Proof of Lemma

Based on Bernstein coefficient range enclosure proof as given by [3], we present here proof of our Lemmas.

Lemma 0.0.1 [1, 2] Let \( p \) be a polynomial of degree \( N \) and let \( \bar{p}(x) \) denotes the range of \( p \) on the given domain \( x \). Then the following property holds for a patch \( D(x) \) of B-spline coefficients:

\[
\bar{p}(x) \subseteq [\min D(x), \max D(x)]
\]

Proof:
Let

\[
p(x) = \sum_{j=1}^{m+k} d_j N_j(x)
\]

be the B-Spline representation for function \( p \) and let \( \bar{p} \) be its range,

\[
\bar{p}([a, b]) \subseteq \left[ \min_{1 \leq j \leq m+k} d_j, \max_{1 \leq j \leq m+k} d_j \right]
\]

that is

\[
\min_{1 \leq j \leq m+k} d_j \leq \sum_{j} d_j N_j(x) \leq \max_{1 \leq j \leq m+k} d_j
\]

which means

\[
\min_{1 \leq j \leq m+k} d_j \leq \min_{I \in S_0} p_{val} \leq \max_{I \in S_0} p_{val} \leq \max_{1 \leq j \leq m+k} d_j
\]

The next Lemma concerned namely when the estimate provided by the B-Spline form is the range.

Lemma 0.0.2 [3] Let \( p \) be a polynomial of degree \( N \) and let \( \bar{p}(x) = [a, b] \). Then

\[
a = \min_{0 \leq I \leq N} d_I(x) \quad \text{iff} \quad \min_{0 \leq I \leq N} d_I(x) = \min_{I \in S_0} d_I(x)
\]

and

\[
b = \max_{0 \leq I \leq N} d_I(x) \quad \text{iff} \quad \max_{0 \leq I \leq N} d_I(x) = \max_{I \in S_0} d_I(x)
\]

Proof:
Let

\[
p(x) = \sum_{j=1}^{m+k} d_j N_j(x)
\]

be the B-Spline representation for function \( p \) and let \( \bar{p} \) be its range,

\[
\bar{p}([0, 1]) = [a, b]
\]
We first note that
\[ p(0) = d_1 \]
and
\[ p(1) = d_{m+k} \]
Suppose now that
\[ a = \min_{1 \leq j \leq m+k} d_j \]
and that \( x \in [0, 1] \), then if \( d_1 = d_2 = \ldots = d_{m+k} \). We have that \( P(0) = d_1 = a = d_{m+k} = P(1) \) and the property is valid. If \( d_1 = d_2 = \ldots = d_{m+k} \) is not satisfied then
\[
\sum_{j=1}^{m+k} d_j N_j(x) > \min_{1 \leq j \leq m+k} d_j
\]
which means minimum cannot occur at an interior point of \([0, 1]\), because
\[ a = \min_{1 \leq j \leq m+k} d_j \]
and
\[ P(0) = a \]
hence it must occur at an end point of the interval. Conversely, suppose that
\[ \min_{1 \leq j \leq m+k} d_j = \min (d_1, d_{m+k}) \]
and assume that
\[ \min_{1 \leq j \leq m+k} d_j = d_1 \]
for simplicity. Then the bound is sharp, i.e.
\[ a = P(0) = d_1 = \min_{1 \leq j \leq m+k} d_j \]
the same argument is valid if
\[ \min_{1 \leq j \leq m+k} d_j = d_{m+k} \]
the second part regarding the sharpness of the upper bound is proven in the same manner.
Bibliography

