Weak positive Poisson stability and Hamiltonian vector fields in mechanical systems

Ramaprakash Bayadi
Ravi N Banavar

Systems and Control Engineering,
IIT Bombay, India.

August 29, 2012
OUTLINE

INTRODUCTION

THE SETTING

DYNAMICS ON THE LEVEL SET

A CONSERVED VOLUME FORM

APPLICATION: SPACECRAFT WITH TWO ROTORS

SUMMARY
Outline

Introduction

The setting

Dynamics on the level set

A conserved volume form

Application: Spacecraft with two rotors

Summary
Motivation

Weak positive Poisson stability (WPPS) is closely related to global controllability.¹

Motivation

Weak positive Poisson stability (WPPS) is closely related to global controllability.\(^1\)

Hamiltonian vector fields are closely associated with mechanical systems.

Motivation

Weak positive Poisson stability (WPPS) is closely related to global controllability.\(^1\)

Hamiltonian vector fields are closely associated with mechanical systems.

We link these concepts for a class of systems.

---

A vector field on a compact manifold is WPPS if it conserves a volume form.\textsuperscript{2}

A vector field on a compact manifold is WPPS if it conserves a volume form.²

A Hamiltonian vector field conserves the canonical two form on the underlying symplectic manifold.

A vector field on a compact manifold is WPPS if it conserves a volume form.\textsuperscript{2}

A Hamiltonian vector field conserves the canonical two form on the underlying symplectic manifold.

For the systems we consider, a component of the Hamiltonian vector field lies on a compact manifold.

\textsuperscript{2}Birtea, P et al. Controllability of Poisson systems, SIAM J. Cntrl and Opt, 2004
A vector field on a compact manifold is WPPS if it conserves a volume form.\(^2\)

A Hamiltonian vector field conserves the canonical two form on the underlying symplectic manifold.

For the systems we consider, a component of the Hamiltonian vector field lies on a compact manifold.

This allows the construction of a volume form using the canonical two form.

Outline

Introduction

The setting

Dynamics on the level set

A conserved volume form

Application: Spacecraft with two rotors

Summary
**Definitions**

**WPPS**

A complete vector field on $Q$ is said to be \textit{WPPS} if for every open set $U \subset Q$ and for every $t > 0$, there exists $T > t$ such that $\phi_T(U) \cap U \neq \emptyset$. 

**Hamiltonian vector field**

For every $\mathcal{C}^\infty$ function $H$ defined on a symplectic manifold $S$, the Hamiltonian vector field $X_H$ corresponding to $H$, is defined as

$$\langle dH(x), v \rangle = \Omega(X_H, v), \quad \forall v \in T_x S.$$ 

The phase space of a mechanical system is $T^*Q$, the cotangent bundle of the configuration space $Q$, is a symplectic manifold.
Definitions

WPPS
A complete vector field on $Q$ is said to be \textit{WPPS} if for every open set $U \subset Q$ and for every $t > 0$, there exists $T > t$ such that $\phi_T(U) \cap U \neq \emptyset$.

Hamiltonian vector field
For every $\mathcal{C}^\infty$ function $H$ defined on a symplectic manifold $S$, the Hamiltonian vector field $X_H$ corresponding to $H$, is defined as

$$\langle dH(x), v \rangle = \Omega(X_H, v), \forall v \in T_xS.$$
**Definitions**

**WPPS**
A complete vector field on $Q$ is said to be **WPPS** if for every open set $U \subset Q$ and for every $t > 0$, there exists $T > t$ such that $\phi_T(U) \cap U \neq \emptyset$.

**Hamiltonian vector field**
For every $C^\infty$ function $H$ defined on a symplectic manifold $S$, the Hamiltonian vector field $X_H$ corresponding to $H$, is defined as

$$\langle dH(x), v \rangle = \Omega(X_H, v), \ \forall v \in T_xS.$$  

The **phase space** of a mechanical system is $T^*Q$, the cotangent bundle of the configuration space $Q$, is a symplectic manifold.
A Lie group action on $T^*Q$ induces a momentum map $J : T^*Q \rightarrow g^*$, where $g^*$ is the dual of the Lie algebra of $G$. This function is a generalization of linear and angular momenta.
A Lie group action on $T^*Q$ induces a momentum map $J : T^*Q \rightarrow g^*$, where $g^*$ is the dual of the Lie algebra of $G$. This function is a generalization of linear and angular momenta.

When $Q = G$, $G$ acts on itself by left and right actions. Thus, we have the left and right momentum maps $J_L$ and $J_R$. 
A Lie group action on $T^*Q$ induces a \textit{momentum map} $J : T^*Q \rightarrow \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of $G$. This function is a generalization of linear and angular momenta.

When $Q = G$, $G$ acts on itself by left and right actions. Thus, we have the left and right momentum maps $J_L$ and $J_R$.

\textbf{Noether’s theorem}

If $H$ is invariant under a $G$-action, then the corresponding momentum map is a constant of motion for $X_H$. Thus, the dynamics evolves on $J^{-1}(\mu)$, the level set of the momentum map.
DYNAMICS ON $J^{-1}_L(\mu) \subset T^*G$

When $H : T^*G \longrightarrow \mathbb{R}$ is left-invariant, the dynamics evolves on $J^{-1}_L(\mu)$. 

Example: For a free rigid body, $X_{\phi H}(R,\mu) = (RS(I - 1)RT\mu, 0)$. This can be interpreted as the equation of motion expressed in the inertial frame of reference.
**Dynamics on** \( J_{L}^{-1}(\mu) \subset T^*G \)

When \( H : T^*G \longrightarrow \mathbb{R} \) is **left-invariant**, the dynamics evolves on \( J_{L}^{-1}(\mu) \).

The diffeomorphism \( \phi := (\pi, J_{L}) \) maps \( J_{L}^{-1}(\mu) \) to \( G \times \{\mu\} \), where \( \pi : T^*G \longrightarrow G \) is the cotangent bundle projection.
Dynamics on $J_L^{-1}(\mu) \subset T^*G$

When $H : T^*G \rightarrow \mathbb{R}$ is left-invariant, the dynamics evolves on $J_L^{-1}(\mu)$.

The diffeomorphism $\phi := (\pi, J_L)$ maps $J_L^{-1}(\mu)$ to $G \times \{\mu\}$, where $\pi : T^*G \rightarrow G$ is the cotangent bundle projection.

**Theorem (Right-trivialized dynamics)**

Define $H^{-} : g^* \rightarrow \mathbb{R}$ by $H^{-}(\eta) = H(g \cdot \eta)$. Let $X^\phi_H := \phi_* X_H$. Then,

$$X^\phi_H(g, \mu) = \left( T_e L_g \frac{\delta H^{-}}{\delta \eta} \bigg|_{\eta = Ad_{g^*}^* \mu}, 0 \right).$$

Example For a free rigid body, $X^\phi_H(R, \mu) = (R \cdot S(I - 1)^T \mu, 0)$. This can be interpreted as the equation of motion expressed in the inertial frame of reference.
Dynamics on $J^{-1}_L(\mu) \subset T^*G$

When $H : T^*G \rightarrow \mathbb{R}$ is left-invariant, the dynamics evolves on $J^{-1}_L(\mu)$.

The diffeomorphism $\phi := (\pi, J_L)$ maps $J^{-1}_L(\mu)$ to $G \times \{\mu\}$, where $\pi : T^*G \rightarrow G$ is the cotangent bundle projection.

**Theorem (Right-trivialized dynamics)**

Define $H^- : \mathfrak{g}^* \rightarrow \mathbb{R}$ by $H^-(\eta) = H(g \cdot \eta)$. Let $X^\phi_H := \phi_* X_H$. Then,

$$X^\phi_H(g, \mu) = \left( T_{eLg} \frac{\delta H^-}{\delta \eta} \bigg|_{(\eta = \text{Ad}^*_g \mu)} , 0 \right).$$

**Example**

For a free rigid body, $X^\phi_H(R, \mu) = (RS (I^{-1} R^T \mu), 0)$. This can be interpreted as the equation of motion expressed in the inertial frame of reference.
Dynamics on $J_L^{-1}(\mu) \subset T^*(G \times S)$

The Hamiltonian $H : T^*G \times T^*S \rightarrow g^*$ induces two parameterized families of functions:

$$H_{\alpha_s} : T^*G \rightarrow \mathbb{R}, \quad H_{\alpha_s}(\alpha_g) := H(\alpha_g, \alpha_s),$$
$$H_{\alpha_g} : T^*S \rightarrow \mathbb{R}, \quad H_{\alpha_g}(\alpha_s) := H(\alpha_g, \alpha_s).$$
Dynamics on $J_L^{-1}(\mu) \subset T^*(G \times S)$

The Hamiltonian $H : T^*G \times T^*S \to g^*$ induces two parameterized families of functions:

$$H_{\alpha_s} : T^*G \to \mathbb{R}, \quad H_{\alpha_s}(\alpha_g) := H(\alpha_g, \alpha_s),$$

$$H_{\alpha_g} : T^*S \to \mathbb{R}, \quad H_{\alpha_g}(\alpha_s) := H(\alpha_g, \alpha_s).$$

Hamiltonian vector field on $T^*G \times T^*S$

Hamiltonian vector field corresponding to $H$ is given by

$$X_H(\alpha_g, \alpha_s) = (X_{H_{\alpha_s}}(\alpha_g), X_{H_{\alpha_g}}(\alpha_s)).$$
Dynamics on $J_L^{-1}(\mu) \subset T^*(G \times S)$

The Hamiltonian $H : T^*G \times T^*S \to \mathfrak{g}^*$ induces two parameterized families of functions:

$$H_{\alpha_s} : T^*G \to \mathbb{R}, \ H_{\alpha_s}(\alpha_g) := H(\alpha_g, \alpha_s),$$
$$H_{\alpha_g} : T^*S \to \mathbb{R}, \ H_{\alpha_g}(\alpha_s) := H(\alpha_g, \alpha_s).$$

Hamiltonian vector field on $T^*G \times T^*S$

Hamiltonian vector field corresponding to $H$ is given by

$$X_H(\alpha_g, \alpha_s) = (X_{H_{1\alpha_s}}(\alpha_g), X_{H_{2\alpha_g}}(\alpha_s)).$$

The diffeomorphism $\varphi := (\phi, \text{id}_{T^*S})$, where $\phi$ is as defined above, maps $J_L^{-1}(\mu)$ to $G \times \{\mu\} \times T^*S$. 
Dynamics on $J_L^{-1}(\mu) \subset T^*(G \times S)$

**Theorem (Right-trivialized dynamics)**

$$(\varphi_*X_H)(g, \mu, \alpha_s) = \left( TeL_g \left. \frac{\delta H_{\alpha_s}}{\delta \eta} \right|_{\eta=Ad_{g^*}^*\mu}, 0, X_{H2\phi^{-1}(g, \mu)}(\alpha_s) \right).$$
Outline

Introduction

The setting

Dynamics on the level set

A conserved volume form

Application: Spacecraft with two rotors

Summary
A conerved volume form on $G$

Let $X = T\pi_1 \left( X_H^\phi \right)$, where $\pi_1$ is the projection to the first factor. That is,

$$X(g) = T_e L_g \left. \frac{\delta H^-_{\alpha_s}}{\delta \eta} \right|_{(\eta=\text{Ad}^*_g \mu)}.$$
A conserved volume form on $G$

Let $X = T \pi_1 \left( X^\phi_H \right)$, where $\pi_1$ is the projection to the first factor. That is,

$$X(g) = TeL_g \frac{\delta H_{\alpha_s}^-}{\delta \eta} \bigg|_{(\eta = \text{Ad}_g^{*}\mu)}.$$

We construct a volume form on $G$ using $\Omega_{T^*G}$, the canonical two form on $T^*G$, that is conserved under $X$.

$$\begin{align*}
(\Omega_{T^*G}) &\xrightarrow{\phi} (\Omega) \\
G &\xrightarrow{i} (i_{(0,\xi)}\Omega) \\
G \times g^* &\xrightarrow{(\Theta_\xi)} G \times g^*
\end{align*}$$
**A conserved volume form on $G$**

Let $X = T\pi_1 \left( X^\phi_H \right)$, where $\pi_1$ is the projection to the first factor. That is,

$$X(g) = T_e L_g \left. \frac{\delta H_{\alpha_s}}{\delta \eta} \right|_{(\eta = \text{Ad}_g^* \mu)}.$$

We construct a volume form on $G$ using $\Omega_{T^*G}$, the canonical two form on $T^*G$, that is conserved under $X$.

$$\begin{array}{c}
(\Omega_{T^*G})_{T^*G} \xrightarrow{\phi} (\Omega)_{G \times g^*} \\
(\Theta_\xi)_{G} \xrightarrow{i} (i_{(0,\xi)} \Omega)_{G \times g^*}
\end{array}$$

Let $\Omega := \phi_* \Omega_{T^*G}$ and $\xi \in g^*$. Define the one-form on $G$:

$$\Theta_\xi := i^* i_{(0,\xi)} \Omega.$$
A conserved volume form on $G$

To construct a volume form on $G$, choose any $n$ linearly-independent set of vectors in $g^*$, \{\xi_1, \xi_2, \ldots, \xi_n\}. Define

$$
\Gamma := \Theta_{\xi_1} \wedge \Theta_{\xi_2} \wedge \cdots \wedge \Theta_{\xi_n} = i^*i_{(0,\xi_1)}\Omega \wedge i^*i_{(0,\xi_2)}\Omega \wedge \cdots \wedge i^*i_{(0,\xi_n)}\Omega.
$$
A conserved volume form on $G$

To construct a volume form on $G$, choose any $n$ linearly-independent set of vectors in $g^*$, $\{\xi_1, \xi_2, \ldots, \xi_n\}$. Define

$$\Gamma := \Theta_{\xi_1} \wedge \Theta_{\xi_2} \wedge \cdots \wedge \Theta_{\xi_n} = i^* i_{(0,\xi_1)} \Omega \wedge i^* i_{(0,\xi_2)} \Omega \wedge \cdots \wedge i^* i_{(0,\xi_n)} \Omega.$$ 

**Theorem**

$$L_X \Gamma = 0.$$
A conserved volume form on $G$

To construct a volume form on $G$, choose any $n$ linearly-independent set of vectors in $g^*$, $\{\xi_1, \xi_2, \ldots, \xi_n\}$. Define

$$\Gamma := \Theta_{\xi_1} \wedge \Theta_{\xi_2} \wedge \cdots \wedge \Theta_{\xi_n} = i^* i_{(0, \xi_1)} \Omega \wedge i^* i_{(0, \xi_2)} \Omega \wedge \cdots \wedge i^* i_{(0, \xi_n)} \Omega.$$ 

**Theorem**

$$L_X \Gamma = 0.$$ 

**Sketch of the proof**

The proof makes use of fact that the canonical two form is conserved under the flow of any Hamiltonian vector field.
If a vector field conserves a volume form on a compact manifold, then the vector field is WPPS.³


If a vector field conserves a volume form on a compact manifold, then the vector field is WPPS.  

**Theorem**

If $G$ is compact, $X$ is WPPS on $G$.

---

\(^3\)Birtea, P et al. *Controllability of Poisson systems*, SIAM J. Cntrl and Opt, 2004

If a vector field conserves a volume form on a compact manifold, then the vector field is WPPS.\textsuperscript{3}

**Theorem**

If $G$ is compact, $X$ is WPPS on $G$.

**Example**

Bhat et al.\textsuperscript{4} prove that the vector field $RS \left(I^{-1}(R^T\mu - l)\right)$, where $l$ is a parameter, is WPPS on $SO(3)$. The above theorem thus generalizes this result.

\textsuperscript{3}Birtea, P et al. *Controllability of Poisson systems*, SIAM J. Cntrl and Opt, 2004

Outline

Introduction

The setting

Dynamics on the level set

A conserved volume form

Application: Spacecraft with two rotors

Summary
CONTROL OF A SPACECRAFT WITH TWO ROTORS

Figure: Rigid body model of a spacecraft with rotors.
The configuration space for the system is \( Q = SO(3) \times \mathbb{T}^2 \).
Control of a spacecraft with two rotors

The configuration space for the system is $Q = SO(3) \times \mathbb{T}^2$.

The Hamiltonian is the kinetic energy, which is symmetric with respect to the standard $SO(3)$ action on $Q$. 
The configuration space for the system is \( Q = SO(3) \times \mathbb{T}^2 \).

The Hamiltonian is the kinetic energy, which is symmetric with respect to the standard \( SO(3) \) action on \( Q \).

The reduced Hamiltonian \( H^- : so(3)^* \times T^*\mathbb{T}^2 \) is given by

\[
H^- (\Pi, p) = \frac{1}{2} (\Pi - \tilde{l})^T \tilde{I} (\Pi - \tilde{l}) + \frac{1}{2} l^T \tilde{I}_r l.
\]

\( p := (\theta, l), \tilde{I} = (I_L - I_r)^{-1}, \tilde{I}_r = \text{diag} (I_{r1}^{-1}, I_{r2}^{-1}), \tilde{l} = i(l), i : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)-canonical inclusion.
Control of a spacecraft with two rotors

The configuration space for the system is $Q = SO(3) \times \mathbb{T}^2$.

The Hamiltonian is the kinetic energy, which is symmetric with respect to the standard $SO(3)$ action on $Q$.

The reduced Hamiltonian $H^- : \mathfrak{so}(3)^* \times T^*\mathbb{T}^2$ is given by

$$H^- (\Pi, p) = \frac{1}{2} (\Pi - \tilde{l})^T \tilde{I}(\Pi - \tilde{l}) + \frac{1}{2} l^T \tilde{I}_r l.$$

$p := (\theta, l), \tilde{I} = (I_L - I_r)^{-1}, \tilde{I}_r = \text{diag} (I_{r1}^{-1}, I_{r2}^{-1}), \tilde{l} = i(l), i : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ - canonical inclusion.

The right-trivialized dynamics is given by

$$\dot{R} = RS \left( \tilde{I}(R^T \mu - \tilde{l}) \right),$$

$$\dot{\theta} = -\pi \left( \tilde{I}R^T \mu \right) + \tilde{I}_r l,$$
Two-rotor control system

Since the variable $\theta$ is of little significance, it is enough to consider the dynamics

$$\dot{R} = RS\left(\tilde{I}(R^T \mu - \tilde{l})\right),$$

$$\dot{l} = u,$$

where $u$ is the control torque.
Two-rotor control system

Since the variable $\theta$ is of little significance, it is enough to consider the dynamics

$$\dot{R} = RS \left( \tilde{I} (R^T \mu - \tilde{l}) \right),$$
$$\dot{l} = u,$$

where $u$ is the control torque.

Define

$$f_0(R, l) := \left( RS \left( \tilde{I} (R^T \mu - \tilde{l}) \right), 0 \right),$$
$$f_1(R, l) = (0, e_1),$$
$$f_2(R, l) = (0, e_2).$$
Two-rotor control system

Since the variable $\theta$ is of little significance, it is enough to consider the dynamics

\[ \dot{R} = R S \left( \tilde{I} (R^T \mu - \tilde{l}) \right), \]
\[ \dot{l} = u, \]

where $u$ is the control torque.

Define
\[ f_0(R, l) := \left( R S \left( \tilde{I} (R^T \mu - \tilde{l}) \right), 0 \right), \]
\[ f_1(R, l) = (0, e_1), \quad f_2(R, l) = (0, e_2). \]

We get a control system over $SO(3) \times \mathbb{R}^2$ of the form

\[ \dot{x} = f_0(x) + \sum_{i=1}^{2} u_i f_i(x). \]
Since $f_0$ is WPPS, global controllability follows if $\{f_0, f_1, f_2\}$ satisfy the Lie algebra rank condition (LARC).\(^5\)

Global controllability

Since $f_0$ is WPPS, global controllability follows if $\{f_0, f_1, f_2\}$ satisfy the Lie algebra rank condition (LARC).\(^5\)

**Proposition**

The set $\{f_0, f_1, f_2\}$ satisfies LARC over $SO(3) \times \mathbb{R}^2$.

---

Global controllability

Since $f_0$ is WPPS, global controllability follows if \{${f_0, f_1, f_2}$\} satisfy the Lie algebra rank condition (LARC).\(^5\)

**Proposition**

The set \{${f_0, f_1, f_2}$\} satisfies LARC over $SO(3) \times \mathbb{R}^2$.

**Theorem**

The-two rotor control system is globally controllable over $SO(3) \times \mathbb{R}^2$.

The reachable set

**Theorem (Bayadi, Banavar and Chang\textsuperscript{6})**

The (left trivialized) reachable set of $TSO(3)$ of a spacecraft with two rotors is $(R, I_L^{-1} (R^T \mu - I_r \Omega_r))$ where $R \in SO(3)$ and $\Omega_r \in \mathbb{R}^3$ is such that $e_3^T \Omega_r = 0$.

---

\textbf{Figure:} Reachable set of $\Omega_b$ at a particular orientation.

---

Outline

Introduction

The setting

Dynamics on the level set

A conserved volume form

Application: Spacecraft with two rotors

Summary
We derive an expression for the Hamiltonian dynamics on the level set of the momentum map for a mechanical system on $Q = G \times S$. 
We derive an expression for the Hamiltonian dynamics on the level set of the momentum map for a mechanical system on $Q = G \times S$.

We construct a volume form on $G$ which is conserved under this dynamics and show that the dynamics is WPPS.
We derive an expression for the Hamiltonian dynamics on the level set of the momentum map for a mechanical system on $Q = G \times S$.

We construct a volume form on $G$ which is conserved under this dynamics and show that the dynamics is WPPS.

Using these results, we analyze the controllability of a spacecraft with two rotors and characterize its reachable set.