Proof_of_ Lemma

Based on Bernstein coefficient range enclosure proof as given by [3], we present here proof of our Lemmas.

Lemma 0.0.1 *[1, 2] Let* p *be a polynomial of degree* N *and let* $\bar{p}(\mathbf{x})$ *denotes the range of* p *on the given domain* $\mathbf{x}$*. Then the following property holds for a patch* $D(\mathbf{x})$ *of B-spline coefficients:*

$$\bar{p}(\mathbf{x}) \subseteq [\min D(\mathbf{x}), \max D(\mathbf{x})]$$

Proof :

Let

$$p(x) = \sum_{j=1}^{m+k} d_j N_j\left(x\right)$$

be the B-Spline representation for function p *and let* $\bar{\mathrm{p}}$ *be its range,*

$$\bar{p}\left([a,b]\right) \subseteq \left[\min_{1\leq j\leq m+k} d_j, \max_{1\leq j\leq m+k} d_j\right]$$

that is

$$\min_{1\leq j\leq m+k} d_j \leq \sum_j d_{\mathrm{j}}\mathrm{N}_{\mathrm{j}}\left(x\right) \leq \max_{1\leq j\leq m+k} d_j$$

which means

$$\min_{1\leq j\leq m+k} d_j \leq \min p_{val} \leq \max p_{val} \leq \max_{1\leq j\leq m+k} d_j$$

The next Lemma concerned namely when the estimate provided by the B-Spline form is the range.

Lemma 0.0.2 *[3] Let* p *be a polynomial of degree* N *and let* $\bar{p}(\mathbf{x}) = [a, b]$*. Then*

$$a = \min_{0\leq I\leq \mathrm{N}} d_I(\mathbf{x}) \qquad \textit{iff} \qquad \min_{0\leq I\leq \mathrm{N}} d_I(\mathbf{x}) = \min_{I\in S_0} d_I(\mathbf{x})$$

and

$$b = \max_{0\leq I\leq \mathrm{N}} d_I(\mathbf{x}) \qquad \textit{iff} \qquad \max_{0\leq I\leq \mathrm{N}} d_I(\mathbf{x}) = \max_{I\in S_0} d_I(\mathbf{x})$$

Proof :

Let

$$p(x) = \sum_{j=1}^{m+k} d_j N_j(x)$$

be the B-Spline representation for function p *and let* $\bar{\mathrm{p}}$ *be its range,*

$$\bar{p}\left([0,1]\right) = [a,b]$$

We first note that

$$p(0) = d_1$$

and

$$p(1) = d_{m+k}$$

Suppose now that

$$a = \min_{1 \leq j \leq m+k} d_j$$

and that $x \in [0,1]$*, then if* $d_1 = d_2 = ... = d_{m+k}$*. We have that* $P(0) = d_1 = a = d_{m+k} = P(1)$ *and the property is valid. If* $d_1 = d_2 = ... = d_{m+k}$ *is not satisfied then*

$$\sum_{j=1}^{m+k} d_j N_j(x) > \min_{1 \leq j \leq m+k} d_j$$

which means minimum cannot occur at an interior point of $[0,1]$*,because*

$$a = \min_{1 \leq j \leq m+k} d_j$$

and

$$P(0) = a$$

hence it must occur at an end point of the interval. Conversely, suppose that

$$\min_{1 \leq j \leq m+k} d_j = \min(d_1, d_{m+k})$$

and assume that

$$\min_{1 \leq j \leq m+k} d_j = d_1$$

for simplicity. Then the bound is sharp, i.e.

$$a = P(0) = d_1 = \min_{1 \leq j \leq m+k} d_j$$

the same argument is valid if

$$\min_{1 \leq j \leq m+k} d_j = d_{m+k}$$

the second part regarding the sharpness of the upper bound is proven in the same manner.

Bibliography

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