants of a flow.$^3$)

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- The equation of motion on a reduced space, given by the principle of least action on a reduced Lagrangian function $l$, is called the Euler-Poincaré equation (EP).

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The Euler-Poincaré framework for the Chaplygin’s sphere where the center of mass coincides with the geometric center of the sphere has been discussed by Schneider\(^4\).

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The Euler-Poincaré equation - with potential energy terms

- Start with the extended configuration space $\tilde{Q}$ and the associated Lagrangian $\tilde{L}$, which is assumed invariant under $G$. 
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- System configuration $Q$ is an immersed submanifold of $\tilde{Q}$ and the system Lagrangian is invariant under the isotropy subgroup - $G_k$.
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- Start with the extended configuration space $\tilde{Q}$ and the associated Lagrangian $\tilde{L}$, which is assumed invariant under $G$.
- System configuration $Q$ is an immersed submanifold of $\tilde{Q}$ and the system Lagrangian is invariant under the isotropy subgroup - $G_k$.
- The velocity constraints expressed as a distribution - $D \subset TQ$ - give rise to a reduced constrained-Lagrangian.
The Euler-Poincaré equation - with potential energy terms

- Start with the extended configuration space $\tilde{Q}$ and the associated Lagrangian $\tilde{L}$, which is assumed invariant under $G$.
- System configuration $Q$ is an immersed submanifold of $\tilde{Q}$ and the system Lagrangian is invariant under the isotropy subgroup - $G_k$.
- The velocity constraints expressed as a distribution - $D \subset TQ$ - give rise to a reduced constrained-Lagrangian.
- Incorporating the advection dynamics, we obtain the Euler-Poincaré equation.
Outlining

Introduction

The Setting

Modeling of spherical robot

Dynamic equation

Equilibrium Configuration

Controllability
**Coordinate frames**

Figure: Coordinate frames for the system
**Coordinate frames**

**Figure:** Coordinate frames for the system

**Configuration space**

\[ Q = SO(3) \times \mathbb{R}^2 \times S^1 \times S^1 \]
**Notations**

- $R_s \in SO(3)$- orientation of the sphere with respect to the inertial frame,
- $R_\alpha$ - orientation of the yoke with respect to the sphere body frame,
- $R_\varphi$ - orientation of the pendulum with respect to the yoke frame,
- $(\omega_s)^s$, $(\omega_\alpha)^Y$, $(\omega_\varphi)^P$- angular velocity of sphere in sphere frame, angular velocity of yoke in yoke frame and angular velocity of pendulum in pendulum body frame respectively.
- $\dot{r}_s$ - linear velocity of the centre of mass of the sphere.
Lagrangian

\[
L = \frac{1}{2} m_s \| \dot{r}_s \|^2 + \frac{1}{2} \left\langle \mathbb{I} \omega_s^s, \omega_s^s \right\rangle + \frac{1}{2} \left[ m_p g l \langle \hat{e}_3, R_s R_\alpha R_\varphi \hat{k}_p \rangle + \langle R_s \omega_s^s + R_s R_\alpha (\omega_\alpha)^Y + R_s R_\alpha R_\varphi (\omega_\varphi)^P \rangle \times R_s R_\alpha R_\varphi \hat{k}_p \rangle \right]^2
\]

\(\text{K.E. of sphere}\)
\(\text{P.E. of pendulum}\)
\(\text{K.E. of pendulum}\)

Rolling constraint: \( \dot{r}_s = (\omega_s^I)^T \times r \hat{e}_3 \quad \Rightarrow \quad \dot{r}_s = (\hat{\omega}_s)^I r \hat{e}_3 \)
Lagrangian

\[ L = \frac{1}{2} m_s \| \dot{r}_s \| ^2 + \frac{1}{2} \langle \omega_s^s, \omega_s^s \rangle + m_p gl \langle \hat{e}_3, R_s R_\alpha R_\varphi \hat{k}_p \rangle \]

\[ \frac{1}{2} m_p \| \dot{r}_s + [R_s \omega_s^s + R_s R_\alpha (\omega_\alpha)^Y + R_s R_\alpha R_\varphi (\omega_\varphi)^P] \times R_s R_\alpha R_\varphi \hat{k}_p \| ^2 \]

Rolling constraint: \( \dot{r}_s = (\omega_s)^I \times r \hat{e}_3 \quad \Rightarrow \quad \dot{r}_s = (\hat{\omega}_s)^I r \hat{e}_3 \)

Symmetry

- Left group action, \( G = SO(3) \ltimes \mathbb{R}^3 \) on manifold \( Q \).
Lagrangian

\[
L = \frac{1}{2} m_s \| \dot{r}_s \|^2 + \frac{1}{2} \langle \hat{\omega}_s^s, \omega_s^s \rangle + m_g l \langle \hat{e}_3, R_s R_\alpha R_\varphi \hat{k}_p \rangle \\
K.E. of sphere \\
P.E. of pendulum
\]

\[
+ \frac{1}{2} m_p \| \dot{r}_s + [\dot{R}_s \omega_s^s + \dot{R}_s R_\alpha (\omega_\alpha)^Y + R_s R_\alpha R_\varphi (\omega_\varphi)^P ] \times R_s R_\alpha R_\varphi \hat{k}_p \|^2
\]

K.E. of pendulum

Rolling constraint: \( \dot{r}_s = (\omega_s)^I \times r\hat{e}_3 \quad \Rightarrow \quad \dot{r}_s = (\hat{\omega}_s)^I r\hat{e}_3 \)

Symmetry

- Left group action, \( G = SO(3) \times \mathbb{R}^3 \) on manifold \( Q \).
- \( L \) and \( D \) are invariant when \( R_1^T \hat{e}_3 = \hat{e}_3 \). (Remain unchanged if we translate the inertial frame anywhere on the XY-plane and rotate it about \( \hat{e}_3 \), the direction of gravity.)
Lagrangian

\[ L = \frac{1}{2} m_s \| \dot{r}_s \|^2 + \frac{1}{2} \langle \mathbb{I} \omega^s_s, \omega^s_s \rangle + m_p g l \langle \hat{e}_3, R_s R_\alpha R_\phi \hat{k}_p \rangle \\
\underbrace{\text{K.E. of sphere}} + \underbrace{\text{P.E. of pendulum}} \\
+ \frac{1}{2} m_p \| \dot{r}_s + [R_s \omega^s_s + R_s R_\alpha (\omega_\alpha)^Y + R_s R_\alpha R_\phi (\omega_\phi)^P] \times R_s R_\alpha R_\phi \hat{k}_p \|^2 \\
\underbrace{\text{K.E. of pendulum}} \\
\]

Rolling constraint: \( \dot{r}_s = (\omega_s)^I \times r \hat{e}_3 \implies \dot{r}_s = (\hat{\omega}_s)^I r \hat{e}_3 \)

Symmetry

- Left group action, \( G = SO(3) \rtimes \mathbb{R}^3 \) on manifold \( Q \).
- \( L \) and \( D \) are invariant when \( R^T_1 \hat{e}_3 = \hat{e}_3 \). (Remain unchanged if we translate the inertial frame anywhere on the XY-plane and rotate it about \( \hat{e}_3 \), the direction of gravity.)
- Symmetry group
  \[ G_{\hat{e}_3} = \{(R_s, b) \in SO(3) \rtimes \mathbb{R}^3 | R^T_s \hat{e}_3 = \hat{e}_3 \} = SO(2) \rtimes \mathbb{R}^2. \]
- The advected quantity here is \( \Gamma(t) = R^T_s \hat{e}_3 \)
Mappings

**Adjoint and Co-adjoint operation for** \( SE(3) = SO(3) \ltimes \mathbb{R}^3 \)

- \( \text{Ad}_{(R,x)}(\xi, \upsilon) = (R\xi R^{-1}, R\upsilon - R\xi R^{-1}x) \)
- \( \text{Ad}^*_{(R,x)}(\mu, \beta) = (R\mu R^{-1} + x \diamond (R\beta), R\beta) \)

where \((\xi, \upsilon) \in \mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3\), \((\mu, \beta) \in \mathfrak{se}^*(3) = \mathfrak{so}^*(3) \ltimes (\mathbb{R}^3)^*\). \( R\beta \) denote the induced left action of \( R \) on \( \beta \) i.e. the left action of \( SO(3) \) on \( \mathbb{R}^3 \) induces a left action of \( SO(3) \) on \( (\mathbb{R}^3)^* \).

**Adjoint and Co-adjoint action of** \( \mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3 \)

- \( \text{ad}_{(\eta, \upsilon)}(\xi, \upsilon) = ([\eta, \xi], \eta \upsilon - \xi \upsilon) \)
  where induced action of \( \mathfrak{so}(3) \) on \( \mathbb{R}^3 \) is denoted by \( \eta \upsilon \).
- \( \text{ad}^*_{(\eta, \upsilon)}(\mu, \beta) = (-[\eta, \mu] + \beta \diamond \upsilon, -\eta \beta) \).
Lagrangian reduction

- Configuration space $S = SO(3) \times \mathbb{R}^3 \times S^1 \times S^1$
Lagrangian reduction

- Configuration space $S = SO(3) \times \mathbb{R}^3 \times S^1 \times S^1$

  Original Lagrangian $\quad L : T(SO(3) \times \mathbb{R}^2 \times S^1 \times S^1) \to \mathbb{R}$

  Reduced Lagrangian $\quad l : t \times M \times S^1 \times S^1 \to \mathbb{R} \quad t \in so(3) \times \mathbb{R}^3$

  Constrained Lagrangian $\quad l_c : \mathfrak{h} \times M \times S^1 \times S^1 \to \mathbb{R} \quad \mathfrak{h} \in so(3)$

Rolling constraint $\bar{\mathcal{Y}} = r^\mathfrak{h} \hat{\omega} s^s \Gamma$. $\hat{\omega} s^s = R^T s \dot{R} s$ is the (left-invariant) sphere-body angular velocity.
**Lagrangian Reduction**

- Configuration space $S = SO(3) \times \mathbb{R}^3 \times S^1 \times S^1$

  **Original Lagrangian**
  $$L : T(SO(3) \times \mathbb{R}^2 \times S^1 \times S^1) \longrightarrow \mathbb{R}$$

  **Reduced Lagrangian**
  $$l : t \times M \times S^1 \times S^1 \longrightarrow \mathbb{R} \quad t \in so(3) \times \mathbb{R}^3$$

  **Constrained Lagrangian**
  $$l_c : \mathfrak{h} \times M \times S^1 \times S^1 \longrightarrow \mathbb{R} \quad \mathfrak{h} \in so(3)$$

- $M$ is the orbit space of $G/G_{\hat{e}_3}$ acting on $\hat{e}_3$ in $\mathbb{R}^3$.

\[
L(R_s, \hat{e}_3, \dot{R}_s, \dot{X}, R_\alpha, R_\varphi, \dot{R}_\alpha, \dot{R}_\varphi) = l(e, R_s^T \dot{R}_s, R_s^T \dot{X}, R_s^T \hat{e}_3, R_\alpha, R_\varphi, \dot{R}_\alpha, \dot{R}_\varphi),
\]

\[
= l(\hat{\omega}_s^s, \bar{Y}, \Gamma, R_\alpha, R_\varphi, \dot{R}_\alpha, \dot{R}_\varphi),
\]

\[
= l_c(\hat{\omega}_s^s, r\hat{\omega}_s^s \Gamma, \Gamma, R_\alpha, R_\varphi, \dot{R}_\alpha, \dot{R}_\varphi).
\]

- Rolling constraint
  $$\bar{Y} = r\hat{\omega}_s^s \Gamma.$$

- $\hat{\omega}_s^s = R_s^T \dot{R}_s$ is the (left-invariant) sphere-body angular velocity.
Outline

Introduction

The Setting

Modeling of spherical robot

Dynamic equation

Equilibrium Configuration

Controllability
The Euler-Poincaré equation

\[
\frac{d}{dt} \left( \frac{\partial l_c}{\partial \omega_s^s} \right) - ad^*_{\omega_s^s} \left( \frac{\partial l_c}{\partial \omega_s^s} \right) = - \left( \frac{\partial l}{\partial \bar{Y}} \diamond \dot{\Gamma} \right) + \left( \frac{\partial l}{\partial \Gamma} \diamond \Gamma \right),
\]

\[
\frac{d}{dt} \left( \frac{\partial l}{\partial \dot{\alpha}} \right) - \frac{\partial l}{\partial \alpha} = 0,
\]

\[
\frac{d}{dt} \left( \frac{\partial l}{\partial \dot{\phi}} \right) - \frac{\partial l}{\partial \phi} = 0,
\]

\[\dot{\Gamma} = -\omega_s^s \times \Gamma.\]

The diamond operator

\[\rho_v : \mathfrak{s}\mathfrak{o}(3) \to \mathbb{R}^3 \quad \rho_v^* : \mathfrak{s}\mathfrak{o}^*(3) \to \mathbb{R}^3^*\]

\[\mathbb{R}^3 \times \mathbb{R}^3^* \to \mathfrak{s}\mathfrak{o}^*(3) : (v, w) \to v \diamond w \triangleq \rho_v^*(w)\]
Carrying out the differentials, the dynamic equation is represented as

\[
M(\Gamma, \alpha, \varphi) \begin{bmatrix}
\dot{\omega}_s \\
\ddot{\alpha} \\
\ddot{\varphi}
\end{bmatrix} = -\frac{d}{dt}(M(\Gamma, \alpha, \varphi)) \begin{bmatrix}
\omega_s \\
\dot{\alpha} \\
\dot{\varphi}
\end{bmatrix} + \begin{bmatrix}
ad^*_{\omega_s} \left( \frac{\partial l_c}{\partial \omega_s} \right) \\
\frac{\partial T(\Gamma, \alpha, \varphi)}{\partial \alpha} \\
\frac{\partial T(\Gamma, \alpha, \varphi)}{\partial \varphi}
\end{bmatrix}
\begin{bmatrix}
\omega_s \\
\dot{\alpha} \\
\dot{\varphi}
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial l}{\delta \Gamma} \times \Gamma \\
-\frac{\partial V(\Gamma, \alpha, \varphi)}{\partial \alpha} \\
-\frac{\partial V(\Gamma, \alpha, \varphi)}{\partial \varphi}
\end{bmatrix}
+ \begin{bmatrix}
-(\frac{\partial l}{\partial Y}) \times \dot{\Gamma} \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
\tau_\alpha \\
\tau_\varphi
\end{bmatrix}.
\]
Outline

Introduction

The Setting

Modeling of spherical robot

Dynamic equation

Equilibrium Configuration

Controllability
Equilibrium Configuration

Figure: Equilibrium configuration manifolds: a) downright position of pendulum b) upright position.

Set \((\omega^s, \dot{\alpha}, \dot{\phi}) \equiv 0\), and assuming constant holding torques \(\tau_\alpha\) and \(\tau_\varphi\),

\[
m_p g l \chi \times \Gamma_e = 0 \implies \chi \times \Gamma_e = 0,
\]

\[
\frac{\partial V(\Gamma, \alpha, \varphi)}{\partial \alpha} = \tau_\alpha; \quad \frac{\partial V(\Gamma, \alpha, \varphi)}{\partial \varphi} = \tau_\varphi.
\]

\[
\chi = R_\alpha R_\varphi \hat{k}
\]
Observations on equilibria

- Vector $\mathcal{X}$ is collinear with the gravity vector $R_s^T \hat{e}_3$ in sphere-body frame.
Observations on equilibria

- Vector $\mathcal{X}$ is collinear with the gravity vector $R_s^T \hat{e}_3$ in sphere-body frame.
- Fix $\alpha$, $\varphi$: all configurations obtained by a rotation around the vertical axis passing through the point of contact, constitute the equilibrium manifold.
Observations on equilibria

- Vector $\mathbf{X}$ is collinear with the gravity vector $R_s^T\hat{e}_3$ in sphere-body frame.
- Fix $\alpha, \varphi$: all configurations obtained by a rotation around the vertical axis passing through the point of contact, constitute the equilibrium manifold.
- If $R_{se}$ is an arbitrary orientation, then any $\alpha \& \varphi$ such that $\mathbf{X}$ is in the downright or upright position constitutes an equilibrium.
**Observations on equilibria**

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- Fix $\alpha, \varphi$: all configurations obtained by a rotation around the vertical axis passing through the point of contact, constitute the equilibrium manifold.
- If $R_{se}$ is an arbitrary orientation, then any $\alpha$ & $\varphi$ such that $\mathcal{X}$ is in the downright or upright position constitutes an equilibrium.
- The control equilibrium of the reduced system is given as

$$\{(R_s, \alpha, \varphi)|\Gamma \times \mathcal{X} = 0\} \Rightarrow \{(R_s, \alpha, \varphi)|R_{\alpha}R_{\varphi} = R_s^T\}$$

where $\Gamma = R_s^T \hat{e}_3$ and $\mathcal{X} = R_{\alpha}R_{\varphi}\hat{k}$. 
Outline

Introduction

The Setting

Modeling of spherical robot

Dynamic equation

Equilibrium Configuration

Controllability
Vector fields on the reduced space

The control vector fields on the reduced space $T(SO(3) \times S^1 \times S^1)$ are

$$\tilde{Y}_i = M^{-1}(\Gamma, \alpha, \varphi) \begin{bmatrix} 0 \\ y_i \end{bmatrix}$$

The potential vector field on the reduced space $T(SO(3) \times S^1 \times S^1)$ is

$$(\text{grad}V)^\sim = M^{-1}(\Gamma, \alpha, \varphi) \begin{bmatrix} m_pg \Gamma \times \mathcal{X} \\ 0 \end{bmatrix}$$

where $i = \alpha, \varphi$ and $y_i$ is a $T^*(S^1 \times S^1)$-valued function.
Computational Procedure

- Calculate the symmetric product $\langle \tilde{Y}_i : \tilde{Y}_j \rangle$.
- Evaluate the iterated symmetric product of $Sym(\mathcal{Y}) = \{\mathcal{Y} \cup (\text{grad}V)\}$.
- The system is local configuration accessible at equilibrium if the rank of $\text{Lie}(Sym(\mathcal{Y})) = \text{dim}(Q)$ at $q_0$.
- Every bad symmetric product from $\{\mathcal{Y} \cup (\text{grad}V)\}$ is the linear combination of lower degree good symmetric products, then the system is small time local configuration controllable.
D. Schneider, *Non-holonomic Euler-Poincaré equations and stability in Chaplygin’s sphere*, Dynamical Systems, 2002, pp.87-130


THANK YOU