CHARACTERIZING REACHABLE SETS IN A SPACECRAFT WITH TWO ROTORS: THE LAGRANGE-ROUTH APPROACH

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Lagrange-Routh equations for a spacecraft with two rotors

Reconstruction

Controllability analysis

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**Attitude control**

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Controlling the orientation of a spacecraft/aircraft with respect to an inertial frame of reference.
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Actuating techniques for attitude control

- Gas jet thrusters - cause a change in angular momentum
- Internal momentum exchange devices - angular momentum is redistributed
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Problems in attitude control

- Stationary spacecraft - changing from an initial orientation to a final one.
- Spinning spacecraft - maintaining the direction of the axis of spin fixed as a vector or changing the axis of spin.
- Tracking a particular orientation trajectory.
**A spacecraft with internal rotors**

**Figure:** Rigid body model of a spacecraft with rotors.
Attitude control of a spacecraft with rotors

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When there are two rotors, the complete attitude dynamics is small time locally controllable, subject to the condition that the inertial angular momentum is zero\textsuperscript{2}.

Attitude control of a spacecraft with rotors

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When there are two rotors, the complete attitude dynamics is small time locally controllable, subject to the condition that the inertial angular momentum is zero\(^2\).

Boyer and Alamir\(^3\) conclude that ‘a five-dimensional subspace of feasible states is potentially reachable’ when the inertial angular momentum is nonzero for a spacecraft with two rotors.

We characterize the reachable sets of a spacecraft with two rotors when the inertial angular momentum is nonzero by making use of Lagrange-Routh reduction\textsuperscript{4}.

The conservation of angular momentum:

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The equations of motion:

$$\dot{R} = R \hat{\Omega}_b, \quad I_s \dot{\Omega}_b = \sum_{i=1}^{2} b_i u_i - \hat{\Omega}_b R^T \mu, \quad \mu = \text{constant}$$
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Constraint on the tangent bundle \( T^2 Q \):

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\langle c, I_s \dot{\Omega}_b + \hat{\Omega}_b R^T \mu \rangle = 0.
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THE STATE SPACE AND PHASE SPACE

LAGRANGIAN MECHANICS

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HAMILTONIAN MECHANICS

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Lagrangian mechanics

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- The equations of motion on $T^*Q$ are given by the Hamiltonian vector field corresponding to a Hamiltonian function.
The Lagrangian and Hamiltonian mechanics represent the same phenomenon, through the Legendre transformation $FL$. 

\[
TQ \quad \quad FL \quad \quad T^*Q
\]
Observation

The kinetic energy of a mechanical system does not depend on the orientation of the frame of reference.
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**Mathematical language**

Mathematically, this fact is equivalent to saying that the Lagrangian function is *invariant under a Lie group action*.

\[ L(g \cdot v) = L(v), \ \forall v \in TQ, \ \forall g \in G, \]

where \( G \) is the Lie group.
The momentum map

When such a symmetry is present, there exist conserved quantities. One such important object which leads to conserved quantities is the momentum map

\[ J : TQ \rightarrow g^*, \]

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Theorem (Noether)

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The value of the momentum map is conserved along the solutions of equations of motion.

The mechanical system thus evolves on a level set of the momentum map, $J^{-1}(\mu) \subset TQ$. 
More technical machinery - The Routhian

Define the *locked inertia tensor* $\mathbb{I}(q) : \mathfrak{g} \rightarrow \mathfrak{g}^*$:

$$
\langle \mathbb{I}(q)\xi, \eta \rangle = \langle \xi_{Q}(q), \eta_{Q}(q) \rangle_{q}.
$$

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The *mechanical connection one form* $A(q) : TQ \to \mathfrak{g}$ as follows,

$$A(q) = \mathbb{I}(q)^{-1}\mathbf{J}.$$  \hfill (2.2)

The *amended potential* $V_\mu(q) : Q \to \mathbb{R}$ is defined as follows

$$\tilde{V}_\mu(q) = V(q) - \langle \mu, \mathbb{I}(q)^{-1}\mu \rangle.$$
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$V_\mu$ is $G_\mu$-invariant.

Using the mechanical connection one form $A$, the Routhian is defined on $J^{-1}(\mu)$ as

$$R^\mu(v_q) := L(v_q) - A_\mu(v_q).$$
Further reduction from $J^{-1}(\mu)$

Treating $J^{-1}(\mu)$ as a manifold and $R^\mu$ as a function over this manifold, the Lie group action by $G_\mu \subset G$ induces further symmetries.
Further reduction from $\mathbf{J}^{-1}(\mu)$

Treating $\mathbf{J}^{-1}(\mu)$ as a manifold and $\mathcal{R}^\mu$ as a function over this manifold, the Lie group action by $G_\mu \subset G$ induces further symmetries.

**Theorem (Marsden, Ratiu and Scheurle 2000)**

The reduced phase space $P_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ is bundle isomorphic to $T(Q/G) \times_{Q/G} Q/G_\mu \to Q/G$.

**Figure:** Fiber product

\[ T(Q/G) \times_{Q/G} Q/G_\mu \to Q/G \]
**FURTHER REDUCTION FROM $J^{-1}(\mu)$**

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Lagrange-Routh equations give the equations of motion over $T(Q/G) \times_{Q/G} Q/G_\mu$.

**Figure:** Fiber product $T(Q/G) \times_{Q/G} Q/G_\mu \longrightarrow Q/G$. 
Given the dynamics on $P_\mu$, how does one reconstruct the dynamics on $J^{-1}(\mu)$?
Reconstruction

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**Figure:** Reconstruction procedure [Taken from *Foundation of Mechanics* by Abraham and Marsden].

Given a curve $y$ on $Q/G_\mu$, find a curve $d$ in $Q$ that projects to $y$. 
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Lagrangian-Routh equations for a spacecraft with two rotors

The configuration space is $Q = SO(3) \times \mathbb{T}^2$. The Lagrangian is

$$L(\dot{R}, \dot{\Theta}_i) = \frac{1}{2} \int_B \| \dot{R}X \|^2 \, dV + \frac{1}{2} \sum_{i=1}^2 \left[ \int_{r_i} \| R\dot{\Theta}_i X + \dot{R}\Theta_i X \|^2 \, dV \right].$$

The Lagrangian is invariant under the Lie group action of $SO(3)$, given by $\phi_P(R, \Theta) = (PR, \Theta)$. 
The reduced phase space $\mathbf{J}^{-1}(\mu)/G_\mu$

For a spacecraft with rotors, $\mathbf{J}^{-1}(\mu) \cong SO(3) \times T(\mathbb{T}^2)$. 
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It follows that $\mathbf{J}^{-1}(\mu)/G_\mu = SO(3)/G_\mu \times T(\mathbb{T}^2) \cong \mathcal{O}_\mu \times T(\mathbb{T}^2)$. 
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The isotropy subgroup $G_\mu$

For a given $\mu \in \mathfrak{so}(3)^* \cong \mathbb{R}^3$, $G_\mu$ is given by

$$G_\mu = \{ R \in SO(3) \mid R\mu = \mu \}.$$

For $\mu \neq 0$, $G_\mu \cong S^1$. 
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The coadjoint orbit $O_\mu$
For a given $\mu \in \mathfrak{so}(3)^* \cong \mathbb{R}^3$, $O_\mu$ is given by

$$O_\mu = \{ \text{Ad}_R^*\mu, \ R \in SO(3) \}.$$ 

Therefore $O_\mu \cong S^2$. 
The reduced phase space $J^{-1}(\mu)/G_{\mu}$

The reduced phase space $J^{-1}(\mu)/G_{\mu}$ is thus diffeomorphic to $S^2 \times T(T^2) \cong (S^2 \times \mathbb{R}^2) \times T^2$.

**Figure:** $(S^2 \times \mathbb{R}^2) \times T^2$ as a fiber bundle over $T^2$
Lagrange-Routh equations for a spacecraft with two rotors

The Lagrange-Routh equations on $S^2 \times T(\mathbb{T}^2)$ are given by

$$\dot{\Pi}(t) = \Pi(t) \times I_L^{-1}\Pi(t) - \Pi(t) \times A_s\Omega_r(t),$$

$$A_s(I_L - I_r)\dot{\Omega}_r(t) = -A_s\dot{\Pi}(t) - u,$$

$$\dot{\Theta} = \Omega_r,$$

where $u$ is the torque on the rotors.
Lagrange-Routh equations for a spacecraft with two rotors

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$$

$$
\dot{\Theta} = \Omega_r,
$$

where $u$ is the torque on the rotors.

Since the rotor angles $\Theta$ are not important for the problem, we restrict our attention to the first two of the above equations, which evolve over $S^2 \times \mathbb{R}^2$. 
Lagrange-Routh equations for a spacecraft with two rotors

Define

\[(\Pi, l) := (\Pi, A_s [\Pi + (I_L - I_r)\Omega_r])\]

The first two of the above equations are the same as

\[
\dot{\Pi} = \Pi \times \tilde{I}(\Pi - l),
\]
\[
\dot{l} = u.
\]
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The first two of the above equations are the same as

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$$\dot{l} = u.$$

Here $l$ represents the angular momentum of the rotors alone.
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Reconstruction procedure

Lagrange-Routh procedure gives equations that describe a curve over the reduced space $S^2 \times T^2$. 

Figure: The curve in $S^2 \times T^2$ obtained through Lagrange-Routh equations and the original curve in $SO(3) \times T^2$ that projects to it.
Reconstruction procedure

Lagrange-Routh procedure gives equations that describe a curve over the reduced space $S^2 \times T^2$.

The curve we are interested is in $SO(3) \times T^2$, that projects to the above curve and has a momentum $\mu$. 

![Diagram showing the curves and the projection](image-url)
Reconstruction procedure

The orientation \( R \in SO(3) \) needs to be determined through the reconstruction equation.
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The projection is given by $\pi : SO(3) \times \mathbb{T}^2 \rightarrow S^2 \times \mathbb{T}^2$, $\pi(R, \Theta) = (R^T \mu, \Theta)$. 
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Given a curve $y = (\Pi, \Theta)$ in $Q/G_\mu \cong S^2 \times \mathbb{T}^2$, have to choose a curve $\bar{q} = (\bar{R}, \bar{\Theta})$ in $SO(3) \times \mathbb{T}^2$ such that $(\bar{R}^T \mu, \Theta) = (\Pi, \Theta)$. 
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We have to choose the curve $\bar{R}$, the lift of $\Pi$, such that $\bar{R}(t)$ rotates $\Pi(t)$ to $\mu$. 
Construction of $\tilde{R}$

$\pi : SO(3) \longrightarrow S^2$ is a principal fiber bundle with $S^1$ as fibers.
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$\pi : SO(3) \rightarrow S^2$ is a principal fiber bundle with $S^1$ as fibers.

Local sections
A map $K : S^2 \rightarrow SO(3)$ which satisfies $\pi \circ K = \text{id}_{S^2}$ is called a local section of the fiber bundle.
Construction of $\tilde{R}$

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Given a curve $\Pi$ on $S^2$ and a local section $K$, $K \circ \Pi$ qualifies to be a lift of $\Pi$. 
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**Example**
$K(\Pi) = \exp \widehat{w_1(\Pi)}$, where

$$w_1(\Pi) = \varphi \frac{\Pi \times \mu}{\| \Pi \times \mu \|}, \quad \cos \varphi = \frac{(\Pi \cdot \mu)}{\| \mu \|^2}.$$
Local sections induce parameterization of $SO(3)$ on the product $S^1 \times S^2$. 
Local sections induce parameterization of $SO(3)$ on the product $S^1 \times S^2$.

Thus, in terms of a parameterization $(\alpha, \Pi, l) \in S^1 \times S^2 \times \mathbb{R}^2$, the following equations give a local representative of the dynamics over $J^{-1}(\mu)$:

\[
\begin{align*}
\dot{\alpha} &= (\Pi - p(\Pi)) \cdot \tilde{I}(\Pi - l) \\
\dot{\Pi} &= \Pi \times \tilde{I}(\Pi - l), \\
\dot{l} &= u,
\end{align*}
\]  

(4.1)

where $p(\Pi)$ is a function dependent on the local section.
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Control affine systems

A control system on a smooth manifold $M$ of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$$

are called control affine systems. The vector field $f_0$ is called the drift vector field and $f_i$ are called the controlled vector fields.
CONTROL AFFINE SYSTEMS

A control system on a smooth manifold $M$ of the form

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are called control affine systems. The vector field $f_0$ is called the drift vector field and $f_i$ are called the controlled vector fields.

NONWANDERING POINTS AND WPPS VECTOR FIELDS

- A point $x \in M$ is called a nonwandering point of a vector field $f$ if for every $T > 0$ and every neighbourhood $V_x$ of $x$, there exists a $t > T$ such that $\phi_t(V_x) \cap V_x \neq \emptyset$. 
CONTROL AFFINE SYSTEMS

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- A point $x \in M$ is called a nonwandering point of a vector field $f$ if for every $T > 0$ and every neighbourhood $V_x$ of $x$, there exists a $t > T$ such that $\phi_t(V_x) \cap V_x \neq \emptyset$.
- A vector field $f$ is said to be weakly positively Poisson stable (WPPS) if the set of nonwandering points of $f$ is the entire manifold.
Weak positive Poisson stability

Recall the vector fields in our case

$$f_0(x) = \begin{pmatrix} (\Pi - p(\Pi)) \cdot \tilde{I}(\Pi - l) \\ \Pi \times \tilde{I}(\Pi - l) \\ 0 \end{pmatrix}, \quad f_i = \begin{pmatrix} 0 \\ 0_{3 \times 1} \\ e_i \end{pmatrix}$$
Weak positive Poisson stability

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\[
f_i = \begin{pmatrix}
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e_i
\end{pmatrix}
\]

The vector field \( X_l := \Pi \times \tilde{I}(\Pi - l) \) on \( S^2 \) is WPPS since it is the Hamiltonian vector field corresponding to the Hamiltonian \( H = 1/2(\Pi - l)^T \tilde{I}(\Pi - l) \).
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The vector field \( X_l := \Pi \times \tilde{I}(\Pi - l) \) on \( S^2 \) is \( WPPS \) since it is the Hamiltonian vector field corresponding to the Hamiltonian \( H = 1/2(\Pi - l)^T \tilde{I}(\Pi - l) \).

\textit{Almost all} of the integral curves of \( X_l \) on \( S^2 \) are periodic.
Lie algebraic rank condition and WPPS

**Proposition**

The vector field

\[
\begin{pmatrix}
(\Pi - p(\Pi)) \cdot \tilde{I}(\Pi - l) \\
\Pi \times \tilde{I}(\Pi - l)
\end{pmatrix}
\]

on \(S^1 \times S^2\) is WPPS.

**Definition**

The control system (5.1) is said to satisfy the LARC if at every \(x \in M\), the smallest Lie algebra generated by \(\{f_0, f_1, \ldots, f_m\}\) spans \(T_x M\), the tangent space at \(x\).
Proposition

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Theorem (WPPS vector fields and global controllability [8])

If the drift vector field is WPPS, then global controllability is equivalent to the Lie algebra rank condition (LARC).
The main result

Proposition

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Theorem (Global controllability)
The control system (4.1) is globally controllable over $S^1 \times S^2 \times \mathbb{R}^2$. 

Recall that $(\alpha, \Pi, l) \in S^1 \times S^2 \times \mathbb{R}^2$ gives a local representation for $(R, l) \in SO(3) \times \mathbb{R}^2$. This means that any combination of the orientation and rotor angular momentum can be achieved.

Corollary
The (left trivialized) reachable sets of $TSO(3)$ of a spacecraft with two rotors is $(R, I - l L(R^T \mu - I r \Omega r))$ where $R \in SO(3)$ and $\Omega r \in \mathbb{R}^2$. 
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**Corollary**

The (left trivialized) reachable sets of $T_{SO(3)}$ of a spacecraft with two rotors is $(R, I_L^{-1}(R^T \mu - I_r \Omega_r))$ where $R \in SO(3)$ and $\Omega_r \in \mathbb{R}^2$. 
Interpreting the result

The set of reachable angular velocities at $R$ is just the translation of $(\Omega_b)_1 - (\Omega_b)_2$ plane by the vector $I_L^{-1}R^T \mu$. 

**Figure**: Reachable set of body angular velocity $\Omega_b$ at a particular orientation.
Parameters of UoSAT-12 [9]:
$I_L = \text{diag}(40.45, 42.09, 40.36)$, $I_r = \text{diag}(8 \times 10^{-3}, 7.7 \times 10^{-3}, 0)$ in kg m$^2$. 
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For $\mu = (0, 0, 10)$ kg ms$^{-1}$, at an $R$ making an angle 45 degrees with the $x$ axis, all possible angular velocities have the third component as 0.1752 rad s$^{-1}$. 
THANK YOU
OUTLINE

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