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# Electromagnetic Waves and Optics, a Linear System Approach

– Monograph –

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*Dedicated to Darshan Lal Khaneja*

# Preface

The origin of this book can be traced back to author's research in nuclear magnetic resonance (NMR) spectroscopy. In 1945, Purcell, Torrey and Pound detected Nuclear Magnetic Resonance (NMR) in condensed matter. They had a tank (cavity/resonator) they filled with radio-frequency (rf) waves and immersed nuclear spins in it, all placed in static magnetic field $B$. When $B$ was tuned, spins absorbed rf and water level in the tank ( rf level in the cavity actually ) went down. How do you fill tanks with rf/microwaves. What matters is we fill it at right frequency, not the shape of the cavity or filling radiator. How do we understand all this. Modern NMR/radio equipment uses tank circuits to amplify weak signals and LC oscillators to generate rf. Radars use special oscillator/cavities to generate microwaves and special pipes called waveguides to channel them. In this book we study all this in the framework of linear systems.

Maxwell equations that govern the evolution of EM waves is a linear system. To solve for the linear system, we have to calculate its eigenvaues and eigenfunctions (modes). In this book, we take this approach and calculate modes for Maxwell equations, both in free space (where they are travelling waves) to space bounded by conductors like cavities and waveguides (where they are standing waves). Central to theme of the book is the phenomenon of *resonance*. When the modes of the linear system have imaginary eigenvalues $j\omega_0$, and we drive the system with a harmonic input $\exp(j\omega_0 t)$, the mode builds up unbounded. We say, we drive the system on resonance and pump energy into it. In this book, we show how we study cavities, resonators and waveguides as systems driven on resonance. We study tank circuits and resonant antennas. We show resonance is central to how we produce so called microwaves in cavities. Another important notion is study of linear systems is the concept of a steady state. When eigenvalues of linear sytem have negative real parts then system is termed lossy and in response to an harmonic input settles down to a steady state satisfying an algebraic equation. This steady state analysis forms the basis of analysis of antenna's and transmission lines as discussed in the text.

In this book, we study analysis of Maxwell equations, waveguides, cavities, transmission lines and antenna's. We study oscillators for producing rf/microwaves and amplifiers and tank circuits for amplifying these signals. The later part of this book is devoted to study of light including optical instruments and pheonomenon of optical resonance in an atom, which explains dielectric response of a medium and absorbtion of light. In writing the book, I have tried to answer all the questions, I had as an electrical engineering undergraduate at IIT Kanpur, taking an electromagnetism course. I have chosen subjects that I feel are conceptual. I have stressed system theoretic ideas. In using analysis tools of electromagnetic theory that can be found elsewhere, I have stressed what system theoretic concept we are invoking. May that be calculation of modes, or steady state of a linear system, notion of resonance and resonant driving, etc. The book is written as a research monograph, but it is more an effort on author's part to see all of electromagnetism as a linear system. The book is intended for a first course in electromagnetism or elctromagnetic theory. The book may also be used in a course on rf/microwave engineering and optics.

It is an opportunity to acknowledge numerous people who I believe directly and indirectly helped with this effort. I would like to thank Professor Steffen Glaser and Niels Nielsen for numerous years of excellent collaboration in NMR spectroscopy that ultimately led me to develop this text. I would like to thank Professor Roger Brockett, who helped me in nurturing my taste in physics. I am grateful to Professor M. Sachidananda at IIT Kanpur whose course on electromagnetism has kept my interest in the subject alive and culminated in this text. I will like to thank the wonderful colleagues and academic environment of SYSCON at IIT Bombay that provided ample opportunity for self development. Finally I like to acknowledge the support of my family which made this effort possible.

IIT Bombay, *Navin Khaneja*
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# Part I
# Electromagnetic waves and linear systems

Electromagnetic (EM) waves are all around us. From the world of light and color, to TV and radio broadcast, to cell phones and dish TV, everywhere. In this book, we study generation and propagation of EM waves as a linear system. The evolution of EM waves is governed by Maxwell equations. We treat Maxwell equations as a linear system, driven by current sources so called antennas. So what is a linear system. To fix ideas, consider a spring mass system of mass $m$ and spring constant $k$, to which force $F$ is applied. The displacement of the system from its equilibrium denoted by $x$ satisfies

$$m\ddot{x} + kx = F, \tag{0.1}$$

or

$$\ddot{x} + \omega_0^2 x = F/m, \tag{0.2}$$

with $x_1 = \omega_0 x$ and $x_2 = \dot{x}$, and $u = \frac{F}{m}$ we find the above system written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u. \tag{0.3}$$

If we denote $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we can write the above system as

$$\frac{dX}{dt} = AX + bu. \tag{0.4}$$

In general, $A$ is a $n \times n$ matrix, $X, b$ are $n \times 1$ vector and $u$ is a time varying input. We call above equation a *linear system*. Maxwell equations are a linear system, where input are current sources that act as antenna. To solve for evolution of linear system, we have to solve for the eigenvales and eigenvectors of the matrix $A$. The eigenvectors are called *modes*. This is what we do in this book, we solve for modes of the Maxwell equations which in free space are travelling waves. We solve for modes of the Maxwell equations when the space is bounded by conductors as in what are called waveguides, cavities and resonators. These are called cavity modes. Central to theme of the book is the phenomenon of *resonance*. When the modes of the linear system have imaginary eigenvalues $j\omega_0$ and we drive the system with a harmonic input $u = \exp(j\omega_0 t)$ or $u = \cos\omega_0 t$, the mode builds up unbounded. We say we drive the system on resonance and pump energy into it. In this book, we show how we study cavities, resonators and waveguides as systems driven on resonance. We study tank circuits and resonant antennas. We show resonance is central to how we produce so called microwaves in cavities. Another important notion is study of linear systems is the concept of a steady state. When eigenvalues of $A$ have negative real parts then system is termed lossy and in response to an input $u = \exp(j\omega_0 t)$ settles down to a steady state vector $X_0 \exp(j\omega_0 t)$ with $X_0$ satisfying the linear equation,

$$j\omega_0 X_0 = aX_0 + b \tag{0.5}$$

This steady state analysis forms the basis of analysis of antenna's and transmission lines as discussed in the text. Even when system is lossless, there is always a small resistance present making it lossy and we can talk about steady state. While the earlier part of the book covers EM waves as low frequencies radio waves (3 kHz-3 GHz) and microwaves (3 GHz-300 GHz) , the last chapters of the book treat optical phenomenon, with EM waves in optical frequencies ($10^{14} - 10^{15}$ Hz).

# Chapter 1
# Linear Systems and Electromagnetic Waves

Suppose a city gets fresh water supply as a fountain or eruption from the ground. How will we use this water. We will build a tank to collect all this water and then use pipes to carry this water to houses and then have taps at end of pipes to get water. Now imagine instead of water, I have electromagnetic (EM) waves, say microwaves (electromagnetic waves with frequency 3-300 GHz) . I have a supply of these microwaves as radiation from electron circulating a magnetic field. How do we collect this radiation. How do we channel this radiation. How do we then use this radiation in say a RADAR that detect flying objects using microwaves or a microwave oven that cooks food using microwaves. As we see subsequently, we really collect this radiation in a tank called cavity, and then use pipes called wave-guides to take it to an antenna (the analogue of tap). which directs these waves towards a flying object. How do we fill a cavity with microwaves. How do we model such electromagnetic waves. What equations govern their evolution is space and time. All this we study in this book. We model generation, storage and transmission of electromagnetic waves as a linear system. We show that such a perspective of linear system lends great mathematical insight is analysis of equations that govern the generation, storage and transmission of electromagnetic waves. So what do we mean by linear systems. Lets have a look.

## 1.1 Linear Systems

Consider a mass $M$ tied by spring of spring constant $k$ to a wall as in Fig. 1.1. The displacement $x$ of the mass from its equilibrium is related to force $F$ as

$$M\ddot{x} + kx = F. \tag{1.1}$$

Using $\omega_0^2 = \frac{k}{M}$ and $u = \frac{F}{M}$, we get for $x_1 = \omega_0 x$ and $x_2 = \dot{x}$,

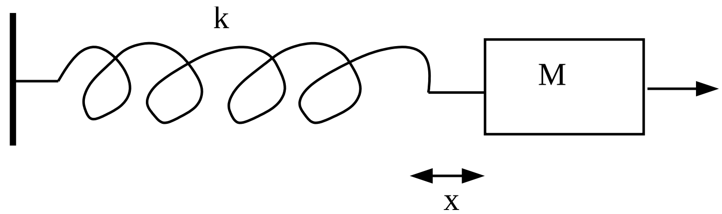



**Fig. 1.1** Fig. shows a mass $M$ tied to wall with a spring with spring constant $k$ being pulled with force $F$. $x$ measures displacement from equilibrium.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u. \tag{1.2}$$

If we denote $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we can write the above system as

$$\frac{dX}{dt} = AX + bu. \tag{1.3}$$

In general, $A$ is a $n \times n$ matrix, $X, b$ are $n \times 1$ vector and $u$ is a time varying input. We call Eq. ( 1.3) a *linear system*.

Without an input the system takes the form

$$\dot{X} = AX. \tag{1.4}$$

Using $X(t + \Delta t) = (I + A\Delta t)X(t)$, we get $X(n\Delta t) = (I + A\Delta t)^n X(0)$, with $t = n\Delta t$, we het $X(t) = (I + \frac{At}{n})^n X(0)$. With $\Delta t \to 0$ and $n \to \infty$, we have $X(t) = \exp(At)X(0)$ where

$$\exp(At) = I + At + \frac{(At)^2}{2!} + \cdots + \frac{(At)^n}{n!} + \ldots. \tag{1.5}$$

Suppose $u$ Eq. (1.3) is nonzero over an interval $\tau$ and $\tau + \Delta$. Then $X(\tau + \Delta t) = (I + A\Delta t)X(\tau) + bu(\tau)\Delta t$, and continuing we get

$$\begin{aligned} X(t) &= \exp(A(t-\tau))X(\tau) + \exp(A(t-\tau))bu(\tau)\Delta t \\ &= \exp(At)X(0) + \exp(A(t-\tau))bu(\tau)\Delta t. \end{aligned}$$

and then in general for non zero $u$, we have

$$X(t) = \exp(At)X(0) + \int_0^t \exp(A(t-\tau))bu(\tau)d\tau. \tag{1.6}$$

This is called the *variation of constant formula*. Although Eq. (1.6) solves Eq. (1.3), we show how we can further simplify.

To solve for evolution of the system Eq. (1.4), we observe that if the initial state $X(0)$ of the system is $e$ an eigenvector of $A$ with eigenvalue $\lambda$, then the solution is simply $X(t) = \exp(\lambda t)e$. Simply substitute in Eq. (1.4). In general, we can decompose $X(0) = \sum_i \alpha_i(0)e_i$ where $e_i$ are eigenvectors of $A$ with eigenvalue $\lambda_i$. Then observe $X(t) = \sum_i \alpha_i(0)\exp(\lambda_i t)e_i$ is the solution as can be seen by substitution.

The solution is understood as simply writing the differential equation in terms of basis $e_i$ with $X(t) = \sum_i \alpha_i(t)e_i$ then the vector equation gets expressed as $n$ scalar equations

$$\dot{\alpha}_i = \lambda_i \alpha_i, \tag{1.7}$$

with solution $\alpha_i(t) = \exp(\lambda_i t)\alpha_i(0)$. In presence of input as in Eq. (1.3), we can express $b = \sum_i b_i e_i$ and equations take form

$$\dot{\alpha}_i = \lambda_i \alpha_i + b_i u(t), \tag{1.8}$$

with solution

$$\alpha_i(t) = \exp(\lambda_i t)\alpha_i(0) + b_i \int_0^t \exp(\lambda_i(t-\tau))u(\tau)d\tau. \tag{1.9}$$

With $\alpha_i(0) = 0$, we simply have $\alpha_i(t) = b_i \int_0^t \exp(\lambda_i(t-\tau))u(\tau)d\tau$. Of particular interest is the case $b_i$ correspond to imaginary eigenvalues $j\omega_i$. Then $\alpha_i(t) = b_i \int_0^t \exp(j\omega_i(t-\tau))u(\tau)d\tau$. If the input $u(t) = \exp(j\omega t)$, and we choose $\omega = \omega_k$, then all modes average only the mode $k$ survives giving $\alpha_k(t) = b_k \exp(j\omega_k t)t$. We say we excite the system on resonance with mode $k$, which builds as shown. This is the main idea of this text. We identify modes and their frequencies and then drive the system at the frequency of a mode to build that mode. We say we pump energy into a mode.

For example in Eq. (1.2), we have eigenvalues as $j\omega_0$ and $-j\omega_0$ with eigenvectors $e_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$, and $e_2 \begin{bmatrix} 1 \\ -j \end{bmatrix}$. We write input $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{i}{2}(e_1 - e_2)$. If input $u(t) = \exp(j\omega_0 t)$, the mode $e_1$ builds up. If input $u(t) = \cos(\omega_0 t) = \frac{\exp(j\omega_0 t)+\exp(-j\omega_0 t)}{2}$, the mode $e_1$ and $e_2$ both build up.

### 1.1.1 Steady State Solution

Given the linear system

$$\frac{dX}{dt} = AX + bu, \tag{1.10}$$

where $A$ is dissipative, i.e, eigenvalues of $A$ have negative real part. Let $u = \exp(j\omega t)$. Then we can simply observe $x = X_0 \exp(j\omega t)$ is a solution to Eq. (1.10), where $X_0$ is solution to algebraic equation

$$j\omega X_0 = AX_0 + b. \tag{1.11}$$

or

$$\underbrace{(j\omega\mathbf{1} \ -A)}_{\mathscr{A}} X_0 = b. \tag{1.12}$$

Since $A$ is dissipative, we have $\mathscr{A}$ invertible and we can solve for $X_0$, We call it steady state solution because if $y$ is the true solution then $z = y - x$ satisfies

$$\frac{dz}{dt} = Az \tag{1.13}$$

with $z(T) = \exp(AT)z(0)$, since $A$ is dissipative, $z(T) \to 0$ as $T \to \infty$ and $y(T) \to x(T)$. In physical problems, one encounters $A$ that have purely imaginary eigenvalues, as in spring mass problem in 1.2. However this is an ideal situation, in practice there is a small amount of friction present, which gives dissipative $A$. Therefore, even in such ideal situtations, it makes sense to talk about steady state solution keeping in mind negligibly small dissipation.

Consider a scalar linear system

$$\dot{x} = -\alpha x + u. \tag{1.14}$$

For $s = j\omega$, and input $u = u(s)exp(st)$, the steady state is $x(s)\exp(st)$, where $x(s)$ satisfies algebraic equation

$$(s+\alpha)x(s) = u(s), \tag{1.15}$$

with

$$G(s) = \frac{x(s)}{u(s)} = \frac{1}{s+\alpha}. \tag{1.16}$$

$G(s)$ is called the input-output transfer function which is the frequency response of the system as shown in Fig. 1.2A.

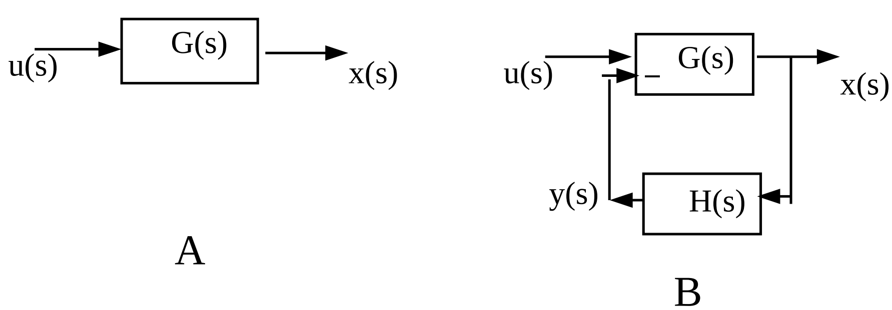



**Fig. 1.2** Fig. shows linear systems represented by transfer functions

Now lets take $x(t)$ and feed it as input to another system

$$\dot{y} = -\beta y + x(t), \tag{1.17}$$

then

$$H(s) = \frac{y(s)}{x(s)} = \frac{1}{s+\beta}. \tag{1.18}$$

Now if we take $y$ and feed it back to $x$ with a negative sign

$$\dot{x} = -\alpha x + u - y, \tag{1.19}$$

as shown in Fig. 1.2B. Then

$$x(s) = G(s)(u(s) - y(s)) = G(u(s) - Hx(s)), \tag{1.20}$$

or

$$C(s) = \frac{x(s)}{u(s)} = \frac{G(s)}{1 + G(s)H(s)}. \tag{1.21}$$

$C(s)$ is called the feedback or closed loop transfer function.

## 1.2 Maxwell Equations

Why do we care about linear systems. Because we want to study electromagnetic waves [18, 21, 22, 31, 17]. We know electromagnetic waves everyday from radio and TV broadcast and light and color around us. These waves propagate as oscillating electric and magnetic field. The equation governing their evolution are the the Maxwell equations. Electromagnetic waves are space and time varying electric and magnetic fields related by Maxwell equations as in

$$\mu \frac{\partial H}{\partial t} = -\nabla \times E \tag{1.22}$$

$$\varepsilon \frac{\partial E}{\partial t} = \nabla \times H - J(r,t), \tag{1.23}$$

where each spatial location $r$ indices a vector entry in linear system as in Eq. ( 2.16). The vector entries are $E$ (Electric) and $H$ (Magnetic) fields. The goal is to calculate $E(r,t)$ and $H(r,t)$ resulting from an current input $J(r,t)$. That is solve a linear system for an input $u(t)$. It means we have to find modes of the system and corresponding eigenvalues. That is we have to solve for $Ax = \lambda x$ in Eq. ( 1.4), i.e solve for

$$-\frac{1}{\mu}\nabla \times E = \lambda H, \tag{1.24}$$

$$\frac{1}{\varepsilon}\nabla \times H = \lambda E. \tag{1.25}$$

Of particular interest is when $\lambda = j\omega$ , imaginary eigenvalues. In which case, the eigenvectors are as follows. For three vector $k = |k|\hat{k}$ , with $\hat{x}, \hat{y}, \hat{k}$ forming a right handed system. For $\frac{E_0}{H_0} = \eta = \sqrt{\frac{\mu_0}{\sigma_0}}$, we have

$$E = E_0 \exp(-jk \cdot r)\hat{x} = E_0 \exp(-jk_x x + k_y y + k_z z)\hat{x}, \tag{1.26}$$

$$H = H_0 \exp(-jk \cdot r)\hat{y} = H_0 \exp(-jk_x x + k_y y + k_z z)\hat{y}. \tag{1.27}$$

is a eigenvector with $c|k| = \omega$. Thus eigenvectors are the Fourier modes.

Now we want to drive Maxwell equations with current sources $J(r,t)$. This current source is an antenna. They can infact be simplified to $J(r)u(t)$. Think of $J(r)$ as $b$ in linear system. These sources are localized and can be expanded into a Fourier basis but these Fourier basis are precisely the eigenfunctions of Maxwell equation with imaginary eigenvalues. Then if we choose $u(t) = exp(j\omega t)$, then as described above, we will excite the modes with frequency $\omega$. The antenna puts energy into the Fourier modes whose frequency is same as the frequency of excitation of antenna. These modes get energy, while other modes don't and we can detect these modes very far from the antenna because they are delocalized and hence we can communicate to someone very far because we pump energy into a mode that is shared simultaneously between the transmitter and receiver. In summary, we care about eigefunctions corresponding to imaginary eigenvalues as their natural evolution constitutes travelling waves and we communicate by pumping energy in these modes.

Till now we talked about Maxwell equations in free space. The eigenmodes with imaginary eigenvalues are Fourier modes. The evolution of these modes is

$$E = E_0 \exp(j(\omega t - k \cdot r))\hat{x}, \tag{1.28}$$

$$H = H_0 \exp(j(\omega t - k \cdot r))\hat{y}. \tag{1.29}$$

This is an EM wave propagating in the $k$ direction.

Another very interesting situation is when Maxwell equations are confined in a cavity. A cavity is free space enclosed with conducting walls. EM waves don't propagate inside conductor, leading to zero tangential electric fields and zero normal magnetic fields at boundaries. We can solve the eigenvectors of the Maxwell equations with these boundary conditions to get what are called modes of the cavities. These modes say

$$\phi_1, \phi_2, \dots, \phi_k, \dots,$$

and have eigenvalues

$$j\omega_1, j\omega_2, \dots, j\omega_k, \dots.$$

When we excite the cavity with a current source, again we can expand the localized source into cavity eigenfunctions. Fig. (1.4) shows a current source as wire inserted in the cavity. If the source has current $\exp(j\omega_k t)$, we pump energy into mode $\phi_k$ and the mode $\phi_k$ will build up and all cavity will get filled with mode $\phi_k$. We show as told now how to fill cavities with microwaves and channel them through pipes

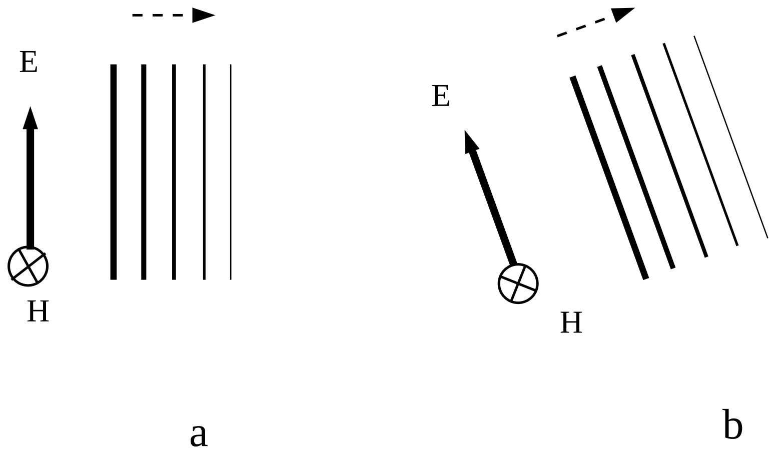



**Fig. 1.3** Fig. a and b shows EM waves propagating along $x$ and $\hat{k}$ direction respectively.

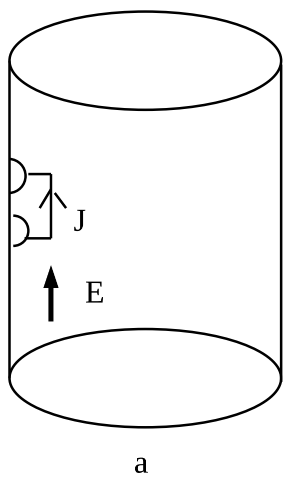



**Fig. 1.4** Fig. a shows how we can excite modes of cavity by putting in a wire carrying oscillating current.

(wave-guides) for many applications. Electromagnetic waves in 30 kHz to 3 GHz are called radio waves. $3-300$ GHz are called microwaves. $10^{12}-10^{14}$ Hz EM waves are called infrared light and $10^{14}-10^{15}$ Hz, visible light and so on. Each frequency type has its own applications. Lets look at some of these applications before we jump into main content of this book.

## 1.3 Applications of Electromagnetic Waves

### 1.3.1 Optical Fibres

These days long range communication between cities and countries is done using fibre optic communication. The light used is Infrared frequency $180-330$ THz. This light is carried by a optical fibre [26] which is a cylindrical dielectric rod made of silica, surrounded by a dielectric shell called cladding as in 1.5A. Light moves in the core. The refractive index $n_1$ of core is greater the refractive index $n_2$ of surrounding air. When light tries to leave the core as in 1.5B, it gets totally internally reflected for small $\theta$. This way fibre confines light and light travels far.

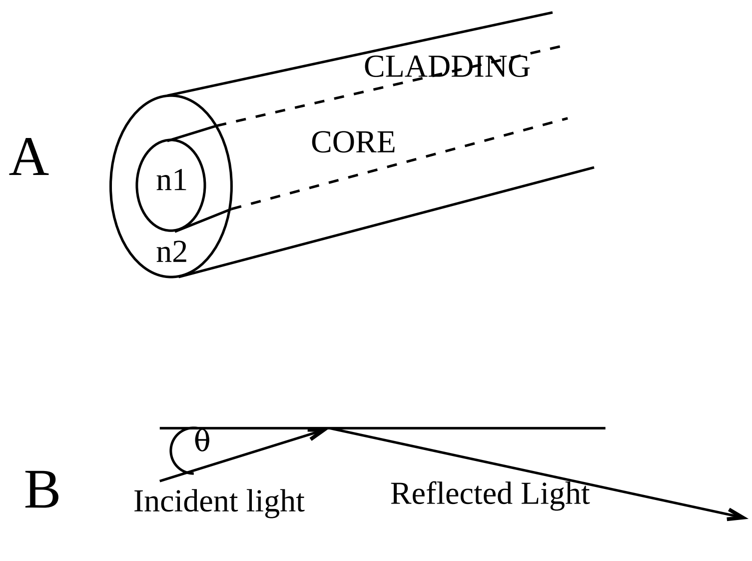



**Fig. 1.5** Fig. A shows an optical fibre with core and cladding. Fig. B shows how light gets totally internally reflected when it tries to leave the fibre.

### 1.3.2 Radio and Television Broadcast

The most traditional application of electromagnetic waves is Radio and Television broadcast. Radio transmission is in low radio frequency $1-2$ MHz in AM radio (amplitude modulated) or in VHF ($30-300$ MHz) for FM radio (frequency modulated). AM broadcast over long distance is done over short wave (HF ($3-30$ MHz). Radio waves in the shortwave band can be reflected or refracted from a layer of electrically charged atoms in the atmosphere called the ionosphere. Therefore, short waves directed at an angle into the sky can be reflected back to Earth at great distances, beyond the horizon. This is called skywave or ”skip” propagation [11, 31]. Thus shortwave radio can be used for very long distance communication, in contrast to radio waves of higher frequency which travel in straight lines (line-of-sight propagation) and are limited by the visual horizon, about 64 km (40 miles). Television broadcast is done VHF frequency. Cell-phones we use every day, work in UHF (300 MHz to 3 GHz). This band is also used for television broadcasting, satellite communication including GPS, personal radio services including Wi-Fi and Bluetooth, walkie-talkies, cordless phones, and numerous other applications [28, 25].

### 1.3.3 Satellite Communications

We all have Dish TV at home where signal is received from a satellite [15]. The signal is first transmitted to satellite using an uplink which operates in Ku microwave band ($12-18$ GHz) frequency. The is is downconverted and broadcast in C-Band ($4-8$ GHz).The signals are received via an outdoor parabolic antenna commonly referred to as a satellite dish.

### 1.3.4 Microwave Links

Pre-fibre optic days, line of sight microwave links provided telephone connectivity between cities. Typical frequencies used were less than 10 GHz. These days terrestrial microwave relay links are used in in telecommunications networks including cellular networks linking base transceiver station (BTS) and base station controller (BSC) and Mobile switching center (MSC). Frequencies from $10-80$ GHz are used.

### 1.3.5 Radar

One of the most prominent application of microwaves is the microwave radar [13, 14] . Radar is a detection system that uses microwaves to determine the range, angle, or velocity of objects. It can be used to detect aircraft, ships, spacecraft, guided missiles, motor vehicles, weather formations, and terrain. A radar system consists of a transmitter, producing electromagnetic waves in the microwave domain, a transmitting antenna, a receiving antenna (often the same antenna is used for transmitting and receiving) and a receiver and processor to determine properties of the object(s). Microwaves (pulsed or continuous) from the transmitter reflect off the object and return to the receiver, giving information about the object’s location and speed. Fig. (1.6) shows a schematic of radar where a oscillator cavity (magnetron, klystron) labelled *a* produces microwaves which are channeled by wave-guides, labelled *b* to Horn antenna, labelled *c*.

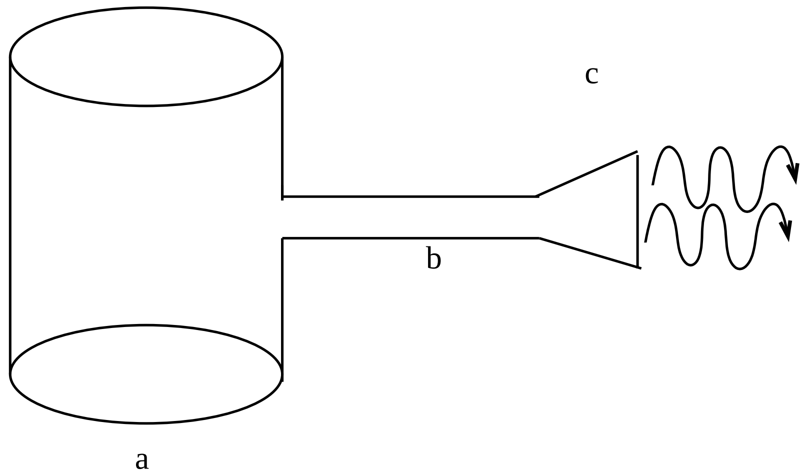



**Fig. 1.6** Fig. shows schematic of a RADAR including oscillator cavity, wave-guide and Horn antenna.

The modern uses of radar are highly diverse, including air and terrestrial traffic control, air-defense systems, antimissile systems, marine radars to locate landmarks and other ships, aircraft anticollision systems, ocean surveillance systems, outer space surveillance and rendezvous systems, meteorological precipitation monitoring, altimetry and flight control systems, guided missile target locating systems, and ground-penetrating radar for geological observations. Radar frequency ranges from 3-30 GHz.

### 1.3.6 Remote Sensing

Electromagnetic radiation in the microwave wavelength region is used in remote sensing to provide useful information about the Earth's atmosphere, land and ocean. A radar mounted on a satellite transmits microwave pulses towards the earth surface. The microwave energy scattered back to the spacecraft is measured. Remote sensing makes use of the radar principle to form an image by utilizing the time delay of the back-scattered signals. In real aperture radar imaging, the ground resolution is limited by the size of the microwave beam sent out from the antenna. Finer details on the ground can be resolved by using a narrower beam. The beam width is inversely proportional to the size of the antenna, i.e. the longer the antenna, the narrower the beam. The microwave beam sent out by the antenna illuminates an area on the ground (known as the antenna's "footprint"). In radar imaging, the recorded signal strength depends on the microwave energy back-scattered from the ground targets inside this footprint. Increasing the length of the antenna will decrease the width of the footprint. It is not feasible for a spacecraft to carry a very long antenna which is required for high resolution imaging of the earth surface. To overcome this limitation, principle of synthetic aperture radar (SAR) is used. SAR capitalizes on the motion of the space craft to emulate a large antenna from the small antenna it actually carries on board. When microwaves strike a surface, the proportion of energy scattered back to the sensor depends on many factors: Physical factors such as the dielectric constant of the surface materials which also depends strongly on the moisture content; Geometric factors such as surface roughness, slopes, orientation of the objects relative to the radar beam direction; The types of land cover (soil, vegetation or man-made objects). Microwave frequency, polarization and incident angle. The ability of microwave to penetrate clouds, precipitation, or land surface cover depends on its frequency. Generally, the penetration power increases for longer wavelength (lower frequency). The SAR back-scattered intensity generally increases with the surface roughness. However, ”roughness” is a relative quantity. Whether a surface is considered rough or not depends on the length scale of the measuring instrument. If a meter-rule is used to measure surface roughness, then any surface fluctuation of the order of 1 cm or less will be considered smooth. On the other hand, if a surface is examined under a microscope, then a fluctuation of the order of a fraction of a millimeter is considered very rough. In SAR imaging, the reference length scale for surface roughness is the wavelength of the microwave. If the surface fluctuation is less than the microwave wavelength, then the surface is considered smooth. For example, little radiation is back-scattered from a surface with a fluctuation of the order of 5 cm if a L-band (15 to 30 cm wavelength, 1 GHz frequency) SAR is used and the surface will appear dark. However, the same surface will appear bright due to increased back-scattering in a X-band (2.4 to 3.8 cm wavelength, 10 GHz frequency ) SAR image. The C band is useful for imaging ocean and ice features. However, it also finds numerous land applications. The L band has a longer wavelength and is more penetrating than the C band. Hence, it is more useful in forest and vegetation study as it is able to penetrate deeper into the vegetation canopy.

### 1.3.7 Radio Astronomy

Radio astronomy is a sub-field of astronomy that studies celestial objects at radio frequencies [**?**, 30]. For example radiation at 21 cm, 1428 MHz frequency is emitted from galactic hydrogen due to what is called hyperfine splitting of its orbital. We can detect this radiation with high angular precision, which gives the angular location of the distant galaxy. The Doppler shift in the frequency gives estimate of how fast Galaxy is receding from us. The angular precision is measured by the ratio $\frac{\lambda}{D}$ where $D$ is the diameter of the antenna. With large wavelength, we need large diameter $D$.

This is made feasible by having a array of such antennas a technique called Synthetic Aperture Radar similar to as in remote sensing.

### 1.3.8 Microwave Heating

Water in the food absorbs microwaves around 2.450 GHz. At this frequency, transitions between rotational states are induced in the water molecule. The molecule in excited rotational state dissipates energy into heat. This forms the basis of what is called a microwave heating in microwave oven. Like Radar in a microwave oven, we have a cavity to produce microwaves, which are carried by a short wave-guide to the main cavity which is filled with microwaves to heat the food.

### 1.3.9 Spectroscopy

One of big applications of electromagnetic waves is spectroscopy. Using electromagnetic waves, we can probe structure of atoms and molecules. Different type of spectroscopies correspond to different frequency electromagnetic waves. Radio-frequency electromagnetic waves can induce transition between the spin states of atomic nucleus placed in a magnetic field $B_0$. The nuclear spin is like a charged spinning top which carries a magnetic moment $\mu$. The difference in the energy of spin (magnetic moment) oriented along and opposite to magnetic field is $E = 2\mu B_0$. Writing this energy in frequency $E = \hbar\omega_0$, we find $\omega_0$ for hydrogen at field of 10 T corresponds to around 400 MHz. Thus radio frequency EM can probe nuclear spins and this field of spectroscopy is called Nuclear Magnetic Resonance (NMR) [10] spectroscopy. Electron spin has $\mu$ around $10^3$ times higher than that of hydrogen nucleus and we can probe electron spin at around same $B_0$ field strength by GHz radiation. This field is called Electron Paramagnetic Resonance (EPR). The energy difference in rotational states of molecules correspond to GHz frequency and resulting spectroscopy goes by name of microwave spectroscopy. The energy difference in the vibrational states of molecules correspond to THz frequency and resulting spectroscopy goes by name of vibrational spectroscopy. The transition of electrons between different orbitals correspond to optical $10^{15}$ Hz frequency and resulting spectroscopy goes by name of optical spectroscopy.

## 1.4 Problems

1. Relative permittivity of medium is 1.4, find velocity of light in the medium.

2. Frequency of EM wave is 1 GHz, find its wavelength.

3. What is the frequency of 1 mm microwave.

4. Relative permittivity of medium is 1.5, find wavelength of 100 GHz microwave.

5. How long is a 1 GHz broad microwave packet in time and space.

# Chapter 2
# Electromagnetic Waves and Propagation

Electromagneic (EM) waves are oscillating electric and magnetic fields. In this chapter, we develop the mathematics to describe their propagation. We begin with Maxwell equations. We show how Maxwell equations constitute a linear system. We show how we can solve for the modes of the linear system. These turn out to be plane waves. We describe the evlotion of these plane waves. We then consider how waves travel across interfaces [11]. We consider interfaces between two dielectrics and interface between a dielectric and a conductor. In the former case, we see how waves change course when crossing an interface and we recover the famous laws of refraction. In the later case, we show how conductor relects EM waves. This reflection forms the basis of mirrors in optics [**?**] and waveguides and cavities in microwave engineering [6, 27] as we see in the subsequent chapters.

## 2.1 Maxwell Equations

Our starting point is the electric field due to a point charge $q_0$ which is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r^2}\,\hat{r}. \tag{2.1}$$

where $\varepsilon_0$ is the permitivity of free space with $\frac{1}{4\pi\varepsilon_0} = 9\times 10^9$ SI units. By use of the Stoke's theorem the above equation can be written as

$$\nabla E = \frac{\rho}{\varepsilon_0}. \tag{2.2}$$

where $\nabla E$ is the divergence of $E$ and $\rho$ charge density. Eq. (2.2) above constitutes our first Maxwell equation.

If we bring a test charge $q$, in presence of electric field $E$, it feels a force $F = qE$. In addition to electric fields we also have magnetic fields. They arise due to moving charges. When charge $q_0$ moves it produces a magnetic field. Consider a wire carrying current $i_0$, then the magnetic field at location $O$ due to infinitesimal current element is given by Biot-Savart law [1, 2] as shown in Fig. 2.1a

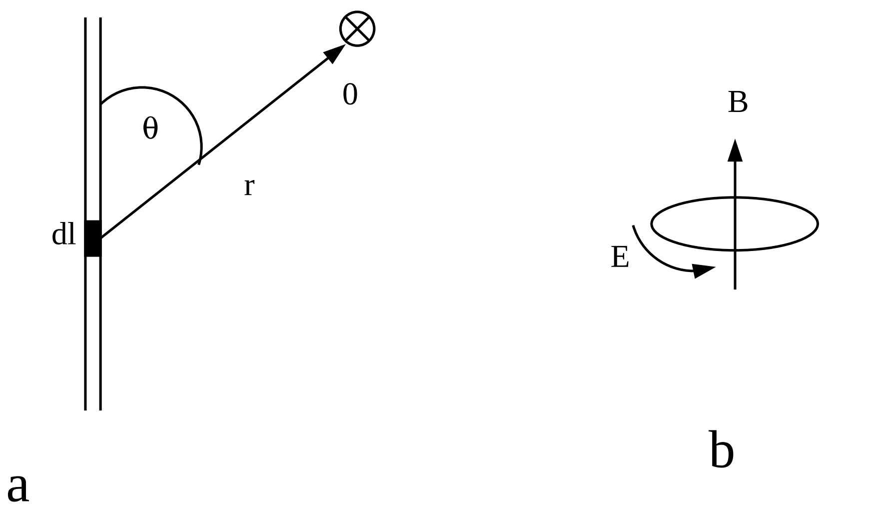



**Fig. 2.1** Fig. a shows magnetic field at point $O$ goint inside the page caused due to infinitesimal current element. Fig. b shown how change of magnetic flux through a loop induces tangential electric field.

$$B = \frac{\mu_0}{4\pi}\frac{i_0 dl \times r}{r^3} = \frac{\mu_0}{4\pi}\frac{i_0 dl \sin\theta}{r^2}\hat{\phi}. \tag{2.3}$$

where $\phi$ is the unit vector at $O$ going inside the page.

As can be seen

$$\nabla \cdot B = 0. \tag{2.4}$$

If we integrate over the whole wire, we get the net field at location $O$ as

$$B = \frac{\mu_0}{2\pi}\frac{i_0}{r}\hat{\phi}. \tag{2.5}$$

If we have current density $J$ at location $x$ and we come infinitely close to this then the density looks infinitely long and we can write $B$ as above in Eq. (2.5). If $A$ is the area of the current density we can write $B(2\pi r) = (\nabla \times B)A$ and $i_0 = JA$, which from Eq. (2.5) gives

$$\nabla \times B = \mu_0 J, \tag{2.6}$$

or

$$\nabla \times H = J. \tag{2.7}$$

Electric field $E$ induces a dipole per unit volume given by the displacement $D = \varepsilon E$. Then $\frac{\partial D}{\partial t}$ tantamounts to a current density and above Eq. 2.7 generalizes to

$$\nabla \times H = J + \frac{\partial D}{\partial t}. \tag{2.8}$$

Finally consider a loop of wire (of radius $r$ and area $A$) with magnetic field $B$ passing through it as shown in Fig. 2.1b. This results in flux $\Phi = BA$ through the loop. Change of flux induces a tangential electric field in the loop whose line integral is given by

$$E(2\pi r) = -\frac{\partial \phi}{\partial t}. \tag{2.9}$$

As we shrink the loop we get for infinitesimal loop $E(2\pi r) = (\nabla \times E)A$, which gives the famous Faraday's law

$$\nabla \times E = -\frac{\partial B}{\partial t}. \tag{2.10}$$

Putting equations 2.2, 2.4, 2.8 and 2.10 together we get the famous Maxwell qquations.

## 2.2 Wave Equation and Linear Systems

The Maxwell equations are

$$\nabla \cdot E = \frac{\rho(x,t)}{\varepsilon_0}, \tag{2.11}$$

$$\nabla \cdot B = 0, \tag{2.12}$$

$$\frac{\partial B}{\partial t} = -\nabla \times E, \tag{2.13}$$

$$\frac{\partial D}{\partial t} = \nabla \times H - J(x,t). \tag{2.14}$$

Suppose we start with $E$ and $B$ satisfying Eq. (2.11 and 2.12). Then evolution under Eq. (2.13 and 2.14) ensures they are staisfied for all $t$. All we have to take notice is the fact and divergence of curl is zero and the equation of continuity

$$\frac{\partial \rho(x,t)}{\partial t} + \nabla \cdot J = 0. \tag{2.15}$$

So we focus on Eq. (2.13 and 2.14). These equations constitute what we call a linear system. In a linear system, we call the state of the system $X$, which evolves according to the equation

$$\dot{X} = AX + bu(t), \tag{2.16}$$

where $X$ is say an $n \times 1$ vector and $A$ an $n \times n$ matrix and $b$ an $n \times 1$ input vector and $u(t)$ input. Without an input the system takes the form

$$\dot{X} = AX. \tag{2.17}$$

To solve for evolution of this system, we observe if the initial state $X(0)$ of the system is $e$, an eigenvector of $A$, with eigenvalue $\lambda$, then the solution is simply $X(t) = \exp(\lambda t)e$. Simply substute in Eq. (2.17). In general, we can decompose $X(0) = \sum_i \alpha_i(0)e_i$, where $e_i$ are eigenvectors of $A$ with eigenvalue $\lambda_i$. Then observe $X(t) = \sum_i \alpha_i(0)\exp(\lambda_i t)e_i$ is the solution as can be seen by substitution.

The solution is understood as simply writing the differential equation in terms of basis $e_i$ with $X(t) = \sum_i \alpha_i(t)e_i$, then the vector equation gets expressed as $n$ scalar equations

$$\dot{\alpha}_i = \lambda_i \alpha_i, \tag{2.18}$$

with solution $\alpha_i(t) = \exp(\lambda_i t)\alpha_i(0)$. In presence of input as in Eq. (2.16), we can express $b = \sum_i b_i e_i$ and equations take the form

$$\dot{\alpha}_i = \lambda_i \alpha_i + b_i u(t), \tag{2.19}$$

with solution

$$\alpha_i(t) = \exp(\lambda_i t)\alpha_i(0) + b_i \int_0^t \exp(\lambda_i(t-\tau))u(\tau)d\tau. \tag{2.20}$$

With $\alpha_i(0) = 0$, we simply have $\alpha_i(t) = b_i \int_0^t \exp(\lambda_i(t-\tau))u(\tau)d\tau$. Of particular interest is the case when $\lambda_i$ correspond to imaginary eigenvalues $j\omega_i$. Then $\alpha_i(t) = b_i \int_0^t \exp(j\omega_i(t-\tau))u(\tau)d\tau$. If the input $u(t) = \exp(j\omega t)$ and we choose $\omega = \omega_k$, then all modes average only the mode $k$ survives giving $\alpha_k(t) = b_k \exp(j\omega_k t)t$. We say we excite the system on resonance with mode $k$ which builds as shown. This is the main idea of this text. We identify modes and their frequencies and then drive the system at the frequency of a mode to build that mode. We say we pump energy into a mode. Lets now go to Maxwell equations in

$$\mu \frac{\partial H}{\partial t} = -\nabla \times E, \tag{2.21}$$

$$\varepsilon \frac{\partial E}{\partial t} = \nabla \times H - J(r,t), \tag{2.22}$$

where each spatial location $r$ indices a vector entry in linear system as in Eq. ( 2.16). The vector entries are $E$ and $B$ fields. The goal is to calculate $E(r,t)$ and $B(r,t)$ resulting from an current input $J(r,t)$. That is solve a linear system for an input $u(t)$. It means we have to find modes of the system and corresponding eigenvalues. That is we have to solve for $Ax = \lambda x$ in Eq. ( 2.17), i.e solve for

$$-\frac{1}{\mu}\nabla \times E = \lambda H, \tag{2.23}$$

$$\frac{1}{\varepsilon}\nabla \times H = \lambda E, \tag{2.24}$$

with $E = E_o \exp(-kx)\ \hat{y}$ and $H = H_o \exp(-kx)\ \hat{z}$ , where $\hat{x},\hat{y},\hat{z}$ are standard unit vectors, we get

$$\frac{k}{\mu}E_o = \lambda H_o, \tag{2.25}$$

$$\frac{k}{\varepsilon}H_o = \lambda E_o, \tag{2.26}$$

which gives $ck = \pm\lambda$, where $c^2 = \frac{1}{\mu\varepsilon}$ and $\frac{E_o}{H_o} = \pm\eta$, where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$. Of particular interest is when $\lambda = j\omega$ is imaginary, then $E = E_o \exp(-jkz)\ \hat{x}$ and $H = H_o \exp(-jkz)\ \hat{y}$ with $ck = \omega$. The modes evolve as $E = E_o \exp(-j(kx-\omega t))\ \hat{y}$, and $H = H_o \exp(-j(kx-\omega t))\ \hat{z}$. This constitues a travelling wave along $x$ direction, as shown in Fig. 2.2a. There is another eigenvector corresponding to $ck = -\omega$, which evolves as $E = E_o \exp(j(kx+\omega t))\ \hat{y}$, and $H = -H_o \exp(j(kx+\omega t))\ \hat{z}$ and constitues a travelling wave along $-x$ direction.

Let $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ be some unit vectors forming a right handed coordinate system such that $\hat{\mathbf{x}} = \frac{k}{|k|}$ with $k = (k_x,k_y,k_z)$ and $E = E_o \exp(-j\ k_x x + k_y y + k_z z)\ \hat{\mathbf{y}}$ also written as $E = E_o \exp(-jk\cdot r)\ \hat{\mathbf{y}}$ and $H = H_o \exp(-jk\cdot r)\ \hat{\mathbf{z}}$ are eigenvectors corresponding to eigenvalue $j\omega$ and $E = E_o \exp(-j(k\cdot r-\omega t))\ \hat{\mathbf{y}}$ and $H = H_o \exp(-j(k\cdot r-\omega t))\ \hat{\mathbf{z}}$ is a travelling wave along $k$ direction as shown in Fig. 2.2b and there is a corresponding wave $E = E_o \exp(j(k\cdot r+\omega t))\ \hat{\mathbf{x}}$ and $H = H_o \exp(j(k\cdot r+\omega t))\ \hat{\mathbf{y}}$ travelling along $-k$ direction.

What have we learnt? The eigefunctions corresponding to imaginary eigenvalues are oscillatory in space and their evolution constitute travelling waves. Eventually we want to drive Maxwell equations with current sources $J(r,t)$. These are antenna's. They can infact be simpified to $J(r)u(t)$. Think of $J(r)$ as $b$ in linear system. These sources are localized and can be expnaded into a Fourier basis but these Fourier basis are precisely the eigenfunctions of Maxwell's equation with imaginary eigenvalues. Then if we choose $u(t) = \exp(j\omega t)$ then as described above we will excite the modes with frequency $\omega$. The antenna puts energy into travelling wave modes whose frequency is same as the frequency of excitation of Antenna. These modes get energy and become much bigger than other modes and we can detect these modes very far from the antenna beacuse they are delocalized and hence we can communicate to someone very far because we pump energy into a mode that is shared simultaneously between the transmitter and receiver. In subsequent chapters, we present other elegant techniques for solving for electric and magnetic fields arising due to antenna excitation. In summary, we care about eigefunctions corre-

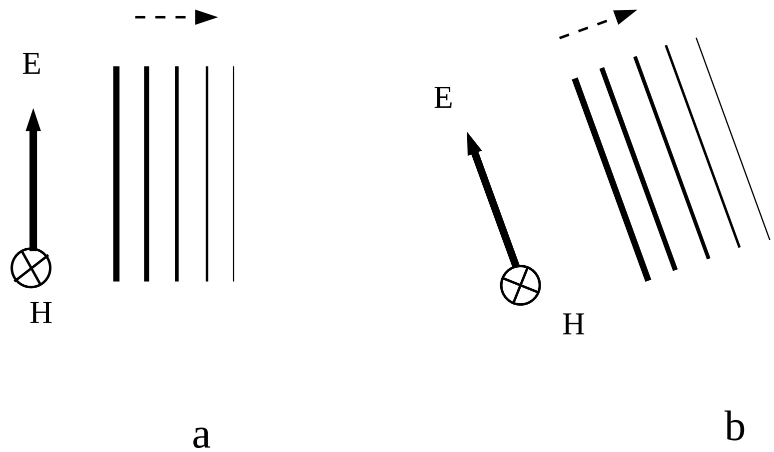



**Fig. 2.2** Fig. a and b shows EM waves propagating along $x$ and $\hat{k}$ direction respectively.

sponding to imaginary eigenvalues as their natural evolution constitutes travelling waves and we communicate by pumping energy in these modes.

## 2.3 Wave Equation and Media Boundary Conditions

We have talked about travelling waves $E = E_0 \exp(-j(k \cdot r - \omega t)\hat{x}$, and $H = H_0 \exp(-j(k \cdot r - \omega t)\hat{y}$, with $\frac{E_0}{H_0} = \eta = \sqrt{\frac{\mu}{\varepsilon}}$. This is transmission in free space. How about transmission across a media interface. Consider the scenario as depicted in figure 2.3. We have a transmitted wave in region I that arrives at media interface, which is say the $y-z$ plane. Region I is $x < 0$ and region 2 is $x > 0$. The permitivity if I is $\varepsilon_1$ and II is $\varepsilon_2$. The incident wave cannot continue unaltered into region II as $\eta$ for region $II$ is different. We will need to alter $E_0$ and $H_0$. We do this through a reflected wave and transmitted wave. We say there is a reflected wave that goes back in region I and transmitted wave that proceeds in region II. This is depicted in figure 2.3.

If we denote the spatial dependence of incident, refelcted and transmiited wave as $\exp(-jk_I \cdot r)$, $\exp(-jk_r \cdot r)$ and $\exp(-jk_t \cdot r)$ , then our first claim is that $k_I, k_r, k_t$ are in the same plain. W.L.O.G, assume $k_I$ in the $x,z$ plain. Then first note that tangential and normal components of $E$ and $H$ is same at the interface. This follows from figure 2.4. In figure 2.4a ,

$$E_2 - E_1 = (\nabla \times E)_\perp \Delta x, \tag{2.27}$$

where $\nabla \times E$ is the curl and $(\nabla \times E)_\perp$ its component perpendicular to the plane. The wave arrangement of incident, transmitted and reflected wave is an eigenfunction of the Maxwell equation satisfying

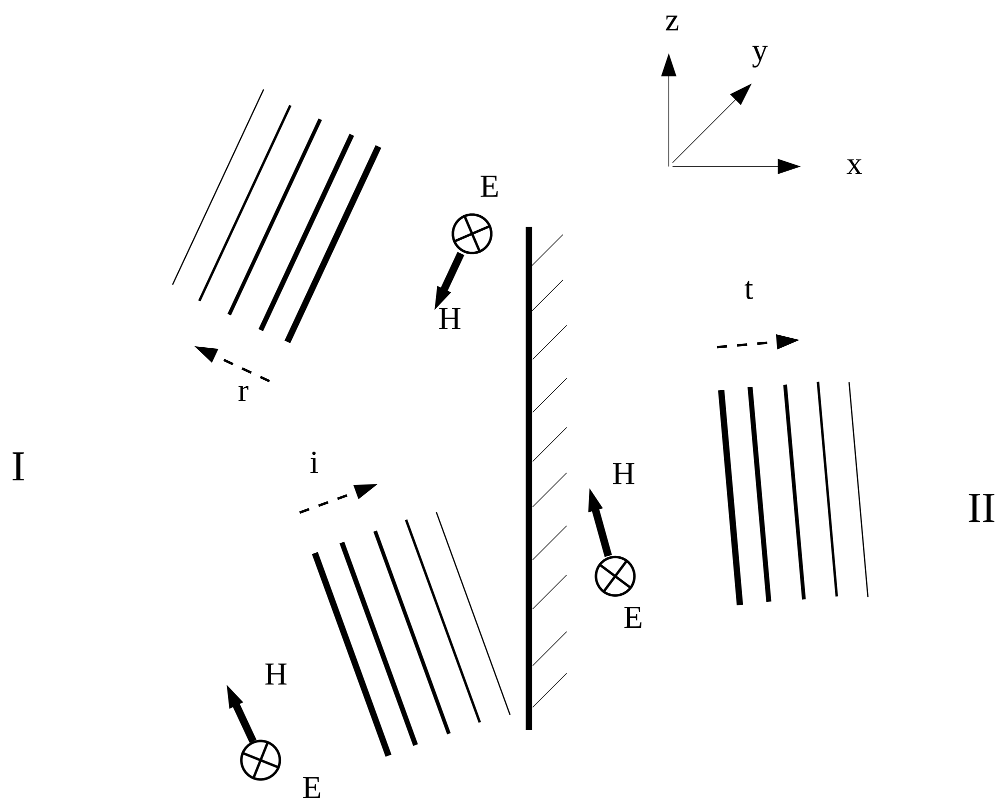



**Fig. 2.3** Figure shows incident, transmitted and reflected waves across a media interface between media *I* and *II*.

$$-\frac{1}{\mu}\nabla \times E = j\omega H \tag{2.28}$$

$$\frac{1}{\varepsilon}\nabla \times H = j\omega E \tag{2.29}$$

hence curl is finite and therefore in Eq. (2.27) as $\Delta x \to 0$ we get $E_2 = E_1$. Furthermore from Eq. (2.28 and 2.29), we find that $\nabla \cdot E = 0$ and $\nabla \cdot H = 0$. Therefore from figure 2.4b we get for $\Delta x \to$ we have

$$E_2 - E_1 = (\nabla \cdot E)\Delta z, \tag{2.30}$$

and since $\nabla \cdot E$ is finite we have $E_1 = E_2$. Same is true for $H$ and we get across the boundary the tangential and normal componets of $E$ and $H$ are the same. Lets compare $y$ component of $E$, we get

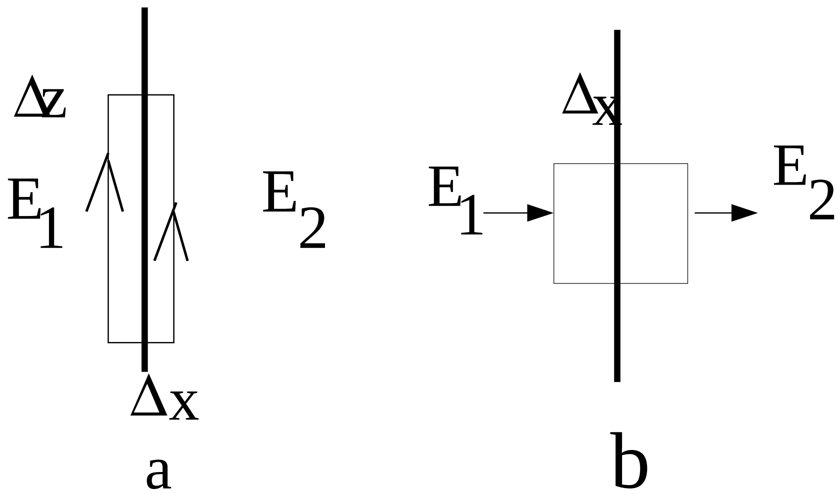



**Fig. 2.4** Fig. a. and b. shows tangential and normal components of the electric field across a media interface.

$$E_{io}\exp(-j(k_{ix}x+k_{iz}z))+E_{ro}\exp(-j(k_{rx}x+k_{ry}y+k_{rz}z))=E_{to}\exp(-j(k_{tx}x+k_{ty}y+k_{tz}z)). \tag{2.31}$$

Lets only vary $y$ to get

$$E_{io}+E_{ro}\exp(-jk_{ry}y)=E_{to}\exp(-jk_{ty}y). \tag{2.32}$$

Differentiaiting we get $k_{ty}=k_{ry}$, which further implies $k_{ty}=k_{ry}=0$. Which means transmitted and refelcted wave is in same plane. Then writing Eq. (2.31), we get from fig. 2.5.

$$E_{io}\exp(-jk_i(\sin\theta_i z+\cos\theta_i x))+E_{ro}\exp(-jk_i(\sin\theta_r z-\cos\theta_r x))=E_{to}\exp(-jk_t(\sin\theta_t z+\cos\theta_t x)). \tag{2.33}$$

Again only vary $z$ to conclude $\sin\theta_i=\sin\theta_r$ and $\frac{\sin\theta t}{\sin\theta i}=\frac{k_i}{k_t}=\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}$. That is angle of incidence is same as angle of relflection. The ratio of sine of transmitted and incident angle obeys what is knows as snell's law of refraction. The transmitted wave bends inwards as it moves into a dense (larger $\varepsilon$) media.

Now equating tangential components of $E$ and $H$ at the origin gives

$$E_{io}+E_{ro} = E_{to}, \tag{2.34}$$

$$(E_{io}-E_{ro})\frac{cos\theta_i}{\eta_1} = E_{to}\frac{\cos\theta_t}{\eta_2}, \tag{2.35}$$

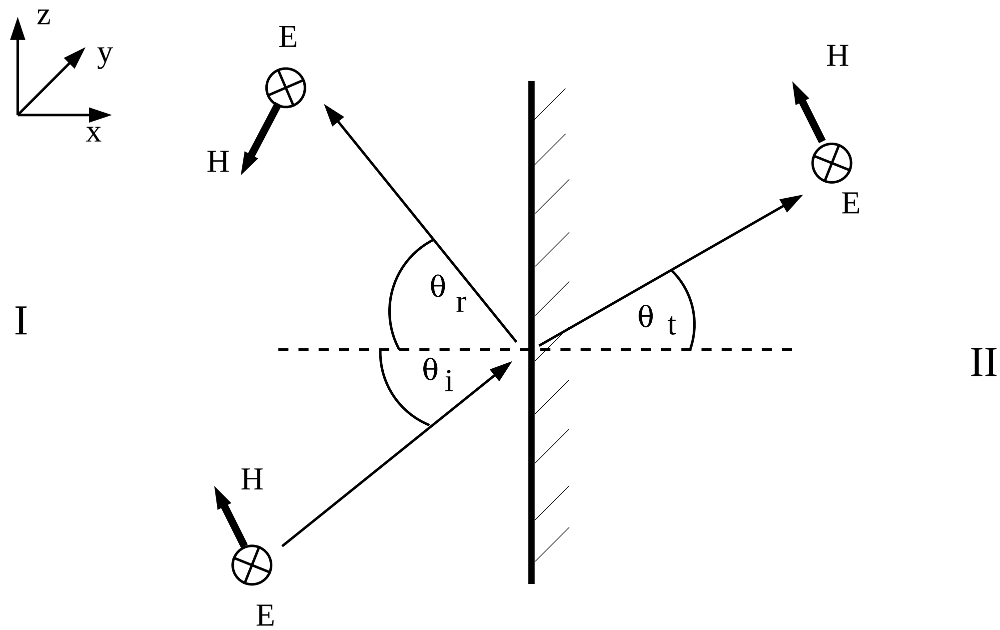



**Fig. 2.5** Fig. shows incident, transmitted and reflected wave at the media interface of medium *I* and *II*. E field is parallel to interface

which gives

$$(E_{io} + E_{ro}) = E_{to}, \tag{2.36}$$

$$(E_{io} - E_{ro}) = \frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i} E_{to}, \tag{2.37}$$

with $a = \frac{\eta_1}{\eta_2}$, and $b = \frac{\cos\theta_t}{\cos\theta_i}$, we have $E_{to} = \frac{2E_{io}}{1+ab}$ and $E_{ro} = \frac{1-ab}{1+ab}E_{io}$, where $b$ is calculated from $\frac{\sin\theta_t}{\sin\theta_i} = a^{-1}$.

For normal incidence, with $\theta_i = \theta_t = 0$, we get $b = 1$ and $E_{ro} = \frac{1-a}{1+a}E_{io}$.

We discussed the case when incident wave has $E$ tangential to interface while $H$ is incident at an angle. The second scenario is when incident wave has $H$ tangential to interface while $E$ is incident at an angle. This is as shown in fig. 2.6.

For this case we get

$$(E_{io} + E_{ro}) = \frac{\eta_1}{\eta_2} E_{to}, \tag{2.38}$$

$$(E_{io} - E_{ro})\cos\theta_i = E_{to}\cos\theta_t, \tag{2.39}$$

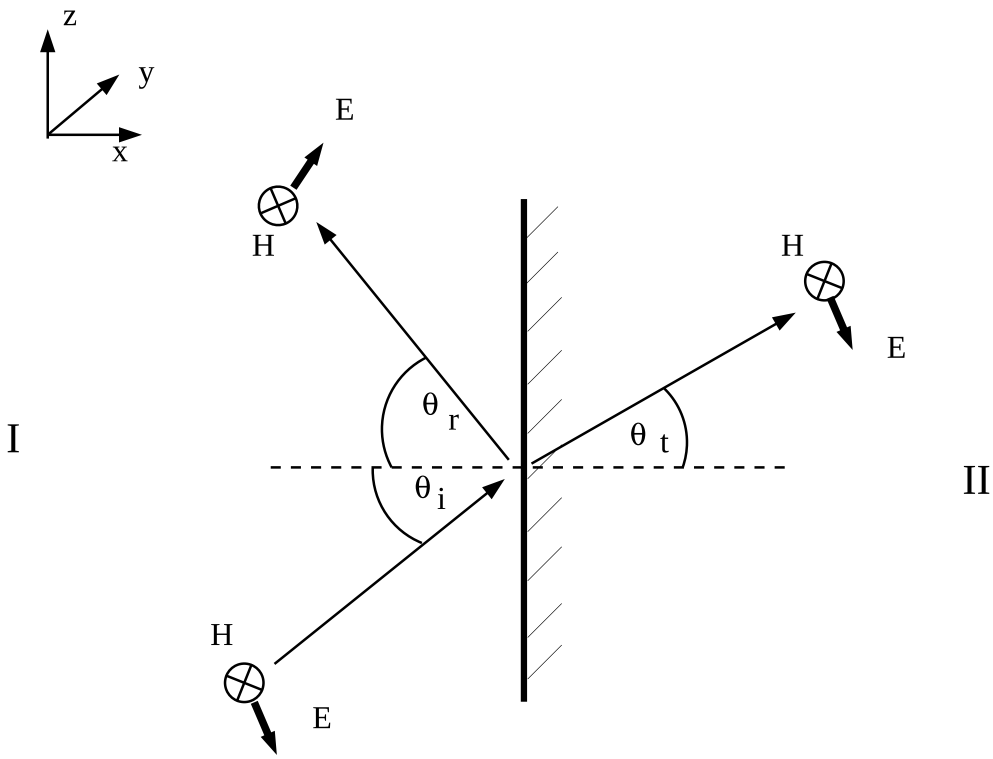



**Fig. 2.6** Fig. shows incident, transmitted and reflected wave at the media interface of medium *I* and *II*. H field is parallel to interface

which gives $E_{to} = \frac{2E_{io}}{a+b}$ and $E_{ro} = \frac{(a-b)E_{i0}}{a+b}$.

For normal incidence with $\theta_i = \theta_t = 0$, we get $b = 1$ and $E_{ro} = \frac{a-1}{a+1}E_{io}$.

## 2.4 Conductor Interfaces

$$-\nabla \times E = j\omega\mu\, H, \tag{2.40}$$

$$\nabla \times H = (j\omega\varepsilon + \sigma)E, \tag{2.41}$$

with $E = E_o \exp(-\gamma x)\ \hat{y}$ amd $H = H_o \exp(-\gamma x)\ \hat{z}$ , where $\hat{x}, \hat{y}, \hat{z}$ are standard unit vectors, we get

$$\gamma E_o = j\omega\mu\, H_o, \tag{2.42}$$

$$\gamma H_o = (j\omega\varepsilon + \sigma)E_o, \tag{2.43}$$

giving

$$\gamma = \sqrt{j\omega\mu(j\omega\varepsilon+\sigma)} = \alpha + j\beta \tag{2.44}$$

and $\frac{E_0}{H_0} = \eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon+\sigma}}$. The wave evolves then $E = E_o \exp(-(\alpha + j\beta)x + j\omega t)\ \hat{y}$ and $H = H_o \exp(-(\alpha + j\beta)x + j\omega t)\ \hat{z}$. This is travelling wave in $x$ direction that decays with $x$ with exponent $\alpha$.

For a typical metal $\sigma = 10^7$ SI units, $\varepsilon = \frac{10^{-10}}{4\pi}$ SI units and $\mu = 4\pi \times 10^{-7}$ SI units and for $\omega = 10^{11}$ SI units we get $\alpha \sim 10^{-6}$ SI units. Therefore within a distance of a micron, the wave attenutates to $e^{-1}$ of its value.

Now consider a media interface where region I is a dielectric with permitivity $\varepsilon_1$ and region II is a conductor with permitivity $\varepsilon_2$ and conductivity $\sigma$. For normal incidence, there is transmitted wave in the conductor that decays rapidly inside the conductor as seen above. For $E$ tangential we get from previous section, we get $E_{ro} = \frac{1-a}{1+a}E_{io}$. Note $a = \sqrt{\frac{\sigma+j\omega\varepsilon_2}{j\omega\varepsilon_1}}$. From numbers quoted above $|a| \gg 1$ and we get $E_{ro} =\sim E_{io}$ and $E_{to} \sim 0$. In limit $a \to \infty$, we say we have perfect conductor which gives $E_{to} = 0$ and $E_{r0} = -E_{io}$, i.e, from a perfect conductor we have perfect reflection. The total electric field goes to zero at the interface, i.e. $E_{io} + E_{ro} = 0$ and $E_{to} = 0$ and $H_{io} - H_{ro} = 2H_{io}$ and $H_{to} = 2H_{io}$.

At a rate $\alpha$ this $H_{to}$ decays to zero such that within a small width $\Delta x \sim \alpha^{-1}$ we drop from $2H_{io}$ to 0. But observe $\nabla \times H = (j\omega\varepsilon_2 + \sigma)E$ . This gives $J\Delta x = \sigma E\Delta x = 2H_{io}$. We call the quantity $J_s = J\Delta x$ as the surface current density at the interface, $J_S = J\Delta x = H$ where $H = 2H_{io}$, the magnetic field at the boundary.

Now consider the case when angle of incidence is not normal. Then we have two cases ,

**a)** As in Fig. 2.5 $E$ is tangential and we get $E_{io} = -E_{ro}$ and the normal componets of $H_{io}$ and $H_{ro}$ cancell and $H$ at interface is $2H_{io}\cos\theta$. The surface current density is then $J_S = 2H_{io}\cos\theta$.

**b)** As in Fig. 2.6 $E$ is tangential, and we get $E_{io} = E_{ro}$. The tangential componets of $E_{io}$ and $E_{ro}$ cancell while $H = 2H_{io}$. The surface current density is then $J_S = 2H_{io}$ along $y$ direction. Furthermore normal componets of $E_{io}$ and $E_{ro}$ add to give us a net normal field of $2E_{io}\sin\theta$ at interface. But there is no normal field inside the conductor. If we allude to fig. 2.4b, this means there is surface charge density of magnitude $\rho_s = 2E_{io}\sin theta\varepsilon_2$ at the interface.

In summary at conductor interface, the tangential component of E field is zero. The normal component of H-field is zero. The tangential component of $H$ field leads to a surface current sensity and normal component of $E$ field leads to a surface charge density. In both cases the fields inside the conductor decay rapidly. It is worthwhile writing the fields after reflection in the region 1. They arise from interference of incident and reflected wave and take the form

**a)**

$$E = -j2E_{io}\exp(-jkz\ \sin\theta)\sin(kx\ \cos\theta)\ \hat{y}, \tag{2.45}$$

$$\begin{aligned} H = {} & 2H_{io}\exp(-jkz\ \sin\theta)(\cos\theta\cos(kx\ \cos\theta)\ \hat{z} \\ & + j\sin\theta\sin(kx\ \cos\theta)\ \hat{x}). \end{aligned} \tag{2.46}$$

**b)**

$$\begin{aligned} E = {} & 2E_{io}\exp(-jkz\ \sin\theta)(\sin\theta\cos(kx\ \cos\theta)\ \hat{x} \\ & + j\cos\theta\sin(kx\ \cos\theta)\ \hat{z}), \end{aligned} \tag{2.47}$$

$$H = 2H_{io}\exp(-jkz\ \sin\theta)\cos(kx\ \cos\theta)\ \hat{y}. \tag{2.48}$$

(2.49)

## 2.5 Conductor Losses

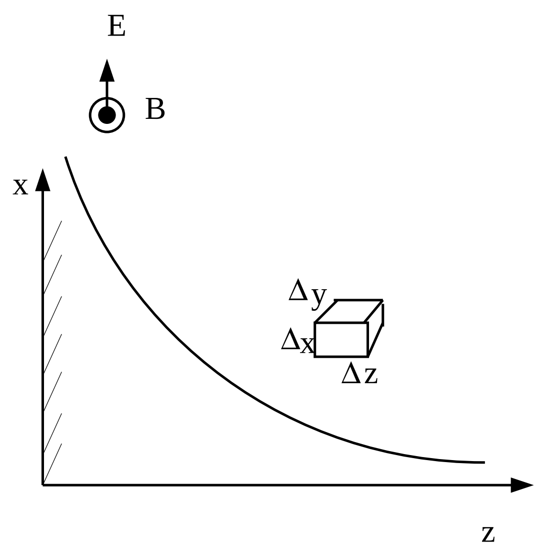



**Fig. 2.7** Fig. shows how electric field decays exponentially inside a conductor

Recall from eq. 2.44 inside a conductor, when $\sigma \gg \omega\varepsilon$

$$\gamma = \sqrt{j\omega\mu(j\omega\varepsilon + \sigma)} = \alpha + j\beta = \sqrt{\frac{\omega\mu}{\sigma}} + j\sqrt{\frac{\omega\mu}{\sigma}} \tag{2.50}$$

Inside the conductor, electric field is $E(z) = E_0 \exp(-\gamma z)$ as shown in Fig. 2.7. Then current density $J(z) = \sigma E(z)$ and integrating it for $z$ from 0 to $\infty$ , we get $J_s = \int J(z)dz = \frac{J(0)}{\gamma} = \frac{\sigma E(0)}{\gamma} = H(0)$. The RMS power loss is a same cubical as shown in Fig. 2.7 is

$$P(z) = \frac{1}{2}|I|^2 R = \frac{1}{2}\underbrace{|J(z)|^2(\Delta z\Delta y)^2}_{|I|^2}\underbrace{(\frac{\rho\Delta x}{\Delta z\Delta y})}_{R}. \tag{2.51}$$

Then $\int P(z)dz = \frac{1}{2}J_s^2 R_s$, where $R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}}$ is termed surface resistance. Alternatively, $J_s = J\Delta z$ and RMS power is

$$P = \frac{1}{2}|I|^2 R = \frac{1}{2}\underbrace{(J\Delta z\Delta y)^2}_{|I|^2}\underbrace{(\frac{\rho\Delta x}{\Delta z\Delta y})}_{R} = \frac{1}{2}\frac{J_s^2\Delta x\Delta y}{\Delta z\sigma}. \tag{2.52}$$

But note $\frac{1}{\sigma\Delta z} = \sigma^{-1}\alpha = \sqrt{\frac{\omega\mu}{2\sigma}} = R_s$. As an example if $H$ at surface of conductor is 2 SI units and $\omega$ is $2\pi \times 10^9$, then for $\sigma = 10^7$ SI units, we get $P = .56W/m^2$.

## 2.6 Problems

1. What is electric potential at 1 $A^\circ$ distance from an electron.

2. What is magnetic field in center a circular loop of 1 cm radius carrying 1 ampere of current.

3. If relative permitivity of a medium is 1.4 , what is the the transmitted angle (with respect to interface), if incident angle of EM wave is $30^\circ$. ($H$ points inside the plance of reflection)

4. In above find the ratio of amplitude of reflected and incident wave.

5. How far will a 1 GHz EM wave travel in a conductor of conductivity $10^7$ SI units.

# Chapter 3
# Waveguides and Resonators

We have to calculate the eigenfunctions of Maxwell equation for eigenvalue $\lambda = j\omega$. In last chapter, we did it for free space and media interface. In this chapter, we look at free space bounded by conductors. This geometry forms what are called, waveguides, cavities and resonators.

## 3.1 Parallel Plane Waveguides

Consider two parallel conductor planes as shown in Fig. 3.1 . Fig. **a** shows the EM mode confined between parallel conductors with $E$ parallel to the conductor plane. Fig. **b** shows the EM mode confined between parallel conductors with $H$ parallel to the conductor plane. From the last chapter, after reflection from the interface $I$, the $E$ and $H$ field in two cases take the form.

**a)**

$$E = -j2E_{io}\exp(-jkz\ \sin\theta)\sin(kx\ \cos\theta)\ \hat{y}, \tag{3.1}$$

$$\begin{aligned} H = {} & 2H_{io}\exp(-jkz\ \sin\theta)(\cos\theta\cos(kx\ \cos\theta)\ \hat{z} \\ & +\ j\sin\theta\sin(kx\ \cos\theta)\ \hat{x}). \end{aligned} \tag{3.2}$$

**b)**

$$\begin{aligned} E = {} & 2E_{io}\exp(-jkz\ \sin\theta)(\sin\theta\cos(kx\ \cos\theta)\ \hat{x} \\ & +\ j\cos\theta\sin(kx\ \cos\theta)\ \hat{z}), \end{aligned} \tag{3.3}$$

$$H = 2H_{io}\exp(-jkz\ \sin\theta)\cos(kx\ \cos\theta)\ \hat{y}. \tag{3.4}$$

$$\tag{3.5}$$

From 3.1 a we have, for tangential $E$ and normal $H$ at the interface $II$ to be zero,

$$\cos\theta d = \frac{m\pi}{kd}, \tag{3.6}$$

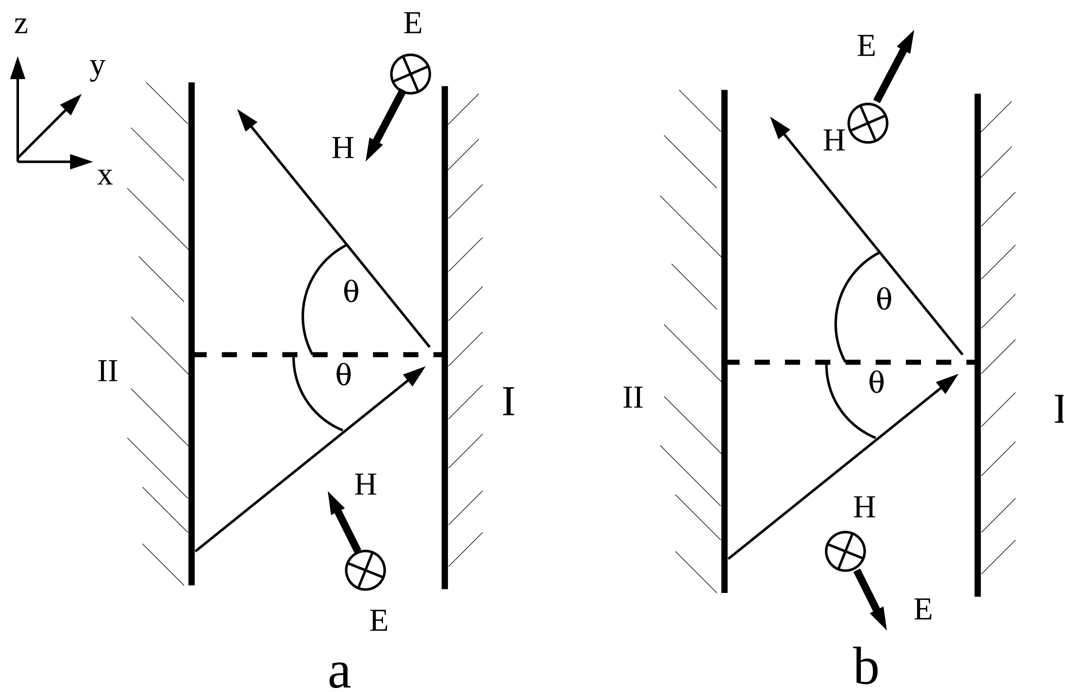



**Fig. 3.1** Fig. a shows EM mode confined between parallel conductors with $E$ parallel to the conductor plane. Fig. b shows EM mode confined between parallel conductors with $H$ parallel to the conductor plane.

and $\beta = k \sin\theta$ satisfies, we have $c^2(\beta^2 + (\frac{m\pi}{d})^2) = \omega^2$ from which we get

$$\beta = \sqrt{(\frac{\omega}{c})^2 - (\frac{m\pi}{d})^2}. \tag{3.7}$$

We have a cut-off frequency when $\beta = 0$. Integer $m$ defines modes and determines $\theta$. Frequency $\omega$ defines how high $m$ can be before we reach cutoff.

From 3.1 b we have, for tangential $E$ and normal $H$ at the interface $II$ to be zero,

$$k \cos\theta d = \frac{m\pi}{kd}, \tag{3.8}$$

and $\beta = k \sin\theta$ satisfies, we have $c^2(\beta^2 + (\frac{m\pi}{d})^2) = \omega^2$ from which we get

$$\beta = \sqrt{(\frac{\omega}{c})^2 - (\frac{m\pi}{d})^2}. \tag{3.9}$$

We have a cut-off frequency when $\beta = 0$. Integer $m$ defines modes and determine $\theta$. Frequency $\omega$ defines how high $m$ can be before we reach cutoff.

## 3.2 Rectangular Waveguides

We now consider a geometry of a hollow rectangular pipe as shown in 3.2. It is called a rectangular waveguide. The $x-y$ directions are called transverse, while $z$ direction longitudinal. When electric field is all transverse, we call it $TE$ mode and when magnetic field is all transverse we call it $TM$ mode.

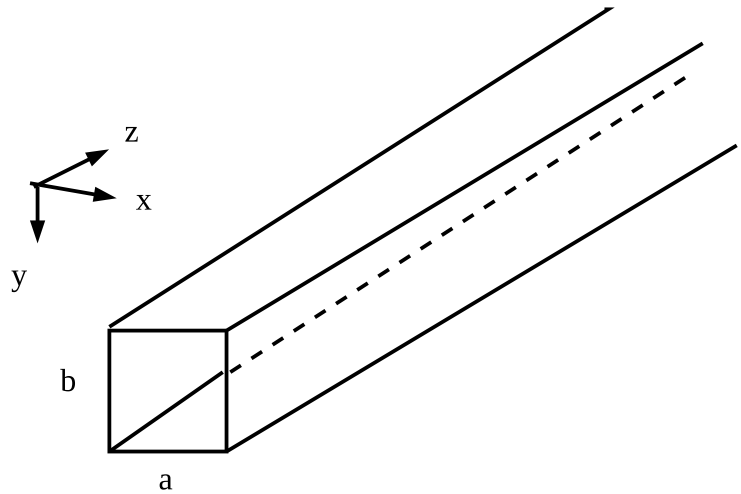



**Fig. 3.2** Fig. shows a hollow rectangular pipe called a rectangular waveguide.

### 3.2.1 TE mode

We propose a TE mode propagating along $z$ direction, where we have $H_z$ and $E_x, E_y, H_x, H_y$, all with a $\exp(-j\beta z)$ dependence, such that using $-\nabla E = j\omega\mu H$ and $\nabla H = j\omega\varepsilon E$, we get

$$j\omega\mu H_x = \frac{\partial E_y}{\partial z}, \tag{3.10}$$

$$j\omega\mu H_y = -\frac{\partial E_x}{\partial z}, \tag{3.11}$$

$$j\omega\varepsilon E_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \tag{3.12}$$

$$j\omega\varepsilon E_y = -\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z}. \tag{3.13}$$

Substituting Eq. 3.10 and 3.11 in Eq. 3.12 and 3.13, we get

$$j\omega(\varepsilon - \frac{\beta^2}{\omega^2\mu})E_x = \frac{\partial H_z}{\partial y}, \tag{3.14}$$

$$j\omega(\varepsilon - \frac{\beta^2}{\omega^2\mu})E_y = -\frac{\partial H_z}{\partial x}. \tag{3.15}$$

Substituting above in Eq. 3.10 and 3.11, we get

$$\frac{\mu}{\beta} j\omega^2(\varepsilon - \frac{\beta^2}{\omega^2\mu})H_x = \frac{\partial H_z}{\partial x}, \tag{3.16}$$

$$\frac{\mu}{\beta} j\omega^2(\varepsilon - \frac{\beta^2}{\omega^2\mu})H_y = \frac{\partial H_z}{\partial y}. \tag{3.17}$$

Observe transverse fields $E_x, E_y, H_x, H_y$ all depend on longitudinal component $H_z$. Observe

$$H_z \propto \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}), \tag{3.18}$$

$$H_x \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}), \tag{3.19}$$

$$H_y \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}), \tag{3.20}$$

$$E_x \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}), \tag{3.21}$$

$$E_y \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}). \tag{3.22}$$

$$\tag{3.23}$$

That's all, at boundary tangential $E$ and normal $H$ are zero all boundary conditions are satisfied. The mode is written as $TE_{mn}$.

From $-\nabla E = j\omega\mu H$ and $\nabla H = j\omega\varepsilon E$, we get Using the fact that $j\omega\mu H_z = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z}$, from which we get $\beta = \sqrt{(\frac{\omega}{c})^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}$. Therefore mode $m, n$ exists only if $\omega \geq c\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$.

### 3.2.2 TM mode

We propose a TM mode propagating along $z$ direction, where we have $E_z$ and $E_x, E_y, H_x, H_y$, all with a $\exp(-j\beta z)$ dependence, such that using $-\nabla E = j\omega\mu H$ and $\nabla H = j\omega\varepsilon E$, we get

$$j\omega\mu H_x = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}, \tag{3.24}$$

$$j\omega\mu H_y = -\frac{\partial E_x}{\partial z} + \frac{\partial E_z}{\partial x}, \tag{3.25}$$

$$j\omega\varepsilon E_x = -\frac{\partial H_y}{\partial z}, \tag{3.26}$$

$$j\omega\varepsilon E_y = \frac{\partial H_x}{\partial z}. \tag{3.27}$$

Substituting Eq. 3.26 and 3.27 in Eq. 3.24 and 3.25, we get

$$j\omega(\mu - \frac{\beta^2}{\omega^2\varepsilon})H_x = -\frac{\partial E_z}{\partial y}, \tag{3.28}$$

$$j\omega(\mu - \frac{\beta^2}{\omega^2\varepsilon})H_y = +\frac{\partial E_z}{\partial x}. \tag{3.29}$$

Substituting above in Eq. 3.26 and 3.27, we get

$$\frac{\varepsilon}{\beta} j\omega^2(\mu - \frac{\beta^2}{\omega^2\varepsilon})E_x = \frac{\partial E_z}{\partial x}, \tag{3.30}$$

$$\frac{\varepsilon}{\beta} j\omega^2(\mu - \frac{\beta^2}{\omega^2\varepsilon})E_y = \frac{\partial E_z}{\partial y}. \tag{3.31}$$

Observe that transverse fields $E_x, E_y, H_x, H_y$ all depend on longitudinal component $E_z$. Observe if

$$H_z \propto \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}), \tag{3.32}$$

$$H_x \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}), \tag{3.33}$$

$$H_y \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}), \tag{3.34}$$

$$E_x \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}), \tag{3.35}$$

$$E_y \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}). \tag{3.36}$$

$$\tag{3.37}$$

That's all, at the boundary tangential $E$ and normal $H$ are zero, all boundary conditions are satisfied. The mode is written as $TM_{mn}$. Again we get $\beta = \sqrt{(\frac{\omega}{c})^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}$. Therefore mode $m, n$ exists only if $\omega \geq c\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$.

## 3.3 Rectangular Cavity

We considered a rectangular waveguide, which is open and indefinite in $z$ direction. If we close its ends and length $z$ length be $z$. Then the $\exp(-j\beta z)$ wave gets reflected an the end and we have two waves $\propto \exp(-j\beta z)$ and $\propto -\exp(-j\beta z)$, adding the two we get something $\propto \sin(\beta z)$. For electric fields to go to zero at the $z$ ends, we have $\beta = \frac{p\pi}{c}$. The resulting mode is called $TE_{mnp}$ or $TM_{mnp}$ depending on if electric or magnetic field is transverse, i.e, if we have $H_z$ or $E_z$. This gives mode frequency

$$\omega = \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 + (\frac{p\pi}{c})^2}. \tag{3.38}$$

Writing explictly the modes for $TE_{mnp}$

$$H_z \propto \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a})\sin(\frac{p\pi z}{c}), \tag{3.39}$$
$$H_x \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a})\cos(\frac{p\pi z}{c}), \tag{3.40}$$
$$H_y \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a})\cos(\frac{p\pi z}{c}), \tag{3.41}$$
$$E_x \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a})\sin(\frac{p\pi z}{c}), \tag{3.42}$$
$$E_y \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a})\sin(\frac{p\pi z}{c}). \tag{3.43}$$

Writing explicitly the modes for $TM_{mnp}$,

$$E_z \propto \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a})\cos(\frac{p\pi z}{c}), \tag{3.44}$$
$$H_x \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a})\cos(\frac{p\pi z}{c}), \tag{3.45}$$
$$H_y \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a})\cos(\frac{p\pi z}{c}), \tag{3.46}$$
$$E_x \propto \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a})\sin(\frac{p\pi z}{c}), \tag{3.47}$$
$$E_y \propto \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a})\sin(\frac{p\pi z}{c}). \tag{3.48}$$

## 3.4 Cylindrical Cavity

Consider a cylindrical cavity of radius $a$ and height $b$, where $b$ direction is longitudinal $z$ direction.

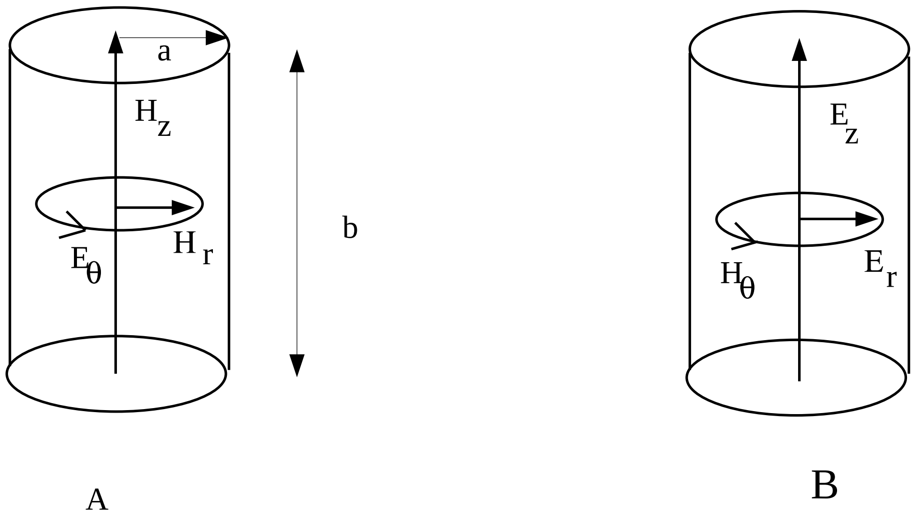



**Fig. 3.3** Fig. shows a cylindrical cavity of radius *a* and height *b*.

### 3.4.1 TE Modes

As shown in Fig. 3.3A we consider $TE$ mode with $H_z$. In cylindrical coordinates (r, $\theta$, z), we have

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \tag{3.49}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \tag{3.50}$$

For TE modes from Eq. (3.14 and 3.15), we can write

$$j\omega(\varepsilon - \frac{\beta^2}{\mu\omega^2})E_r = \frac{1}{r}\frac{\partial H_z}{\partial \theta}, \tag{3.51}$$

$$j\omega(\varepsilon - \frac{\beta^2}{\mu\omega^2})E_\theta = -\frac{\partial H_z}{\partial r}. \tag{3.52}$$

from Eq. (3.16 and 3.17), we get

$$\frac{\mu}{\beta} j\omega^2(\varepsilon - \frac{\beta^2}{\omega^2\mu})H_r = \frac{\partial H_z}{\partial r}, \tag{3.53}$$

$$\frac{\mu}{\beta} j\omega^2(\varepsilon - \frac{\beta^2}{\omega^2\mu})H_\theta = \frac{1}{r}\frac{\partial H_z}{\partial \theta}. \tag{3.54}$$

By choosing $H_z$ as function of $r$ we get $E_\theta$ and $H_r$ that vanishes at the boundary.

Using the fact that

$$j\omega\mu H_z = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} \propto \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2},$$

which gives

From

$$\underbrace{(\frac{\omega^2}{c^2} - \beta^2)}_{l^2} H_z = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) H_z.$$

The solution is $J_0(lr)$. If $Z_1, Z_2, Z_n$ are zeros of the $J_0'(x)$ (other than trivial zeros at 0)Then modes are defined as $la = Z_i$ . This ensures that $E_\theta$ and $H_r$ that vanishes at the boundary.

This defines $\beta^2 = \sqrt{(\frac{\omega^2}{c^2} - (\frac{Z_i}{a})^2}$. We have a wave along $z$ with dependence $\exp(-j\beta x)$. Superimposing on it a reflected wave $\exp(-j\beta x)$ we get electric field has $z$ dependence $\sin(\beta z)$ and magnetic field $\cos(\beta z)$. By choosing $\beta = \frac{p\pi}{b}$, we get a standing wave mode indexed by $p$ and $Z_i$, with

$$\omega = c\sqrt{(\frac{p\pi}{b})^2 + (\frac{Z_i}{a})^2}, \tag{3.55}$$

The resulting fields are shown in Fig. 3.3A.

We choose $H_z$ as function of $r$ only. This is not the only solution. We can choose

$$H_z = f(r)\cos n\theta. \tag{3.56}$$

Then we get

$$-\underbrace{(\frac{\omega^2}{c^2} - \beta^2)}_{l^2} H_z = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}) H_z.$$

$$(-l^2 + \frac{n^2}{r^2}) f(r) = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) f(r).$$

The solution is $J_n(lr)$. If $Z_1, Z_2, \ldots, Z_n$ are zeros of the $J_n'(x)$ (other than trivial zeros at 0)Then modes are defined as $la = Z_i$ . This ensures that $E_\theta$ and $H_r$ that vanishes at the boundary.

This defines $\beta^2 = \sqrt{(\frac{\omega^2}{c^2} - (\frac{Z_i}{a})^2}$. We have a wave along $z$ with dependence $\exp(-j\beta z)$. Superimposing on it a reflected wave $\exp(-j\beta z)$ we get electric field has $z$ dependence $\sin(\beta z)$ and radial magnetic field $\cos(\beta z)$. By choosing $\beta = \frac{p\pi}{b}$, we get a standing wave mode indexed by $p$ and $Z_i$, with

$$\omega = c\sqrt{(\frac{p\pi}{b})^2 + (\frac{Z_i}{a})^2}. \tag{3.57}$$

The resulting fields are shown in Fig. 3.3A.

### 3.4.2 TM modes

As shown in Fig. 3.3B we consider $TM$ mode with $E_z$. For TM modes from Eq. (3.30 and 3.31), we can write

$$j\omega(\mu - \frac{\beta^2}{\varepsilon\omega^2})H_r = \frac{1}{r}\frac{\partial E_z}{\partial \theta}, \tag{3.58}$$

$$j\omega(\mu - \frac{\beta^2}{\varepsilon\omega^2})H_\theta = \frac{\partial E_z}{\partial r}. \tag{3.59}$$

from Eq. (3.28 and 3.29), we get

$$\frac{\varepsilon}{\beta} j\omega^2(\mu - \frac{\beta^2}{\omega^2\varepsilon})E_r = \frac{\partial E_z}{\partial r}, \tag{3.60}$$

$$\frac{\varepsilon}{\beta} j\omega^2(\mu - \frac{\beta^2}{\omega^2\varepsilon})E_\theta = \frac{1}{r}\frac{\partial E_z}{\partial \theta}. \tag{3.61}$$

By choosing $E_z$ as function of $r$ we get $E_\theta$ and $H_r$ that vanishes at the boundary.
Using the fact that

$$j\omega\varepsilon E_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \propto \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2},$$

which gives

From

$$\underbrace{(\frac{\omega^2}{c^2} - \beta^2)}_{l^2} E_z = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})H_z.$$

The solution is $J_0(lr)$. If $Y_1, Y_2, Y_n$ are zeros of the $J_0(x)$. Then modes are defined as $la = Y_i$ . This ensures that $E_z$ vanishes at the boundary.

This defines $\beta^2 = \sqrt{(\frac{\omega^2}{c^2} - (\frac{Y_i}{a})^2}$. We have a wave along $z$ with dependence $\exp(-j\beta x)$. Superimposing on it a reflected wave $\exp(-j\beta x)$ we get radial electric field has $z$ dependence $\sin(\beta z)$, $z$ electric field has dependence $\cos(\beta z)$ and magnetic field $\cos(\beta z)$. By choosing $\beta = \frac{p\pi}{b}$, we get a standing wave mode indexed by $p$ and $Y_i$, with

$$\omega = c\sqrt{(\frac{p\pi}{b})^2 + (\frac{Y_i}{a})^2}. \tag{3.62}$$

The resulting fields are shown in Fig. 3.3B. We choose $E_z$ as function of $r$ only. This is not the only solution. We can choose

We choose

$$E_z = f(r)\cos n\theta. \tag{3.63}$$

$$j\omega\varepsilon E_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \propto \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2},$$

which gives

From

$$\underbrace{(\frac{\omega^2}{c^2} - \beta^2)}_{l^2} E_z = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}) E_z.$$

$$(-l^2 + \frac{n^2}{r^2}) f(r) = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}) f(r).$$

The solution in the annulus is $J_n(lr)$. If $Y_1, Y_2, \ldots, Y_n$ are zeros of the $J_n(x)$ (other than trivial zeros at 0)Then modes are defined as $la = Y_i$. This ensures that $E_\theta$ and $H_r$ that vanishes at the boundary.

This defines $\beta^2 = \sqrt{(\frac{\omega^2}{c^2} - (\frac{Y_i}{a})^2}$. We have a wave along $z$ with dependnece $\exp(-j\beta z)$. Superimposing on it a reflected wave $\exp(-j\beta z)$ we get transverse electric field has $z$ dependence $\sin(\beta z)$ and transverse magnetic field $\cos(\beta z)$. By choosing $\beta = \frac{p\pi}{b}$, we get a standing wave mode indexed by $p$ and $Y_i$, with

$$\omega = c\sqrt{(\frac{p\pi}{b})^2 + (\frac{Y_i}{a})^2}. \tag{3.64}$$

## 3.5 Quality Factor of Resonator

We talked about cavities and their modes, say mode $\phi$ with eigenvalue $\lambda = j\omega$. The mode will just evolve as $\exp(j\omega t)\phi(r)$. It means the mode will live in the cavity for ever. That is not true. The mode corresponds to $J$ on the walls which means resistive losses. This means, we really have $\lambda = -\alpha + j\omega$, where $\lambda$ is a decay constant. How do we calculate $\alpha$. We can calculate how much power $P$ is dissipated on the walls using the formula $P$ per unit area as $\frac{1}{2}J_s^2 R_s$ and we know the energy $E$ of the mode decreases by $\Delta E = P\Delta t$ in unit time $\Delta t$. If $A$ is amplitude of mode then in time $\Delta t$ we have

$$\frac{\Delta A}{A} = \alpha \Delta t. \tag{3.65}$$

Energy $E \propto A^2$ and change of energy $\Delta E = P\Delta t \propto 2A\Delta A$ ,which gives $2\alpha = \frac{P}{E}$. We define the quality factor $Q$ of the cavity as

$$Q = \frac{\alpha}{\omega}. \tag{3.66}$$

## 3.6 Problems

1. For a Waveguide of cross section 1 $cm^2$, find the $\beta$ for lowest TM mode at 30 GHz frequency.

2. For a Waveguide of cross section 1 $cm^2$, find the $\beta$ for lowest TE mode at 30 GHz frequency.

3. Find the resonance frequency of $TE_{112}$ mode in a rectangular cavity 1 $cm^3$

4. Find the resonance frequency of $TM_{111}$ mode in a rectangular cavity 1 $cm^3$

5. In a cyclindrical cavity of 1 cm radius and height, find the lowest resoant frequency of TE and TM modes.

# Chapter 4
# Antennas

## 4.1 Vector Potential

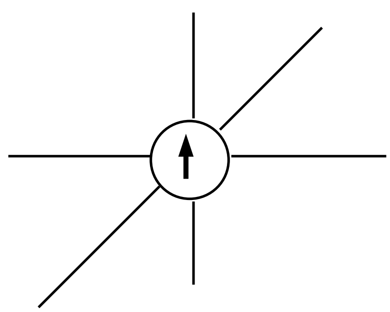


**Fig. 4.1** Fig. shows a point current source called Hertz dipole.

In this chapter, we study Maxwell equations, when they are driven by a current source also called an antenna [16, 19, 20, 24, **?**]. Recall in the presence of current source, the Maxwell equations take the form

$$\mu \frac{\partial H}{\partial t} = -\nabla \times E, \tag{4.1}$$

$$\varepsilon \frac{\partial E}{\partial t} = \nabla \times H - J(r,t), \tag{4.2}$$

where each spatial location $r$, indices a vector entry in the linear system as in Eq. ( 2.16). The vector entries are $E$ and $B$ fields. The goal is to calculate $E(r,t)$ and $B(r,t)$ resulting from an current input $J(r,t)$. That is solve a linear system (2.16) for an input $u(t)$. For instance an input of the type $u(t) = \exp(j\omega t)$. Earlier we said one way to approach this is to find modes of the system and corresponding eigenvalues. That is we have to solve for $Ax = \lambda x$ in Eq. ( 2.17), i.e solve for

$$-\frac{1}{\mu}\nabla \times E = \lambda H, \tag{4.3}$$

$$\frac{1}{\varepsilon}\nabla \times H = \lambda E. \tag{4.4}$$

In this section, we adopt a more direct approach to solve for the Maxwell equations given antenna excitation. The antenna we consider is pint current source, with current density $J_0$ located at origin with current in $z$ direction as shown in fig. 4.1 . It has x-y area $A_e$ and $z$ length $dl$ such that $A_e J = i_0$. We first observe that since $\nabla \cdot B = 0$, means it can be written as $B = \nabla \times \mathbf{A}$. From 4.1, we get

$$\nabla \times (\frac{\partial \mathbf{A}}{\partial t} + E) = 0, \tag{4.5}$$

which implies that

$$\frac{\partial \mathbf{A}}{\partial t} + E = -\nabla V, \tag{4.6}$$

$$E = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V. \tag{4.7}$$

We still have some independence in the choice of $V$ and $\mathbf{A}$ that result in same $E$ and $B$ $V \to V - \frac{\partial \phi}{\partial t}$ and $\mathbf{A} \to \mathbf{A} + \nabla \phi$ doesnot change $E$ and $B$. We call these Gauge degrees of freedom. We use these degree of freedom to choose $\mathbf{A}, V$ such that

$$\frac{\partial V}{\partial t} + \frac{\partial \mathbf{A}_x}{\partial x} + \frac{\partial \mathbf{A}_y}{\partial y} + \frac{\partial \mathbf{A}_z}{\partial z} = 0. \tag{4.8}$$

This is called a Lorentz gauge. This gives from 4.2 that for $i \in \{x, y, z\}$.

$$\nabla^2 \mathbf{A}_i - \frac{\partial^2 \mathbf{A}_i}{c^2 \partial t^2} = \mu_0 J_i(x,t). \tag{4.9}$$

Recall we assume current density is in the $z$ direction and we take a point current source stationed at the origin such that $J_z(x)$ is constant at $J(t)$ over a infinitesimal volume we take it as a $J(t) = \exp(j\omega t) J_0$, as shown in fig. 4.2a. The current source is also called a **Hertz dipole**.

$$\frac{\partial^2 \mathbf{A}_z}{c^2 \partial t^2} - \nabla^2 \mathbf{A}_z - = \mu_0 J_z(x,t). \tag{4.10}$$

We first solve the above equations when current source is stationary and solve for

$$-\nabla^2 \mathbf{A}_z = \mu_0 J_z(x). \tag{4.11}$$

We can see that $\mathbf{A}_z(r) = \frac{C}{r}$ satisfies the equation $\nabla^2 \mathbf{A}_z = 0$. How do we determine $C$, we get $\nabla^2 \mathbf{A}_z = \nabla \cdot (\nabla \mathbf{A}_z)$, where $\nabla \mathbf{A}_z = -\frac{C\hat{r}}{r^2}$. Using stokes theorem, we get $-\nabla \mathbf{A}_z(4\pi r^2) = \mu_0 J_0 V_0$, which gives $C = \frac{\mu_0 J_0 V_0}{4\pi}$ and we get

$$\mathbf{A}_z(r) = \frac{\mu_0 J_0 V_0}{4\pi r}\,\hat{\mathbf{z}}. \tag{4.12}$$

Now we claim that in presence of time varying $J(t)$, we have

$$\mathbf{A}_z(r,t) = \frac{\mu_0 J(t - \frac{r}{c}) V_0}{4\pi r}\,\hat{\mathbf{z}}. \tag{4.13}$$

It is just verified that Eq. (4.13) satisfies (4.10). Writing $J_0 V_0 = idl$ where $i$ is the current and $dl$ the length of the current element we can write [3]

$$\mathbf{A}_z(r,t) = \frac{\mu_0\, i(t - \frac{r}{c}) dl}{4\pi r}\,\hat{\mathbf{z}}. \tag{4.14}$$

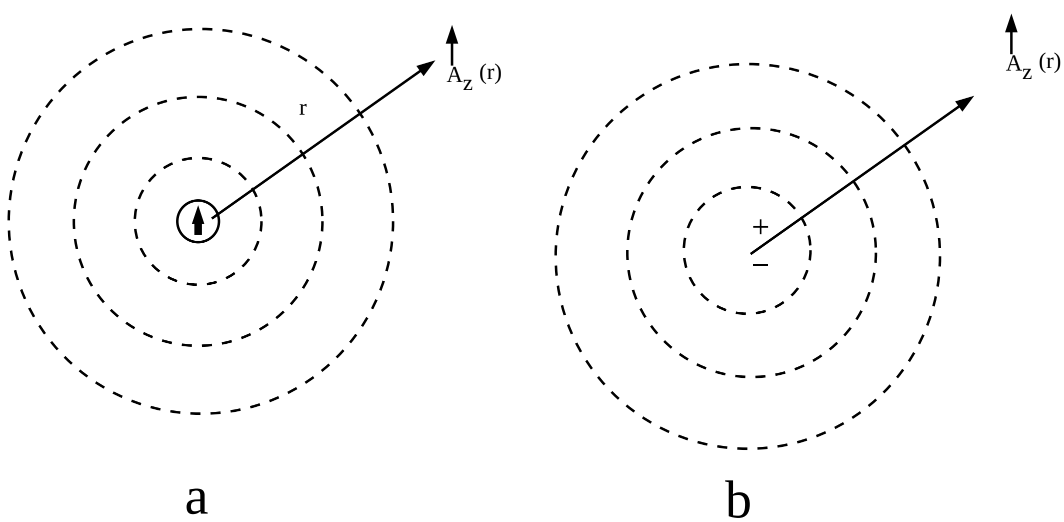



**Fig. 4.2** Fig. a and b show vector potential **A** due to Hertz dipole.

We can calculate $B$ by using $B = \nabla \times \mathbf{A}$ this gives using $i(t - \frac{r}{c}) = i_0 \exp(j\omega(t - \frac{r}{c})) = i_0 \exp(j\omega(t - kr))$, where $k = \frac{\omega}{c}$.

$$B(r,t) = \frac{\mu_0 dl}{4\pi}\left(\frac{i(t-\frac{r}{c})}{r^2} + \frac{i'(t-\frac{r}{c})}{cr}\right)\hat{\phi}, \tag{4.15}$$

$$B(r,t) = \exp(j\omega t)\underbrace{\frac{\mu_0 i_0 dl\exp(-jkr)\sin\theta}{4\pi r}\left(\frac{1}{r}+jk\right)\hat{\phi}}_{B(r)}. \tag{4.16}$$

Thus we get steady state value $B(r)$. To get steady state value $E(r)$ observe

$$j\omega\mu H(r) = -\nabla\times E(r), \tag{4.17}$$

$$j\omega\varepsilon E(r) = \nabla\times H(r) - J(r). \tag{4.18}$$

From (4.18) we have for $r$ away from origin

$$\frac{j}{c^2}\omega E = \nabla\times B, \tag{4.19}$$

which gives

$$E(r,t) = \exp(j\omega t)\underbrace{\frac{c^2\mu_0 i_0 dl\exp(-jkr)}{4\pi r\omega}\left\{\left(-jk^2+\frac{3k}{r}-\frac{3j}{r^2}\right)\sin\theta\ \hat{\theta}+\left(\frac{2k}{r}-\frac{2j}{r^2}\right)\hat{\mathbf{z}}\right\}}_{E(r)}. \tag{4.20}$$

At large distances $r \gg \lambda$, we only keep terms of order $r^{-1}$ giving,

$$B(r) = \frac{j\ k\ \mu_0\ i_0\ dl\ \exp(-jkr)\ \sin\theta}{4\pi r}\ \hat{\phi}, \tag{4.21}$$

$$E(r) = \frac{-j\ k^2\ i_0\ dl\ \exp(-jkr)\ \sin\theta}{4\pi r\omega\varepsilon}\ \hat{\theta}. \tag{4.22}$$

### 4.1.1 Dipole Radiation

We considered a current source and radiation due to that. Now consider an oscillating dipole, where two opposite charges with charge $q$ go back and forth along say $z$ axis with a maximum separation of $d$. The dipole moment

$$D(t) = qd\cos(\omega t). \tag{4.23}$$

This is as shown in Fig. (4.2b), where $\frac{dD}{dt}$ constitutes a current. For simplicity take

$$D(t) = qd\exp(j\omega t). \tag{4.24}$$

Then everything we did for point current source carries over and we make the correspondence $i_0\, dl = jqd\omega$. Then in the expression for $\mathbf{A}_z(r,t), B(r,t), E(r,t)$ in Eq. (4.14, 4.21, 4.22), we just substitute

$$i_0\, dl = jqd\ \ \omega. \tag{4.25}$$

This gives us radiation due to an oscillating dipole.

## 4.2 Radiated Power and Antenna Resistance

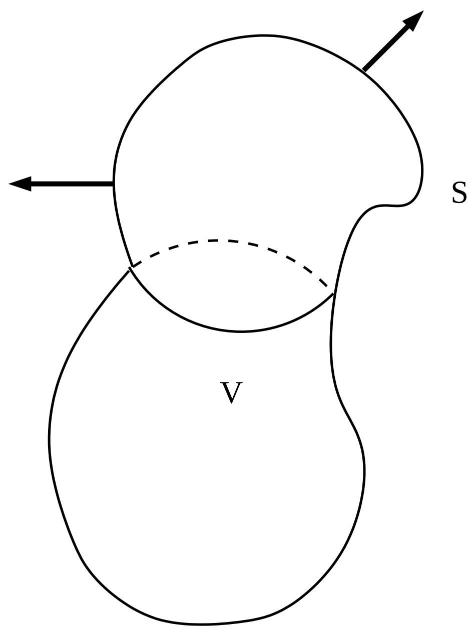



**Fig. 4.3** Figure shows power outflux from a enclosed volume $V$ and surface $S$.

Consider an enclosed volume $V$ as in 4.3. The volume contains electric and magnetic fields and the enclosed energy is

$$\mathscr{E} = \frac{1}{2}\int_V \varepsilon_0 |E|^2 + \mu_0 |H|^2. \tag{4.26}$$

Then the change of energy with time

$$\frac{\partial \mathscr{E}}{\partial t} = \int_V Re(\varepsilon_0 \frac{\partial E}{\partial t} \cdot E^* + \mu_0 \frac{\partial H}{\partial t} \cdot H^*), \tag{4.27}$$

$$= \int_V Re(\nabla \times H \cdot E^* - \nabla \times E \cdot H^*), \tag{4.28}$$

$$= \frac{1}{2} \int_V Re(\nabla \times (H^* \times E)), \tag{4.29}$$

$$= \frac{1}{2} \int_S Re(H^* \times E).dA, \tag{4.30}$$

where we use Maxwell equations to substitute for $\frac{\partial E}{\partial t}, \frac{\partial H}{\partial t}$. We use the identity $a \cdot (b \times c) = \frac{1}{2}(c \cdot (a \times b) - b \cdot (a \times c))$. The last equation uses Stoke's theorem. The quantity

$$P = \frac{1}{2} Re(E \times H^*), \tag{4.31}$$

denotes power leaving the surface per unit area. It is also called Poynting vector.

If we integrate $P$ for $E$ and $B$ as in Eq. (4.21 and 4.22) for a Hertz dipole over a large sphere, we get

$$P = \frac{1}{2} \frac{(ki_0 dl)^2}{6\pi c \varepsilon} = i_0^2 R. \tag{4.32}$$

where

$$R = \frac{(kdl)^2}{6\pi c \varepsilon}. \tag{4.33}$$

If $dl = \lambda/2$, then $R \sim \sqrt{\frac{\mu_o}{\varepsilon_0}} \sim 50$ SI units. It is called radiation resistance of antenna. The power loss due to radiation is seen as a resistance by antenna circuit.

## 4.3 Antenna Arrays

Now consider an array of Hertz dipoles as shown in Fig. (4.4). Let total be $N+1$. Let $d$ be the separation between them. Then the $\mathbf{A}_z(r,t), B(r,t), E(r,t)$ due to each dipole carries a factor $\propto \exp(ikr)$. Where $r$ between successive dipoles differs by $\Delta r = d\cos\phi$ as shown in Fig. (4.4). This phase factor between successive dipoles differs by a factor $Z = \exp(ik\theta)$, where $\theta = dcos\phi = d\sin\upsilon$, where $\upsilon = \frac{\pi}{2} - \phi$. Then adding all the phase factors for a point far than size of array we get for $p = kd\cos\phi$,

$$\eta = \sum_{k=0}^{N} Z^k = \exp(j\frac{(N-1)p}{2}) \frac{\sin\frac{Np}{2}}{\sin\frac{p}{2}}. \tag{4.34}$$

The factor $f(p) = \frac{\sin\frac{Np}{2}}{\sin\frac{p}{2}}$ gives the dependency of $|\eta|$ as function of $p$. It peaks at $p = 0$ where its value is $N$. Then

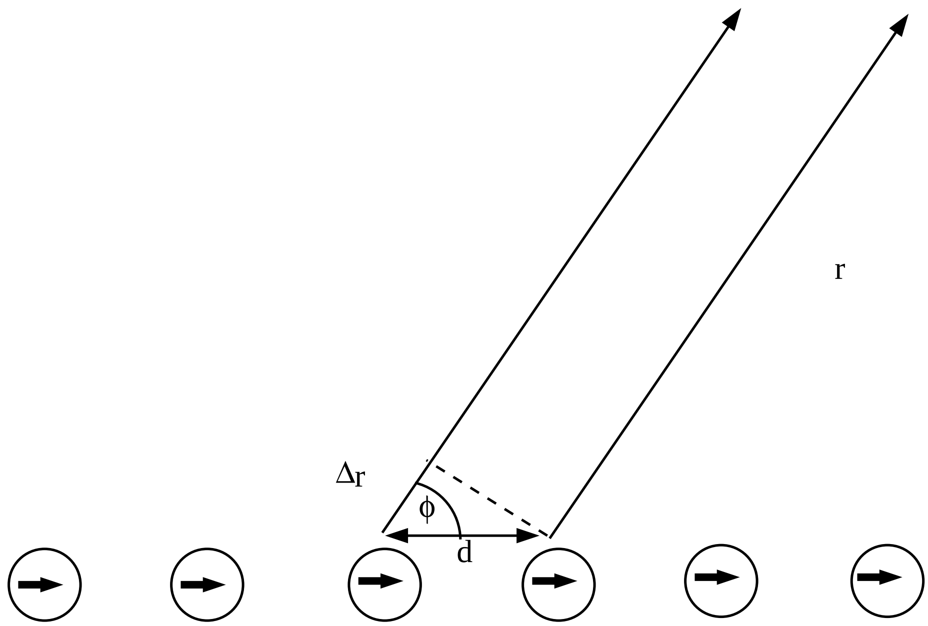



**Fig. 4.4** Fig. shows a 1D array of Hertz dipoles with a separation $d$ between them.

$$\frac{|\eta|}{N} = |\frac{\sin\frac{Np}{2}}{N\sin\frac{p}{2}}| \sim |\,\mathrm{sinc}(\frac{Np}{2})|, \tag{4.35}$$

where $|\,\mathrm{sinc}(p)|$ is as in fig. 4.5. Few lobes down from origin the function significantly diminishes. Lets say the array is 1000 wavelengths long which means $Np = 1000\cos\phi$. If we are ten lobes down means $\frac{Np}{2} = 10\pi$ or $\cos\phi = \frac{\pi}{50}$ or $\sin\upsilon = \frac{\pi}{50}$ a small angle. The antenna array fires vertically up as in $A$ in fig 4.6.

Now suppose we introduce a phase difference in the current in the successive elements of array the form $\exp(i\delta)$, then writing $\delta = -kd\sin\upsilon_0$ we get $p = kd(\sin\upsilon - \sin\upsilon_0)$. Now $p = 0$ when $\upsilon = \upsilon_0$. Therefore this array will not fire vertically but rather at an angle $\upsilon_0$. When $\upsilon_0 = \frac{\pi}{2}$, we get an array that fires at the end called an endfire array as as in $B$ in fig 4.6.

### 4.3.1 2D arrays

Fig. 4.7 A shows a 2D array of Hertz dipoles. Fig. 4.7 B shows the array laid on $x-y$ plane and beam direction making angle $\theta$ with $z$ axis and $\phi$ with $x$ axis. Let $d$ be spacing between dipoles. let

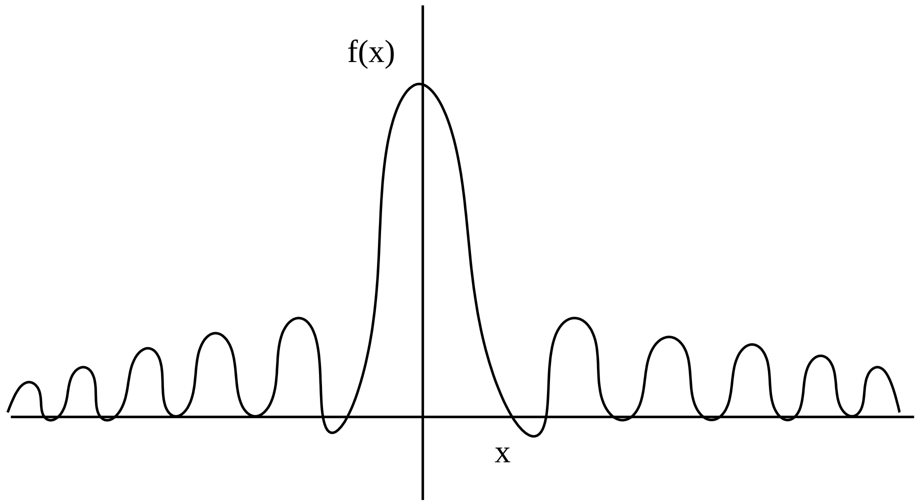



**Fig. 4.5** Fig. depicts the function $f(x) = |\operatorname{sinc}(x)|$

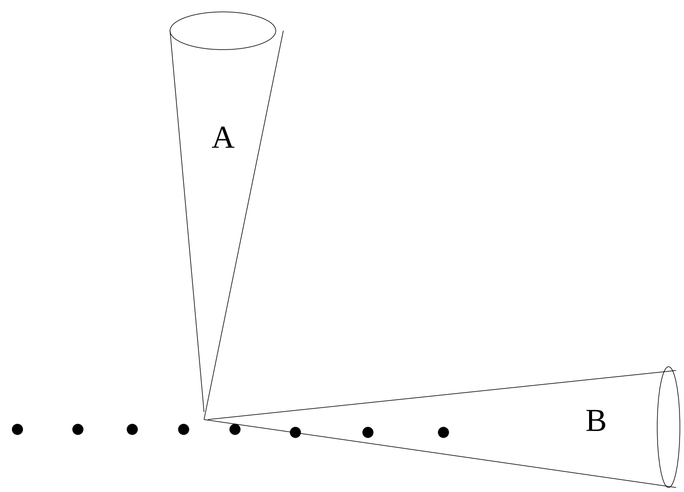



**Fig. 4.6** Fig. depicts how antenna array fires vertically $A$ or horizontally $B$ depending on phase difference of the current in array elements.

$$p_1 = d \sin\theta \cos\phi, \tag{4.36}$$

$$p_2 = d \sin\theta \sin\phi. \tag{4.37}$$

$Z_1 = \exp(jp_1)$ and $Z_2 = \exp(jp_2)$. Then adding phases over antennas we get

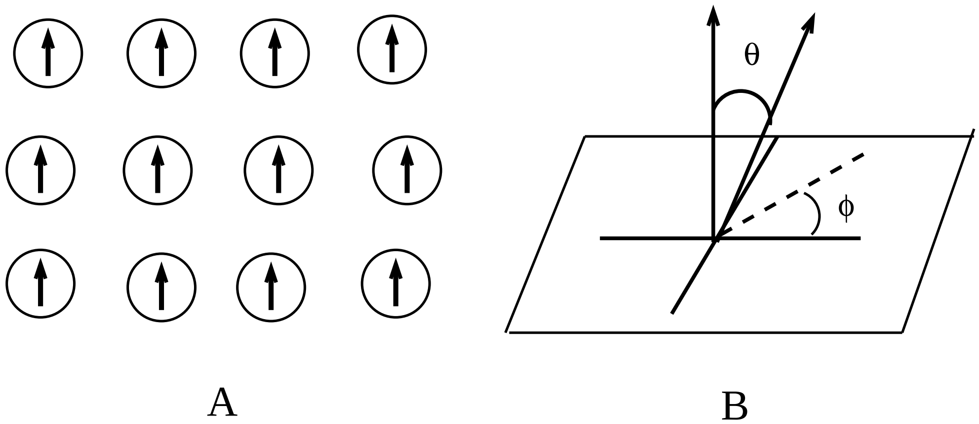



**Fig. 4.7** Fig. A shows a 2D array of Hertz dipoles. Fig. B shows the array laid on $x-y$ plane and beam direction making angle $\theta$ with $z$ axis and $\phi$ with $x$ axis.

$$\eta = \sum_{k,l} Z_1^k Z_2^l = \exp\left(j\frac{(N-1)p_1 + (N-1)p_2}{2}\right)\frac{\sin\frac{Np_1}{2}}{\sin\frac{p_1}{2}}\frac{\sin\frac{Np_2}{2}}{\sin\frac{p_2}{2}}. \tag{4.38}$$

Lets say the array is $1000 \times 1000$ wavelengths in area which means $Np_1 = 1000\sin\theta\cos\phi$ and $Np_2 = 1000\sin\theta\sin\phi$. If we have $\sin\theta = \frac{\pi}{50}$, then one of the $\frac{Np_1}{2}$ or $\frac{Np_1}{2}$ is $\sim 10\pi$ or ten lobes down and $\eta$ diminishes significantly. Therefore all intensity is concentrated in $\theta$ very small, antenna array fires vertically up.

### 4.3.2 Antennas and finite interfaces

In previous chapters, we studied how EM waves reflect of conducting interface. For ease of calculation, we took the conductor plane to be infinite. How about finite conductor interfaces as in mirrors. To see how mirror reflects EM waves say light (optical frequency $10^{15}$ Mhz), which is sub-micrometer wavelength. Imagine an infinite plane. Incident light induces current density $J_S$ on the plane, which acts as an antenna array. If we partition the plane into say rectangles of say $1000 \times 1000$ wavelengths in area (this means $mm^2$ sub-blocks for light) then these rectangles will fire vertically discussed in previous section and shown in fig. 4.8A. Then suppose the real interface we are given is say region $II$ in fig. 4.8A. Then if we remove remaining regions, one by one, we find that since their respective antenna did not interfere (at moderate distances upto meter) with the radiation of region $II$, the radiation in the vertical line of sight of region $II$ is uninterrupted. All we are saying is that pretend the plane is infinite, calculate, the reflected wave and only take the reflected wave in vertical line of sight of region $II$. That is to say we can draw ray diagrams as 4.8B, which is starting point of geometric optics. Instead of light if we had radio waves then a $mm^2$ size reflector will act as a point antenna and send waves in all directions. Then we cannot follow rules of geometric optics.

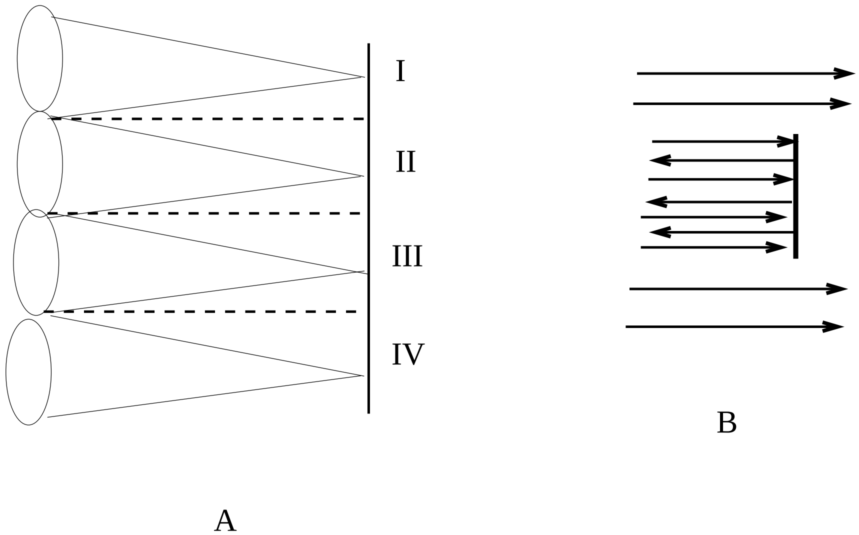



**Fig. 4.8** Fig. A shows reflection of a conducting plane as finite size antenna arrays firing vertically. Fig. B shows ray diagram for incident and reflected light.

## 4.4 Problems

1. Show that $\nabla \times (\nabla \times E) = -\nabla^2 E + \nabla(\nabla \cdot E)$

2. Show that $\nabla \cdot (A \times B) = A \cdot (\nabla \times B) - B \cdot (\nabla \times A)$

3. Write Maxwell equations in terms of vector potential. Use Lorentz gauge.

4. Find $E$ and $B$ for 10 GHz current dipole (at origin) at the point $(r, 0, 0)$, where $r = 1$ km, for 1 watt of radiated power.

5. For an antenna array 1000 wavelength long find the beamwidth.

# Chapter 5
# Transmission Lines

## 5.1 Coaxial Transmission Lines

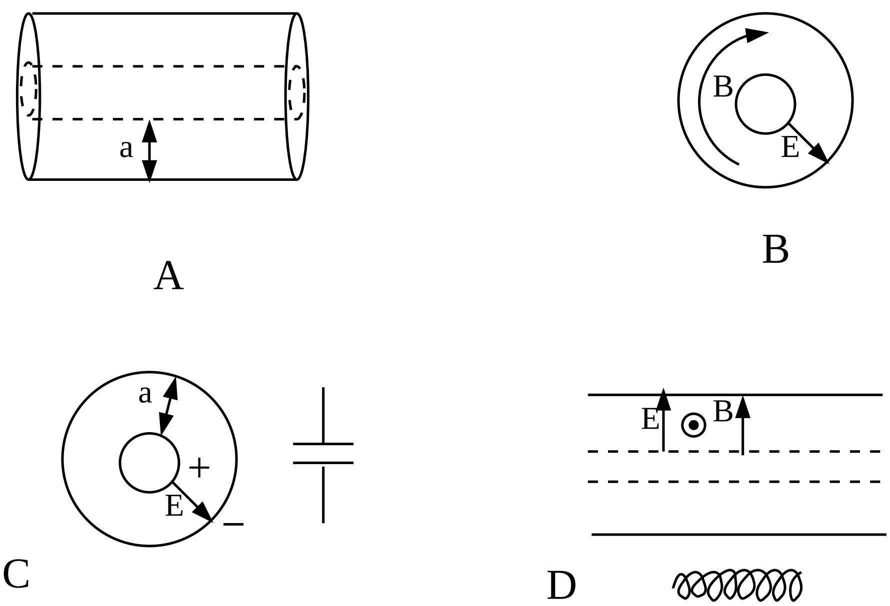



**Fig. 5.1** Fig. A shows a coaxial cable. Fig. B shows a cross-section with radial *E* and circular *B* fields. Fig. C shows how radial *E* gives capacitance. Fig. D shows how circular B gives inductance.

In the previous chapter we taked about a special kind of waveguide, a hollow pipe. In this chapter we talk of a hollow pipe containing a conductor inside it as shown in Fig. 5.1A. This is called a coaxial cable [11, 12] , or just coax. Such a structure supports a novel EM mode called the TEM mode, which was absent in a hollow pipe. TEM means transverse electric and magnetic. The electric field is radial and magnetic field circular as shown in Fig. 5.1B, which shows cross section of coax.

The radial electric field produces electric charge $Q$ of opposite polarity in the inner and outer conductors such that resuting voltage $V(x) = E(x)a$ can be related to the charge $Q$. Then we can relate $V$ and $Q$ as $V(x)C\Delta x = Q(x)$ where $C$ is the capacitance per unit length.

Further more $(E(x+\Delta x) - E(x))a = -\frac{\partial B}{\partial t}A_r$, where $A_r = a\Delta x$ is the area. But $BA_r = L\Delta xI$, where $L$ is the inductance per unit length and $I(x)$ is the current in the inner conductor. This is written as

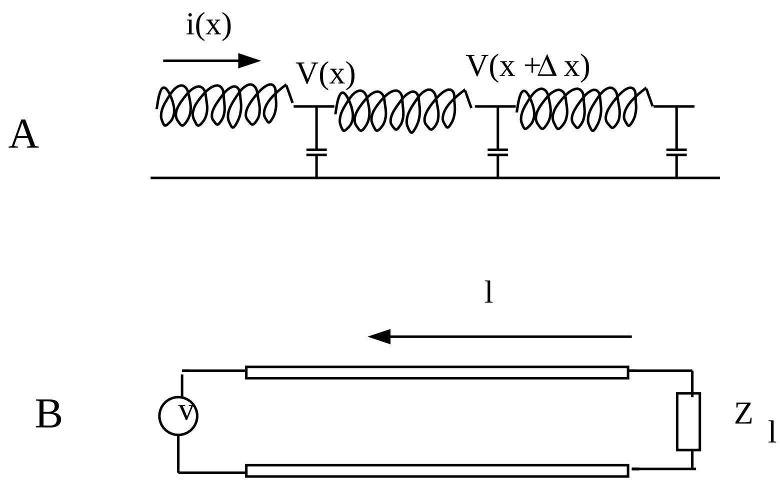



**Fig. 5.2** Fig. A shows electrical equivalent of a transmission line with distributed inductors and capacitors. Fig. B shows voltage source at one end of the line (called generator end) and load at other end.

$$V(x+\Delta x) - V(x) = L\Delta x \frac{dI}{dt}, \tag{5.1}$$

$$-\frac{\partial V}{\partial x} = L\frac{dI(x)}{dt}. \tag{5.2}$$

Similarly

$$I(x) - I(x+\Delta x) = \frac{dQ(x)}{dt}, \tag{5.3}$$

$$\frac{dQ(x)}{dt} = C\Delta x \frac{dV(x)}{dt}, \tag{5.4}$$

$$-\frac{\partial I}{\partial x} = C\frac{dV(x)}{dt}. \tag{5.5}$$

Thus we get two equations of the transmission line

$$-\frac{\partial V}{\partial x} = L\frac{dI(x)}{dt}, \tag{5.6}$$

$$-\frac{\partial I}{\partial x} = C\frac{dV(x)}{dt}. \tag{5.7}$$

Now we drive the transmission line with a voltage source $V = V_0 \exp(j\omega t)$, see fig. 5.2B. Then that gives steady state values $V(x)\exp(j\omega t)$ and $I(x)\exp(j\omega t)$ which satisfy,

$$-\frac{\partial V}{\partial x} = j\omega L I(x), \tag{5.8}$$

$$-\frac{\partial I}{\partial x} = j\omega C V(x). \tag{5.9}$$

See fig. 5.2B. Where we take load at right and voltage generator at left. Lets take $x$ going from right to left. Then above equations read,

$$\frac{\partial V}{\partial x} = j\omega L I(x), \tag{5.10}$$

$$\frac{\partial I}{\partial x} = j\omega C V(x). \tag{5.11}$$

$$\frac{\partial^2 V}{\partial x^2} = -\omega^2 LCV(x) = -\gamma^2 V(x), \tag{5.12}$$

where $\gamma = j\beta$ where $\beta = \omega\sqrt{LC}$ and let $Z_0 = \sqrt{\frac{L}{C}}$.

$$V(x) = V_+ \exp(j\beta x) + V_- \exp(-j\beta x), \tag{5.13}$$

$$I(x) = \frac{1}{Z_0}(V_+ \exp(j\beta x) - V_- \exp(-j\beta x)) \tag{5.14}$$

$$= \frac{1}{Z_0}(I_+ \exp(j\beta x) - I_- \exp(-j\beta x)). \tag{5.15}$$

## 5.2 Impedance

Then for $\Gamma = \frac{V_-}{V_+}$,

$$Z(0) = Z_L = \frac{V(0)}{I(0)} = Z_0 \frac{1+\Gamma}{1-\Gamma}, \tag{5.16}$$

$$Z(x) = \frac{V(x)}{I(x)} = Z_0 \frac{1+\Gamma\exp(-2j\beta x)}{1-\Gamma\exp(-2j\beta x)}. \tag{5.17}$$

Using $\Gamma = \frac{Z_l - Z_0}{Z_l + Z_0}$, we get

$$Z(x) = Z_0 \frac{Z_L \cos\beta x + jZ_0 \sin\beta x}{Z_0 \cos\beta x + jZ_L \sin\beta x}. \tag{5.18}$$

Until now we assumed perfect conductors and the dielectric seprating them as perfect insulator. In practice conductor has small resistance $R$ per unit length and dielectric small admittance $G$ per unit length.

Thus we get two equations of the transmission line change to

$$-\frac{\partial V}{\partial x} = L\frac{dI(x)}{dt} + RI, \tag{5.19}$$

$$-\frac{\partial I}{\partial x} = C\frac{dV(x)}{dt} + GV. \tag{5.20}$$

With steady state,

$$-\frac{\partial V}{\partial x} = (j\omega L + R)I(x), \tag{5.21}$$

$$-\frac{\partial I}{\partial x} = (j\omega C + G)V(x). \tag{5.22}$$

See fig. 5.2B. Where we take load at right and voltage generator at left. Lets take $x$ going from right to left. Then above equations read,

$$\frac{\partial V}{\partial x} = (j\omega L + R)I(x), \tag{5.23}$$

$$\frac{\partial I}{\partial x} = (j\omega C + G)V(x). \tag{5.24}$$

$$\frac{\partial^2 V}{\partial x^2} = \gamma^2 V(x), \tag{5.25}$$

where $\gamma = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta$ and let $Z_0 = \sqrt{\frac{j\omega L + R}{j\omega C + G}}$.

To fix some typical numbers, $R = .01\Omega/m$, $L = .01\mu H/m$, $C = 100pF/m$ and $G = .01\mho/m$ giving at 2 GHz, $Z_0 = 10 + j.04\Omega$.

$$V(x) = V_+ \exp(\gamma x) + V_- \exp(-\gamma x), \tag{5.26}$$

$$I(x) = \frac{1}{Z_0}(V_+ \exp(\gamma x) - V_- \exp(-\gamma x)) \tag{5.27}$$

$$= (I_+ \exp(\gamma x) - I_- \exp(-\gamma x)). \tag{5.28}$$

Then for $\Gamma = \frac{V_-}{V_+}$,

$$Z(0) = Z_L = \frac{V(0)}{I(0)} = Z_0 \frac{1+\Gamma}{1-\Gamma}, \tag{5.29}$$

$$Z(x) = \frac{V(x)}{I(x)} = Z_0 \frac{1+\Gamma \exp(-2\gamma x)}{1-\Gamma \exp(-2\gamma x)}. \tag{5.30}$$

Using $\Gamma = \frac{Z_l - Z_0}{Z_l + Z_0}$, we get

$$Z(x) = Z_0 \frac{Z_L \cosh \gamma x + Z_0 \sinh \gamma x}{Z_0 \cos \gamma x + Z_L \sinh \gamma x}. \tag{5.31}$$

For low loss transmission line where $\gamma = j\beta$, we get $Z(x)$ as in Eq. (5.18). There is no loss in the transmission line then the power transferred to the load is as follows.

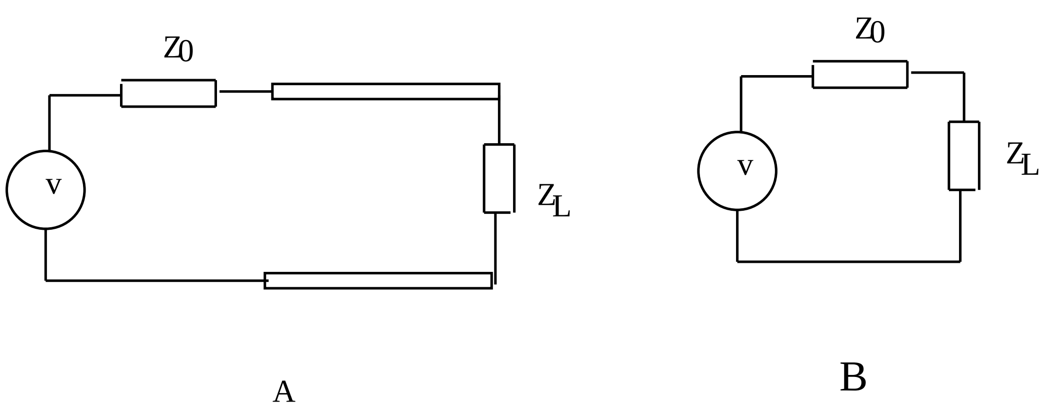



**Fig. 5.3** Fig. A shows a voltage source $V$ with impedance $Z_0$, a transmission line and a load $Z_L$. Fig. B shows load as seen from generator end of transmission line.

## 5.3 Impedance Matching and Smith Charts

Consider a voltage source $V$ with source impedance $Z_0$. It is attached to transmission line with characteristic impedance $Z_0$ and a load at the end of transmission line $Z_L$. We want to know what should be the value of load impedance $Z_L$ so that there is maximum power transferred to load.

Then voltage $V_e$ at the end of the transmission line is

$$V_e = V_+ + V_- = V_+(1+\Gamma). \tag{5.32}$$

Then voltage $V_b$ and current $i_b$ at the beginning of the transmission line is

$$V_b = V_+ \exp(j\beta l) + V_- \exp(-j\beta l), \tag{5.33}$$

and

$$i_b = \frac{1}{Z_0}(V_+ \exp(j\beta l) - V_- \exp(-j\beta l)). \tag{5.34}$$

$$V_b = V_0 - i_b Z_0, \tag{5.35}$$

giving

$$V_+ = \frac{V_0}{2}\exp(-j\beta l). \tag{5.36}$$

The power transferred is $P = Re(V_e I_e^*)$, which gives

$$P = \frac{|V_+|^2}{Z_0}(1 - |\Gamma_L|^2), \tag{5.37}$$

where

$$\Gamma = \frac{1 - \frac{Z_0}{Z_L}}{1 + \frac{Z_0}{Z_L}}. \tag{5.38}$$

For maximum transfer $\Gamma = 0$ or $Z_L = Z_0$.

what if $Z_L \neq Z_0$, what should we do. We can move back from the load towards the voltage source. What we show is after some distance $x$ the resistance $Z(x)/Z_0$ becomes $1 + jv$ . We can arrange to cancel the reactive part $v$ by using an additional piece of transmission line leaving the resulting load $Z(x) = Z_0$ which allows for maximum power transfer.

We can graphically find this distance $x$ using what is called a smith chart. Given

$$r + jx = \frac{Z(x)}{Z_0} = \frac{1+\Gamma}{1-\Gamma}. \tag{5.39}$$

Then for $\Gamma = u + jv$, we can plot the locus of $\Gamma$ for fixed $r$ and fixed $x$. These are called constant resistance and reactance circles. These are given by

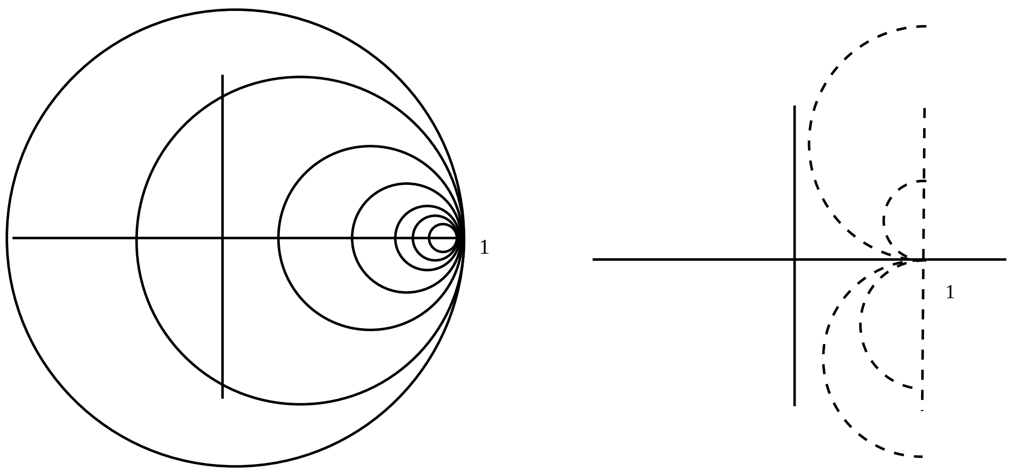
$$(u - \frac{r}{r+1})^2 + v^2 = (r+1)^{-2} \tag{5.40}$$

a circle centered at $(\frac{r}{r+1}, 0)$ with radius $\frac{1}{r+1}$. This is called constant resistance circle. These are depicted in 5.4A, with smaller circle corresponding to larger $r$.

Similarly constant reactance circle is

$$(u-1)^2 + (v - \frac{1}{x})^2 = \frac{1}{x^2}, \tag{5.41}$$

which is centered at $(1, \frac{1}{x})$ with radius $\frac{1}{|x|}$. These are depicted in 5.4B, with smaller circle corresponding to larger $x$.




A B

**Fig. 5.4** Fig. shows constant resistance $A$ and reactance circle $B$.

We start with a given load corresponding to given $(r,x)$. The intersection of constant $r$ and $x$ circles uniquely determine $\Gamma$. As we move back on transmission line $\Gamma(x) \to \Gamma \exp(-j2\beta x)$ that is we rotate $\Gamma$ counterclockwise keeping its magnitude constant, till it hits the constant resistance $r = 1$ circle. We stop and note the reactance value there and this is the reactance we need to annul. All this is depicted in 5.5. Where we located the normalized load $r + jx$ as intesection of constant resistance circle $r$ (shown in solid) and constant reactance circle $x$ (shown in dashed) as point $A$. This point is rotated on cirlce centered at origin (shown in dotted) till we hit constant unit resistance circle (shown in bold) at point B. We can read the reactance of point $B$. The angle $\theta$ of rotation of $A$ to $B$ is given as $\Theta = 2\beta x$.

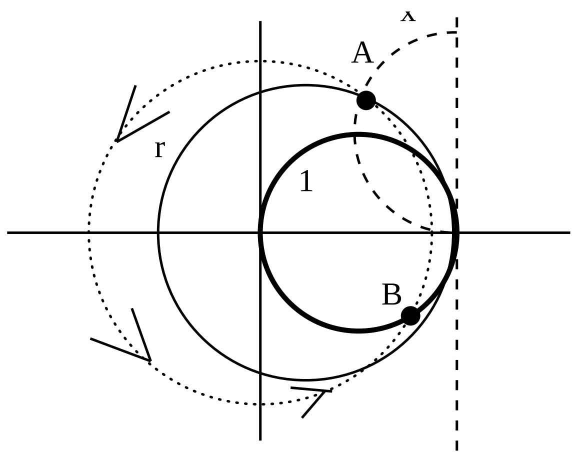



**Fig. 5.5** Fig. shows how to calculate using a Smith chart the distance travelled up a transmission line to reach unity normalized resistance.

How do we cancel the reactance. One way is to add a segment transmission line in parallel (shunt) as shown in 5.7A.

The admittance of the shunt adds to admittance of the load at $x$. With $Y(x) = Z^{-1}(x)$ and $Y_0 = Z_0^{-1}$ we find for $\Gamma' = -\Gamma$, we have

$$g + jy = \frac{Y(x)}{Y_0} = \frac{1+\Gamma'}{1-\Gamma'}. \tag{5.42}$$

Now everything is same proceduraly as before. We start with a given load corresponding to admittance $g + jy$. The intersection of constant $g$ and $y$ circles uniquely determine $\Gamma'$. As we move back on transmission line $\Gamma'(x) \to \Gamma' \exp(-j2\beta x)$ that is we rotate $\Gamma'$ counterclockwise keeping its magnitude constant, till it hits the constant conductance $g = 1$ circle. We stop and note the susceptance value there and this is the susceptance we need to annul. All this is same as depicted in 5.5. Except now our starting point $A$ is normalized admittance which is different from normalized impedance. The susceptance at point $B$ is removed by adding a transmission line in shunt.

What is the impedance and admitance of a transmission line pf length $l$. If it is closed at other end as in fig. 5.7B, then it means $Z_L = 0$ and from Eq. (5.18), $Z(l) = j\tan\beta l$ or $Y(l) = -j\cot\beta l$. If it is open at other end as in fig. 5.7C, then it means $Z_L = \infty$ and from Eq. (5.18), $Z(l) = -j\cot\beta l$ or $Y(l) = j\tan\beta l$. Therefore to cancel a positive susceptance, we should use closed-ended transmission line of $0 \leq l \leq \frac{\lambda}{4}$ and to cancel a negative susceptance, we should use open-ended transmission line of $0 \leq l \leq \frac{\lambda}{4}$.

Observe $\Gamma(x)$ repeats every $2\beta x = 2\pi$ i.e., every $x = \frac{\lambda}{2}$. Therefore load resistance is the same when we step back $x = \frac{\lambda}{2}$. Infact the maximum voltage on transmission

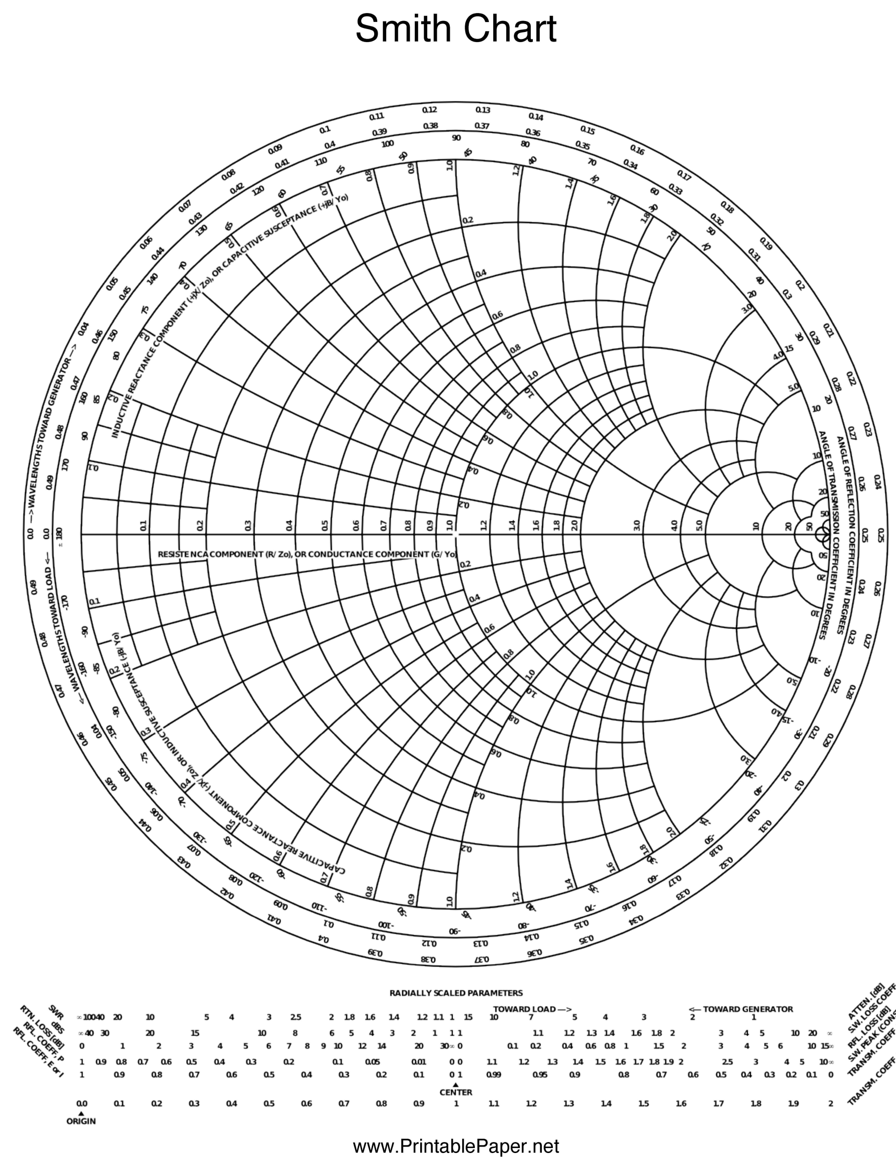



**Fig. 5.6** Fig. shows a smith chart

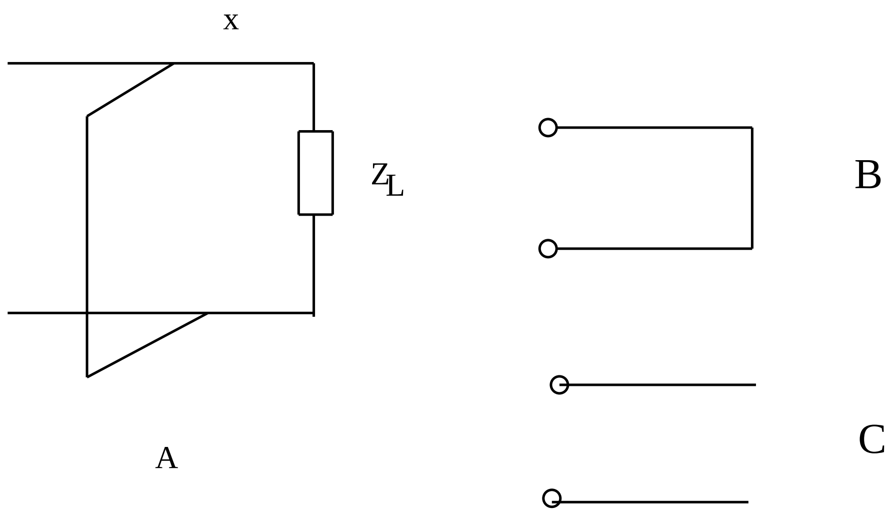



**Fig. 5.7** Fig. A shows how impedance matching is done by adding admittance in parallel to transmission line also called stubs. Fig. B shows a short circuited transmission line. Fig. C shows a open circuited transmission line.

$$V_{max} = V_{+}(1 + |\Gamma|), \tag{5.43}$$

and

$$V_{min} = V_{+}(1 - |\Gamma|). \tag{5.44}$$

and the ratio $\frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$ is called **VSWR**, voltage standing wave ratio. When **VSWR** $= 1$, implies $\Gamma = 0$ and there is no reflected wave. To check if we have matched the load perfectly we measure **VSWR** to find if **VSWR** $= 1$.

## 5.4 Problems

1. For a coaxial cable of inner and outer diameter 1 and 2 cm, find inductance and capacitance per unit length.

2. Find velocity of wave on the coax.

3. Find the wavelength if 10 GHz wave travels on this transmission line.

4. If line is 1 m long, find the input impedance if load impedence is open circuit and closed circuit.

5. Find the characteristic impedence for the line (you may neglect resistance per unit length).

# Chapter 6
# Rf and Microwave Engineering

## 6.1 Microwave Oscillators

We first discuss how to produce microwaves ($3-300$ GHz) frequencies. These are produced in cavities called microwave oscillators. The basic principle is to excite the modes of a cavity resonantly by motion of an electron. It is discusses in the following.

### 6.1.1 Coaxial Cylindrical Cavity

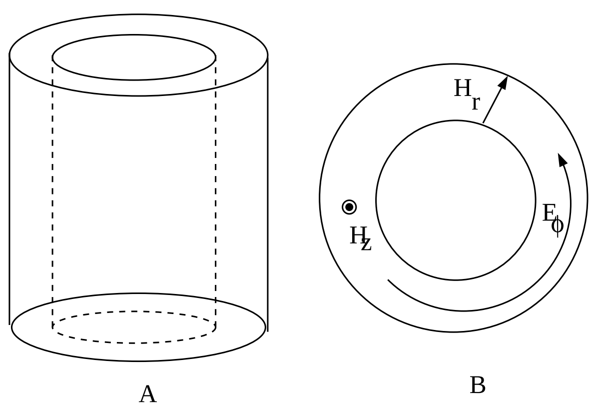



**Fig. 6.1** Fig. A shows a cylindrical cavity with inner and outer conductors. Fig. B shows its cross-section containing TE mode.

Consider a cylindrical cavity as shown in fig. 6.1A. Fig. 6.1A shows cross section. The cavity has an outer conductor of radius $r_2$ and an inner conductor of radius $r_1$ with $r_2 - r_1 = a$. The cavity supports both $TE$ and $TM$ modes. We discuss both.

### 6.1.2 TE Modes

In cylindrical coordinates (r, $\theta$, z), we have

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \tag{6.1}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \tag{6.2}$$

For TE modes from Eq. (3.14 and 3.15), we can write

$$j\omega(\varepsilon - \frac{\beta^2}{\mu\omega^2})E_r = \frac{1}{r}\frac{\partial H_z}{\partial \theta}, \tag{6.3}$$

$$j\omega(\varepsilon - \frac{\beta^2}{\mu\omega^2})E_\theta = -\frac{\partial H_z}{\partial r}, \tag{6.4}$$

from Eq. (3.16 and 3.17), we get

$$\frac{\mu}{\beta} j\omega^2(\varepsilon - \frac{\beta^2}{\omega^2\mu})H_r = \frac{\partial H_z}{\partial r}, \tag{6.5}$$

$$\frac{\mu}{\beta} j\omega^2(\varepsilon - \frac{\beta^2}{\omega^2\mu})H_\theta = \frac{1}{r}\frac{\partial H_z}{\partial \theta}. \tag{6.6}$$

We choose

$$H_z = f(r)\cos n\theta \tag{6.7}$$

Then we get

$$-\underbrace{(\frac{\omega^2}{c^2} - \beta^2)}_{l^2} H_z = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})H_z.$$

$$(-l^2 + \frac{n^2}{r^2})f(r) = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})f(r).$$

The solution in the annulus is $J_n(lr + o)$. If $Z_1, Z_2, \ldots, Z_n$ are zeros of the $J_n'(x)$ (other than trivial zeros at 0)Then mode $lm$ is defined as

$$lr_1 + o = Z_l, \tag{6.8}$$
$$lr_2 + o = Z_m. \tag{6.9}$$

This ensures that $E_\theta$ and $H_r$ that vanishes at the boundary. We solve for $l = \frac{Z_m - Z_l}{a}$.

This defines $\beta^2 = \sqrt{(\frac{\omega^2}{c^2} - (\frac{Z_m - Z_l}{a})^2}$. We have a wave along $z$ with dependence $\exp(-j\beta z)$. Superimposing on it a reflected wave $\exp(-j\beta z)$ we get transverse electric field has $z$ dependence $\sin(\beta z)$ and transverse magnetic field $\cos(\beta z)$. By choosing $\beta = \frac{p\pi}{b}$, we get a standing wave mode indexed by $p$ and $Z_m, Z_l$, with

$$\omega = c\sqrt{(\frac{p\pi}{b})^2 + (\frac{Z_m - Z_l}{a})^2}. \tag{6.10}$$

## 6.2 TM Modes

For TM modes from Eq. (3.30 and 3.31), we can write

$$j\omega(\mu - \frac{\beta^2}{\varepsilon\omega^2})H_r = \frac{1}{r}\frac{\partial E_z}{\partial\theta}, \tag{6.11}$$
$$j\omega(\mu - \frac{\beta^2}{\varepsilon\omega^2})H_\theta = \frac{\partial E_z}{\partial r}, \tag{6.12}$$

from Eq. (3.28 and 3.29), we get

$$\frac{\varepsilon}{\beta}j\omega^2(\mu - \frac{\beta^2}{\omega^2\varepsilon})E_r = \frac{\partial E_z}{\partial r}, \tag{6.13}$$
$$\frac{\varepsilon}{\beta}j\omega^2(\mu - \frac{\beta^2}{\omega^2\varepsilon})E_\theta = \frac{1}{r}\frac{\partial E_z}{\partial\theta}. \tag{6.14}$$

We choose

$$E_z = f(r)\cos n\theta. \tag{6.15}$$

$$j\omega\varepsilon E_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \propto \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2},$$

which gives

From

$$\underbrace{(\frac{\omega^2}{c^2} - \beta^2)}_{l^2} E_z = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})E_z.$$

$$(-l^2 + \frac{n^2}{r^2})f(r) = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})f(r).$$

The solution in the annulus is $J_n(lr+o)$. If $Y_1, Y_2, \ldots, Y_n$ are zeros of the $J_n(x)$ (other than trivial zeros at 0)Then mode $lm$ is defined as

$$lr_1 + o = Y_l, \tag{6.16}$$

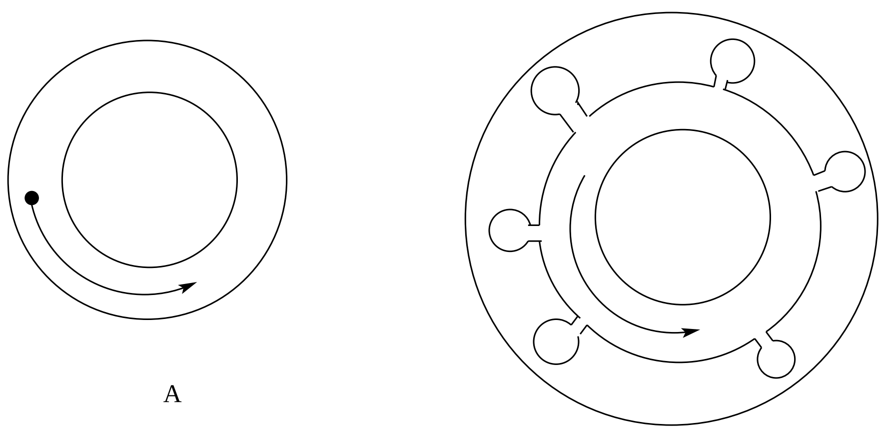
$$lr_2 + o = Y_m \tag{6.17}$$

This ensures that $E_\theta$ and $H_r$ that vanishes at the boundary. We solve for $l = \frac{Y_m - Y_l}{a}$.

This defines $\beta^2 = \sqrt{(\frac{\omega^2}{c^2} - (\frac{Y_m - Y_l}{a})^2}$. We have a wave along $z$ with dependence $\exp(-j\beta z)$. Superimposing on it a reflected wave $\exp(-j\beta z)$ we get transverse electric field has $z$ dependence $\sin(\beta z)$ and transverse magnetic field $\cos(\beta z)$. By choosing $\beta = \frac{p\pi}{b}$, we get a standing wave mode indexed by $p$ and $Y_m, Y_l$, with

$$\omega = c\sqrt{(\frac{p\pi}{b})^2 + (\frac{Y_m - Y_l}{a})^2}. \tag{6.18}$$

### 6.2.1 Magnetron Oscillator



A I

**Fig. 6.2** Fig. A shows the cross-section of a cylindrical cavity with electron circulating the space between inner and outer conductor. Fig. B shows how inner cylindrical cavity is aperture coupled to outer cavities with same resonant frequency.

In fig. 6.2A, we have a coaxial cylindrical cavity as shown. Consider $TE$ mode $\phi$ that has angular dependence $\cos\theta$ and frequency $\omega_0$. If we introduce an electron in the cavity and a magnetic field $B_0$ along the $z$ direction. Then this electron will presses around the magnetic field at the angular velocity $\omega = \frac{eB}{m}$. If we choose $B_0$ such that $\omega = \omega_0$, then the electron orbits in the cavity at angular frequency $\omega_0$. This comprises a circulating current density that is nonzero at angular position $\theta = \omega t$ and hence its overlap with the $TE$ mode ($E_\phi$ part of it) is of the form $\cos\omega_0 t$. Thus we have a mode at frequency $\omega_0$ and we are exciting it as $\cos\omega_0 t = \frac{\exp(j\omega_0 t)+\exp(-j\omega_0 t)}{2}$. Thus we excite a linear system at resonance, we will build up this mode $\phi$ . Rapidly the cavity will be filled up with this mode and if $\omega_0$ is in the GHz range we produce microwaves.

We talked about the $TE$ mode. But we also have $TM$ modes. Let $\phi_1$ be a TM mode that has angular dependence $E_\phi \propto \sin\theta$ and some frequency $\omega_0$. If we introduce an electron in the cavity and a magnetic field $B_0$ along $Z$ direction. Then this electron will presses around the magnetic field at the angular velocity $\omega = \frac{eB}{m}$. If we choose $B_0$ such that $\omega = \omega_0$, then the electron orbits in the cavity at angular frequency $\omega_0$. This comprises a circulating current density that is nonzero at angular position $\theta = \omega t$ and hence its overlap with the $TM$ mode ($E_\phi$ part of it) is of the form $\sin\omega_0 t$. Thus we have a mode at frequency $\omega_0$ and we are exciting as $\sin\omega_0 t = \frac{\exp(j\omega_0 t)-\exp(-j\omega_0 t)}{2j}$. Thus we excite a linear system at resonance, we will build up this mode $\phi$ . Rapidly the cavity will be filled up this mode and if $\omega_0$ is in GHZ range we produce microwaves.

Thus we are saying that if the precession frequency of electron matches the frequency of either the $TE$ and $TM$ mode with $\cos\theta$ dependence we will excite these modes. We also have modes with angular dependence $\cos n\theta$ and frequency $\omega_0$. Then it is excited with electron with precession frequency $\frac{\omega_0}{n}$. The electron is injected into cavity from inner conductor that acts as cathode. The electrons are injected radially outwards and under the Lorentz force of the magnetic field acquire a circular orbit as shown in fig. (6.2)A. The strength of magnetic field determines the angular velocity $\omega$ as $\omega = \frac{eB}{m}$.

Fig. (6.2)B shows cartoon of a microwave oscillator called Magnetron. The inner conductor is a cathode that injects electrons. The Central conductor is anode that attracts these electrons radially and makes them acquire a velocity which under magnetic field gets turned into a circular orbit. Between center conductor and outer conductor are circular chambers/cavities with resonate at frequency $\omega_0$ and can store microwaves. Without these cavities, the volume of central cavity is small, and as we fill the cavity with microwaves it offers increased resistance to motion of electron. Observe the circulating electron creates mode $-\phi$ and resulting $E_\phi$ opposes electron motion. By adding more room, in terms of outer cavities, we store same energy a lower $E_\phi$ value and reduced resistance. Thus we have an arrangement where we have central cavity as in Fig. (6.2)A aperture coupled to outer cavities all of which resonate at same frequency. We excite the central cavity by cyclotron motion of electron, the outer cavities provide more storage room.

### 6.2.2 Klystron and Gyrotron

In magnetron, we have a cylindrical mode with $\cos theta$ dependence and frequency $\omega_0$. We traverse the mode with an electron rotating circularly at angular frequency

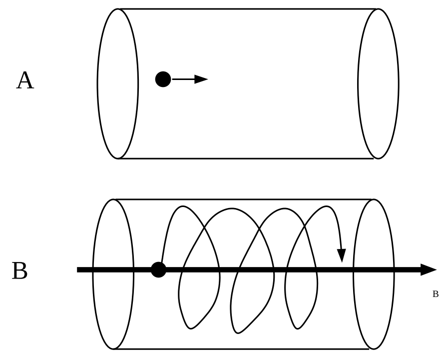



**Fig. 6.3** Fig. a and B. show cylindrical cavity being resonantly excited by linear and gyrating motion of electron respectively.

$\omega_0$, which excites the mode resonantly as $\cos \omega_0 t$. Now consider cylindrical cavity again (without central conductor). We have TM modes in the cavity with $E_z$ having $\cos kz$ dependence. If we make an electron travel along the $z$ direction with velocity $\upsilon$ as shown in Fig. 6.3. Then we are exciting this mode as $\sin k\upsilon t$ with $\omega = k\upsilon$, we can write the excitation as $\sin \omega t$. Now if $\omega$ matches $\omega_0$ the resonant frequency of the mode then we have excited the mode with an rectilinear motion. This is the principal of a Klystron a device used to produce microwaves especially for the Radar application.

In magnetron, we made the electron move in circles which talked to $\cos\theta$ dependence of the mode. In Klystron, we made the electron move in $z$ direction and it talked to $\cos kz$ dependence of the mode. Now we can do both. We can make the electron gyrate as shown in Fig. 6.3B, where we travel vertically along $z$ and also in loops. The loopy motion comes from an applied magnetic field. If $\omega_1$ is the angular (loopy) velocity of the electron and electron travels along the $z$ direction with velocity $\upsilon$, such that $k\upsilon = \omega_2$. Then we have excitation of the mode as $\cos \omega_1 t \cos \omega_2 t$, i.e. we excite the mode at frequency $\omega_1 \pm \omega_2$. If the frequency of mode is say $\omega_0 = \omega_1 + \omega_2$, we obtain resonant excitation of the mode. Furthermore as we we add two frequencies $\omega_1 + \omega_2$, we can get high resulting frequency $\omega_0$. This mechanism of producing microwaves by gyrating motion is realized in device called Gyrotron.

### 6.2.3 Microwave systems and networks

We have talked about microwave cavities. We have talked about waveguides. Now we talk about microwave networks made out of these systems. See Fig. 6.4, which shows a cylindrical cavity (source cavity) acting as a microwave source (a) coupled to a waveguide (b), which is terminated in an microwave cavity (load cavity) (c),

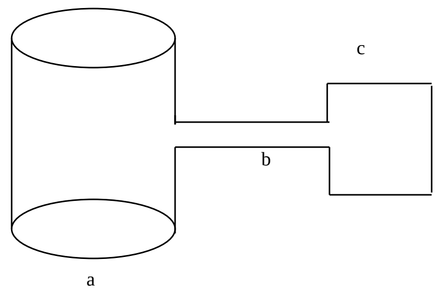



**Fig. 6.4** Fig. shows a microwave network with source and load cavity connected by a waveguide.

which could be a cavity as in microwave oven where we heat our food. The modes of the source are appropriately excited and we build these modes.The EM radiation is then channelled through a waveguide to resonator. How to analyze such networks.

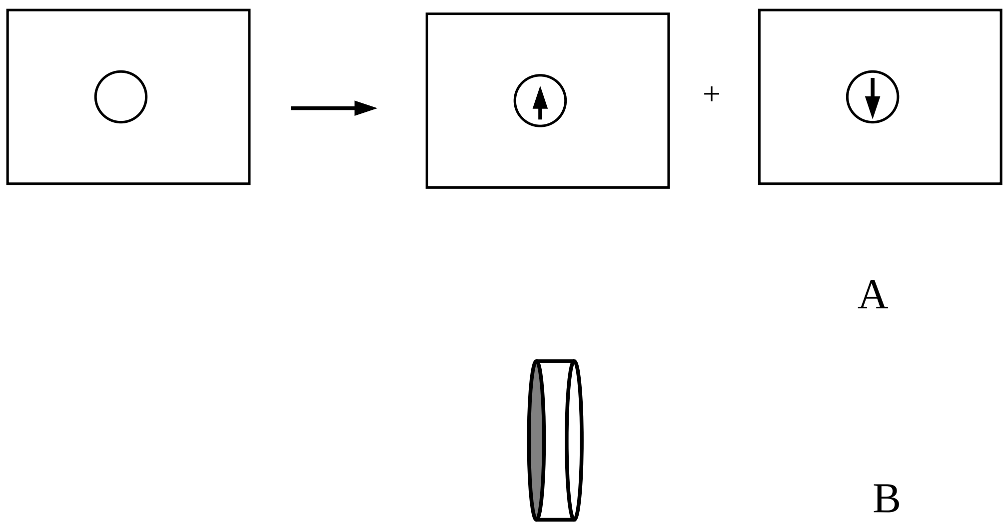



**Fig. 6.5** Fig. A shows how an aperture is modelled as mode current with no aperture and a fictitious current that opposes this current. Fig. B shows electric field in infinitesimal volume inside the wall where it exponentially decays.

Lets say source cavity has a mode $\phi_0$ at frequency $\omega_0$. Lets normalize the mode such that $\langle \phi_0, \phi_0 \rangle = 1$, $(\langle \phi_0, \phi_0 \rangle = \int \varepsilon |E|^2 + \mu |H|^2)$. When we excite the source cavity mode resonantly its amplitude $x$ builds up linearly as time as $\dot{x} = u$. Now suppose we make a small aperture of area $A_e$ in the wall of the source cavity to let microwaves out. We can analyze the aperture by assuming that there is no aperture, in which case the mode will cause certain current on the walls of the waveguide and now we subtract this current by assuming a fictitious current $i_0$, opposite to mode current on the wall. This is as shown in 6.5A. This fictitious current $i_0$ radiates inside and outside the cavity. The radiation inside produces $H$ in proximity to current that opposes the $H$ due to mode current and hence the fictitious current radiates inwards to de-excite the mode. Since fictitious mode is proportional to mode amplitude, we can write

$$\dot{x} = -\alpha_0^2 x + u. \tag{6.19}$$

How much is $\alpha^2$. Let $E$ be the electric field value of $\phi_0$ at the aperture. If the wall was real this $E$ will exponentially decay in the wall and penetrate length $l$ inside the wall. Inner product of current $j = \sigma E$ with this $\phi_0$ gives,

$$\alpha_0^2 = \frac{\omega_0 l A}{V}, \tag{6.20}$$

where $A$ is aperture area, and $V$ is mode volume and $\omega_0$ mode frequency,

$$\dot{x} = -\alpha_0^2 x + u, \tag{6.21}$$

So we are exciting the mode but it leaks out of aperture and decays at rate $\alpha_0^2$. We can solve the above equation and we get steady amplitude of mode in the cavity as $x = \frac{u}{\alpha_0^2}$.

Forget loss for a moment. What if we excited the cavity away from resonance frequency $\omega_0$ at $\omega$ . Then we can describe the the mode amplitude at frequency $s = j\omega$ as $x(s) = \frac{1}{s-a}u(s)$ where $a = j\omega_0$. The quantity $G(s) = \frac{1}{s-a}$ is called the cavity transfer function. At $s = j\omega_0$ it becomes infinite as on resonance the cavity amplitude grows unbounded. This is as shown in Fig. 6.6A. In presence of loss now The transfer function is modified as feedback shown in Fig. 6.6B and the the new transfer function $G_1(s)$ is

$$G_1(s) = \frac{G(s)}{1+\alpha^2 G(s)} = \frac{1}{s-a+\alpha^2}. \tag{6.22}$$

Now we connect source cavity to the waveguide which is coupled to load cavity. How do we analyze this interconnect. Lets simplify and say we have two resonant cavities connected by a small aperture as shown in Fig. 6.7a.

Let $\phi_A$ be mode of cavity $A$ and $\phi_B$ be mode of cavity $B$ both with frequency $\omega_0$ before we create an aperture. Both modes will create currents on cavity wall. If modes are say identical then $\phi_A$ and $\phi_B$ will create same current where aperture is. If we consider $\phi = \phi_A - \phi_B$, the currents due to $\phi_A$ and $\phi_B$ will cancel $\phi$ has no current at the boundary where aperture is and we get a mode of the cavity with aperture.

In above we assumed identical cavities. Suppose they are different. Consider a resonant cavity connected to a waveguide as shown in Fig. 6.7b. Let $\phi_A$ be mode of cavity and $\phi_B$ be mode of waveguide both with frequency $\omega_0$ (say both modes normalized) . The aperture current distribution due to $\phi_A$ and $\phi_B$ are different in general. Lets close cavity $A$ by introducing fictitious current $i_A$ and close cavity $B$

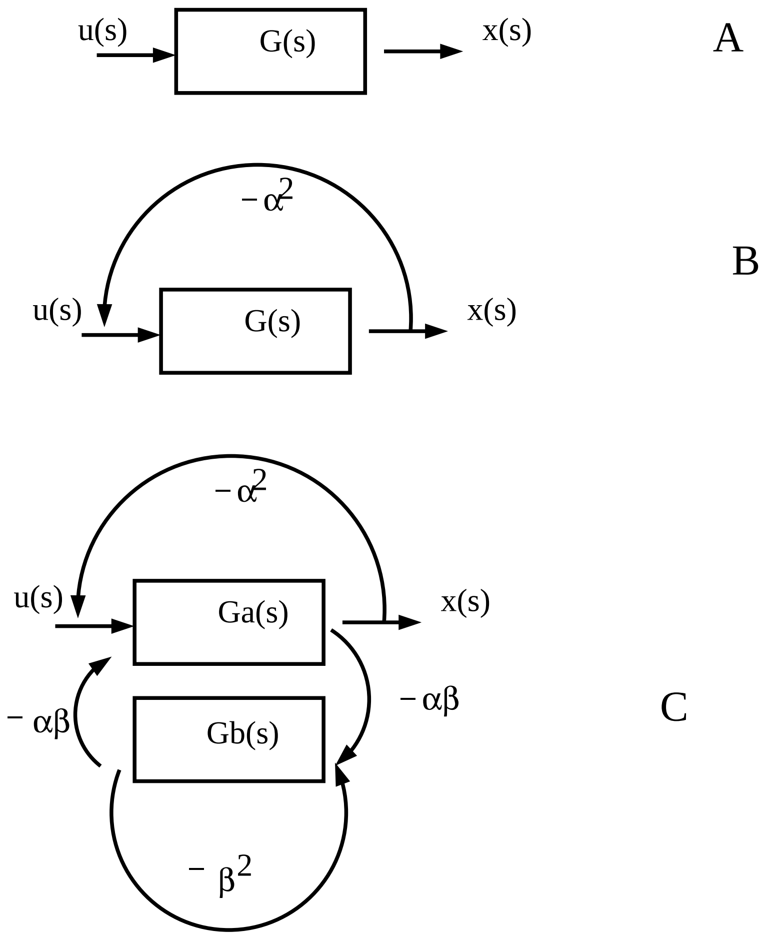



**Fig. 6.6** Fig. A shows how off-resonant excitation to a cavity is modelled as a transfer function. Fig. B shows how loss through aperture is modelled as negative feedback. Fig. C shows interconnect of two cavities through an aperture modelled as a feedback loop.

by introducing fictitious current $i_B$. If $x$ and $y$ are the amplitudes of modes $\phi_A$ and $\phi_B$ we can write

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\alpha_0^2 & -\alpha\beta \\ -\alpha\beta & -\beta^2 y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ 0 \end{bmatrix}. \tag{6.23}$$

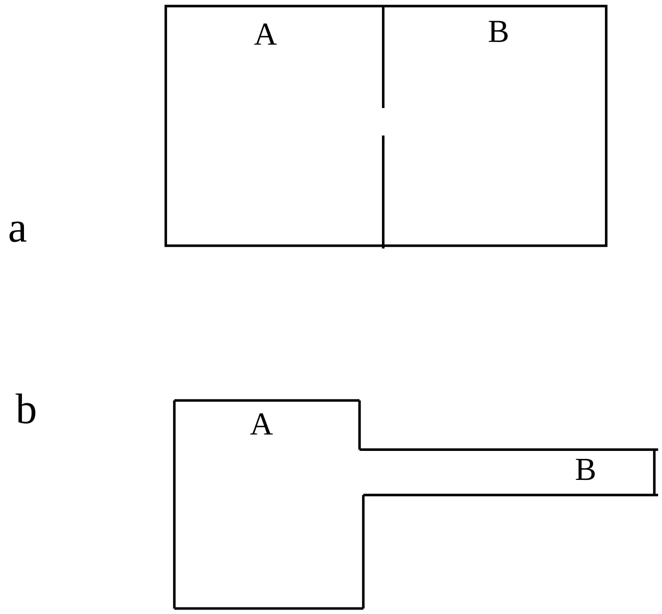



**Fig. 6.7** Fig. A and B shows aperture coupling of two identical and different cavities respectively.

Since the two cavities have different volumes, we find $\alpha^2 \neq \beta^2$. from Eq. (**??**), we find smaller cavity has larger loss. Furthermore $\beta\phi_A - \alpha\phi_B$ is the joint mode that doesnot dissipate. When we excite, this mode builds up. Now suppose the two cavities donot share same resonance. Let resonances be $a$ and $b$. Then the the joint transfer function as in fig. 6.6C is

$$G(s) \propto \frac{1}{1-\frac{\alpha^2\beta^2}{(s-a+\alpha^2)(s-b+\beta^2)}} \propto \frac{1}{(s-a)(s-b)+\beta^2(s-a)+\alpha^2(s-b)} \tag{6.24}$$

Now see two limits, when $\alpha \gg \beta$ then $G(s) \propto \frac{1}{(s-a)(s-b)+\alpha^2(s-b)}$ i.e, we have resonance at $a$. When $\alpha \ll \beta$, we have $G(s) \propto \frac{1}{(s-a)(s-b)+\beta^2(s-a)}$ i.e, we have resonance at $b$. For intermediate values it is a weighted sum. If we assume cavity volume is much larger than the waveguide volume then the final resonance is more weighted towards cavity resonance. If both waveguide and cavity resonance $a$ then as expected the joint system gas has resonance at $a$ irrespective of $\alpha, \beta$ values.

Now return to Fig. 6.4. Suppose source and cavity have resonance at $a$ and suppose waveguide is off, may be its length is not correct. The length is decided between where source and destination are and not much can be done then. Then the joint resonance of waveguide and load cavity will be different and we donot do a good job of exciting this system by source. How can we change the waveguide resonance without changing its length so that everything is on resonance, i.e., impedance matched. This is called impedance matching.

Other scenario is that the load cavity resonance goes off a bit because we put something in it. Then we can change the waveguide resonance so that jointly they come back to the source resonance. In either case, we have to study how to vary the resonance of the waveguide. There are many ways to do it and we study them under impedance matching.

## 6.3 Impedance matching waveguides

### 6.3.1 Taper

Recall of a waveguide with $TE_{mn}$ mode we have

$$\beta = \sqrt{\frac{\omega^2}{c^2} - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}. \tag{6.25}$$

Lets focus on $TE_{11}$ mode with

$$\beta = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2} - \frac{\pi^2}{b^2}}. \tag{6.26}$$

If $l$ is the length of the waveguide then if $\beta l = p\pi$, then waveguide is resonant with $\omega$. If $(p-1)\pi < \beta l < p\pi$, then we can increase the length and bring waveguide into resonance with $\omega$. $l$ is fixed so we can change the dimension of waveguide and make say $a$ and $b$ as functions of $z$ as shown in fug. 6.8, where in 6.8a we change $b$

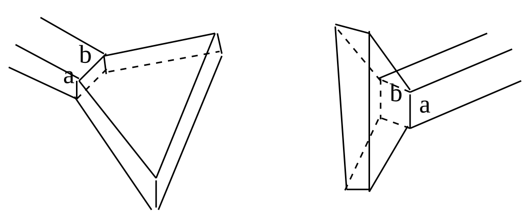



**Fig. 6.8** Fig. a shows how we change $b$ dimension of waveguide. Fig. b shows how we change $a$ dimension of waveguide

as function of $z$ and in 6.8b we change $a$ as function of $z$. Then

$$\beta(z) = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2(z)} - \frac{\pi^2}{b^2(z)}}. \tag{6.27}$$

As we increase say $a(z)$, $\beta$ increases and now $\int_0^l \beta(z)dz = p\pi$ and we bring waveguide into resonance. ofcourse this requires changing structure of waveguide permanently. We prefer instead to not do permanent change to waveguide structure and instead introduce alterations, that change the impedance as described below. For this we first develop a transmission line model of waveguide.

### 6.3.2 Transmission line models

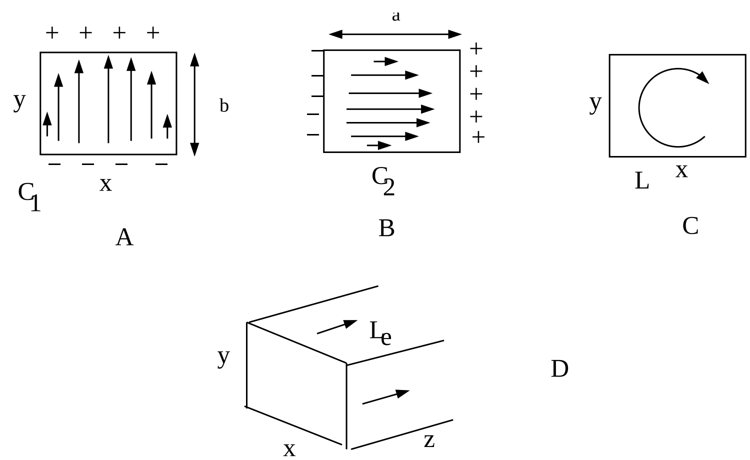



**Fig. 6.9** Fig. shows capacitors and inductors in TE mode.

To fix ideas consider a waveguide with $TE_{11}$ mode. Cross section of the waveguide shows electric field $E_y$ in the $y$ and $E_x$ in the $x$ direction as in Fig. 6.9A and B respectively. The electric field deposits charges on the boundary and the cross section acts as capacitor. Lets see the field $E_y$, it has the form $E_0 \sin(\frac{\pi x}{a})\cos(\frac{\pi y}{b})$ and it is simply $E_0 \sin(\frac{\pi x}{a})$ at the top and bottom boundary, which gives a charge density $\frac{\sigma}{2\varepsilon} = E_y$. We can integrate the charge density to get total charge $Q$ and then using the fact that energy stored in electric field $\int \varepsilon_0 E_y^2 = \frac{Q^2}{C}$, we get the capacitance as $\varepsilon_0 \frac{a\Delta x}{\pi b}$ where $\Delta x$ is the small length in the $z$ direction, or capacitance per unit length is $C_1 = \varepsilon_0 \frac{a}{\pi b}$. Similarly $E_x$ gives $C_2 = \varepsilon_0 \frac{b}{\pi a}$. We remark if we have $TE_{mn}$ mode instead of $TE_{11}$ then the capacitance $C_1 = \varepsilon_0 \frac{a}{\pi m b}$ and $C_2 = \varepsilon_0 \frac{b}{\pi n a}$.

The energy oscillates between electric field $E_x, E_y$ and magnetic fields. The magnetic field $H_z$ corresponds to circular currents as shown in 6.9C, which gives inductance $L$, the value of the inductance is just the area of the loop $\mu_0 ab$. For thickness $\Delta x$, we get $L \to \frac{L}{\Delta x}$. $H_z$ is not the only magnetic field, we also have magnetic field $H_x, H_Y$, which correspond to longitudinal currents as shown in Fig. 6.9D. Lets see the field $H_y$, it has the form $H_0 \sin(\frac{\pi y}{b})\cos(\frac{\pi x}{z})$, and it is simply $H_0 \sin(\frac{\pi y}{b})$ at the side wall, which gives a current density $\frac{j}{2} = H_y$. We can integrate the current density to get total current $I$, and then using the fact that energy stored in the magnetic field $\int \mu_0 H_y^2 = LI^2$, we get the inductance as $\mu_0 \frac{a\Delta x}{\pi b}$ where $\Delta x$ is the small length in the $z$ direction, or inductance per unit length is $L_1 = \mu_0 \frac{\pi a}{b}$. Similarly $H_x$ gives $L_2 = \mu_0 \frac{\pi b}{a}$. We remark, if we have $TE_{mn}$ mode instead of $TE_{11}$ then the inductance $L_1 = \mu_0 \frac{\pi an}{b}$ and $L_2 = \mu_0 \frac{\pi bm}{a}$. Let

$$L_e = L_1 + L_2. \tag{6.28}$$

Then we have a transmission line model as shown in Fig. 6.10, with line length signifying longitudinal $z$ direction. Then we can write transmission line equations

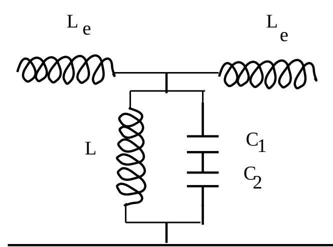



**Fig. 6.10** Fig. shows transmission line model for TE mode.

$$-j\beta V(z) = j\omega L_e I(z), \tag{6.29}$$

$$-j\beta I(z) = (j\omega C_e + \frac{1}{j\omega L})V(z), \tag{6.30}$$

where $C_e^{-1} = C_1^{-1} + C_2^{-1}$. On multiplying the above two equations, this gives with

$$L_e C_e = c^{-2}. \tag{6.31}$$

$$\omega^2 = c^2\beta^2 + \frac{1}{LC_e} = c^2\beta^2 + \frac{1}{L}(\frac{1}{C_1} + \frac{1}{C_2}), \tag{6.32}$$

$$\frac{\omega^2}{c^2} = \beta^2 + (\frac{\pi}{a})^2 + (\frac{\pi}{b})^2. \tag{6.33}$$

or

$$\beta = \sqrt{\frac{\omega^2}{c^2} - (\frac{\pi}{a})^2 - (\frac{\pi}{b})^2}. \tag{6.34}$$

We now show we can change the capacitance of the cross section and get

$$\beta(z) = \frac{1}{c}\sqrt{\omega^2 - \frac{1}{L}(\frac{1}{C_1} + \frac{1}{C_2})}. \tag{6.35}$$

There are various ways to change the capacitance of cross-sections, we study them below.

Before we move forward, we develop a transmission line model for $TM$ mode.

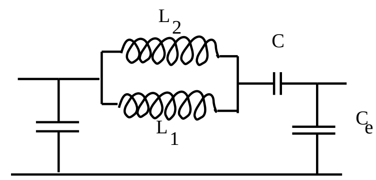



**Fig. 6.11** Fig. shows transmission line model for TM mode.

To fix ideas, consider a waveguide with $TM_{11}$ mode. Cross section of the waveguide shows magnetic field field $H_y$ in the $y$ and $H_x$ in the $x$ direction. These fields give longitudinal currents on side wall and top wall of the waveguide. The energy oscillates between magnetic field $H_x, H_y$ and the electric fields. The electric field $E_z$ corresponds to charge along longitudinal direction. These currents and charge along longitudinal direction are represented by two inductors $L_1$ and $L_2$ (for side and top wall) in parallel and a series capacitor $C$ as shown in Fig. 6.11. $L_1$ and $L_2$ are same as in TE mode. The capacitance $C = \frac{\varepsilon_0 ab}{\Delta}$. The transverse electric fields deposits charge on top and side wall of waveguide cross section and represented by capacitor

$$C_e = C_1 + C_2, \tag{6.36}$$

where $C_1$ and $C_2$ are same as in TE mode.

Then we have a transmission line model as shown in Fig. 6.11, with line length signifying longitudinal $z$ direction. Then we can write transmission line equations

$$-j\beta V(z) = (j\omega L_e + \frac{1}{j\omega C})I(z), \tag{6.37}$$

$$-j\beta I(z) = j\omega C_e V(z), \tag{6.38}$$

where $L_e^{-1} = L_1^{-1} + L_2^{-1}$. This gives with $C_e L_e = c^{-2}$.

$$\omega^2 = c^2\beta^2 + \frac{1}{CL_e} = c^2\beta^2 + \frac{1}{C}(\frac{1}{L_1} + \frac{1}{L_2}), \tag{6.39}$$

$$\frac{\omega^2}{c^2} = \beta^2 + (\frac{\pi}{a})^2 + (\frac{\pi}{b})^2, \tag{6.40}$$

or

$$\beta = \sqrt{\frac{\omega^2}{c^2} - (\frac{\pi}{a})^2 - (\frac{\pi}{b})^2}. \tag{6.41}$$

### 6.3.3 Iris

Now focus on TE modes.

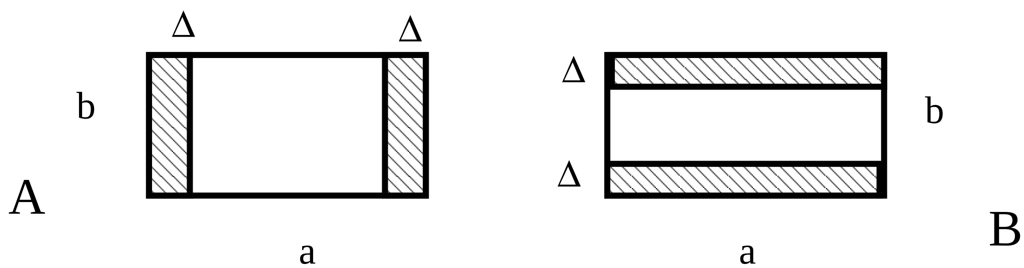



**Fig. 6.12** Fig. shows how we change cross-section of a waveguide using metal strip called iris.

We change the cross section as in fig. 6.12A. The two capacitance $C_1$ and $C_2$ change from $\frac{\varepsilon_0 a}{b}$ and $\frac{\varepsilon_0 b}{a}$ to $\frac{\varepsilon_0(a-\Delta)}{b}$ and $\frac{\varepsilon_0 b}{(a-\Delta)}$ respectively and we get

$$\beta = \sqrt{\frac{\omega^2}{c^2} - (\frac{\pi}{a})^2 - (\frac{\pi}{b})^2 + \frac{\Delta}{a}((\frac{\pi}{a})^2 - (\frac{\pi}{b})^2)}. \tag{6.42}$$

Since $a > b$ we find that $\beta$ decreases.

We change the cross section as in fig. 6.12B. The two capacitance $C_1$ and $C_2$ change from $\frac{\varepsilon_0 a}{b}$ and $\frac{\varepsilon_0 b}{a}$ to $\frac{\varepsilon_0 a}{b-\Delta}$ and $\frac{\varepsilon_0(b-\Delta)}{a}$ respectively and we get

$$\beta = \sqrt{\frac{\omega^2}{c^2} - (\frac{\pi}{a})^2 - (\frac{\pi}{b})^2 - \frac{\Delta}{b}((\frac{\pi}{a})^2 - (\frac{\pi}{b})^2)}. \tag{6.43}$$

Since $a > b$ we find that $\beta$ increases.

Thus using iris as above we can increase or decrease $\beta$ for a cross-section and impedance match.

### 6.3.4 Post and Screws

Focus on TE modes.

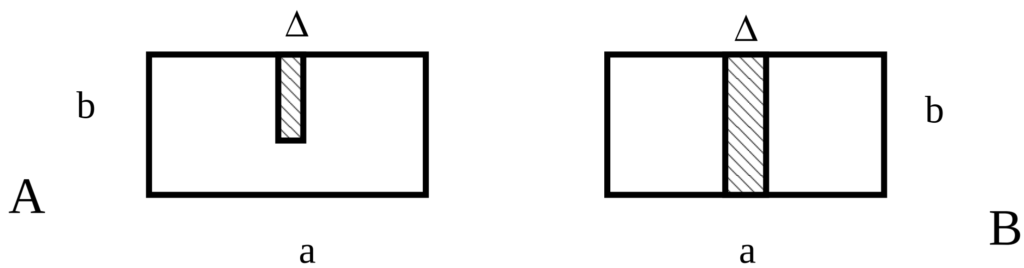



**Fig. 6.13** Fig. shows how we change cross section of a wave guide using a tuning screw.

In addition to using a waveguide iris, post or screw can also be used to give a similar effect and thereby provide waveguide impedance matching. Fig. 6.13A shows a screw coming down partly. Its effect is to increase both $C_1$ and $C_2$ and hence $\beta$ increases.

To decrease $\beta$ screw or post should extend through the waveguide completely making contact with both top and bottom walls as in Fig. 6.13B. The effect is same as iris in Fig. 6.12A and hence $\beta$ decreases.

Screw comes from top to bottom. Posts go bottom to up.

### 6.3.5 Plungers

As discussed before in discussion of tapered transmission lines we can change $a$ and $b$ locally and change $\beta(z)$ by the equation.

$$\beta(z) = \sqrt{\frac{\omega^2}{c^2} - (\frac{\pi}{a(z)})^2 - (\frac{\pi}{b(z)})^2}. \tag{6.44}$$

This can be achieved by aid of plungers as shown in Fig. 6.14A and B.

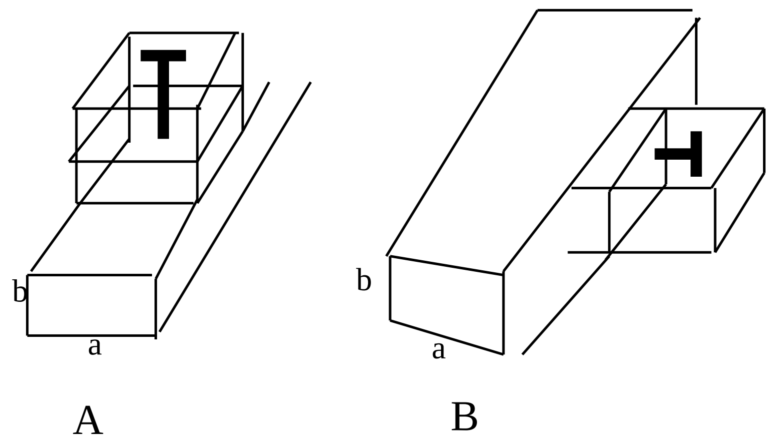



**Fig. 6.14** Fig. shows how cross section of a waveguide is changed using plungers.

### 6.3.6 Rf-oscillators

Rf oscillators can be made with inductors and capacitors in series or parallel. Fig. 6.15a shows the series version where $v_0$ is the applied voltage and $v_1$ voltage across the capacitor. The current through the capacitor is $c\frac{dv_1}{dt}$ and hence the voltage across the inductor is $L\frac{d}{dt}(C\frac{dv_1}{dt})$ giving

$$v_0 - v_1 = LC\frac{d^2v_1}{dt^2}, \tag{6.45}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$,

$$\omega_0^2 v_0 = \frac{d^2 v_1}{dt^2} + \omega_0^2 v_1. \tag{6.46}$$

let $x = v_1$ and $y = \frac{1}{\omega_0}\frac{dv_1}{dt}$.

Then

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_0 v_0 \end{bmatrix}. \tag{6.47}$$

Using *variation of constant formula*,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \omega_0 v_0 \int_0^t \begin{bmatrix} \sin\omega_0\sigma \\ \cos\omega_0\sigma \end{bmatrix} d\sigma = v_0 \begin{bmatrix} 1-\cos\omega_0 t \\ \sin\omega_0 t \end{bmatrix}. \tag{6.48}$$

We find the voltage $v_1$ is oscillating.

Fig. 6.15b shows the parallel version of rf-oscillator where we use a current source. Fig. 6.15c shows implementation of a current source using a transistor circuit. The current through inductor is $i_2$ and therefore voltage across it is $L\frac{di_1}{dt}$ which

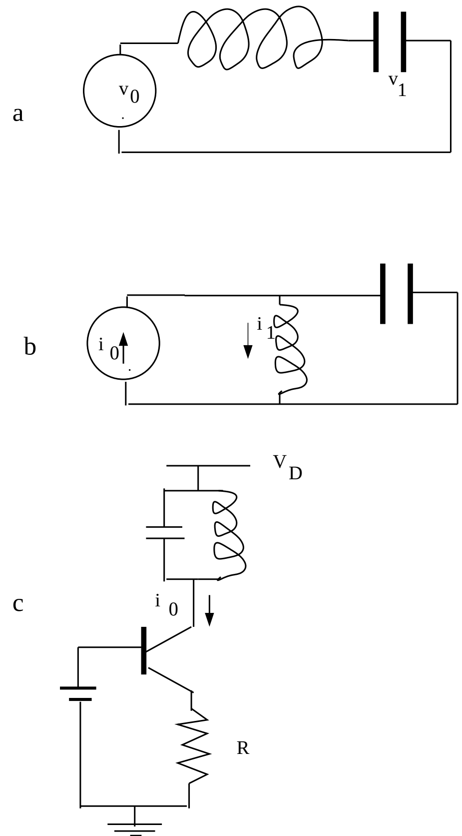



**Fig. 6.15** Fig. a shows an oscillator that uses a voltage source. Fig. b shows an oscillator that uses a current source. Fig. c shows a transistor implementation of Fig. b.

is also the voltage across capacitor and hence current through capacitor is $C\frac{d}{dt}(L\frac{di_1}{dt})$, which gives

$$i_0 - i_1 = LC\frac{d^2 i_1}{dt^2}, \tag{6.49}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$,

$$\omega_0^2 i_0 = \frac{d^2 i_1}{dt^2} + \omega_0^2 i_1. \tag{6.50}$$

Let $x = i_1$ and $y = \frac{1}{\omega_0}\frac{di_1}{dt}$.

Then

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_0 i_0 \end{bmatrix}. \tag{6.51}$$

Using *variation of constant formula*,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \omega_0 i_0 \int_0^t \begin{bmatrix} \sin \omega_0 \sigma \\ \cos \omega_0 \sigma \end{bmatrix} d\sigma = i_0 \begin{bmatrix} 1 - \cos \omega_0 t \\ \sin \omega_0 t \end{bmatrix}. \tag{6.52}$$

We find the voltage across inductor $y$ is oscillating.

Ofcourse, oscillators have resistance, so we need some positive feedback to overcome the loss. In addition, we need to rectify the positive feedback, to have a signal dependent gain. Some common oscillators based on this principle are Collpitt oscillator, Hartley oscillator.

## 6.4 Tank Circuits

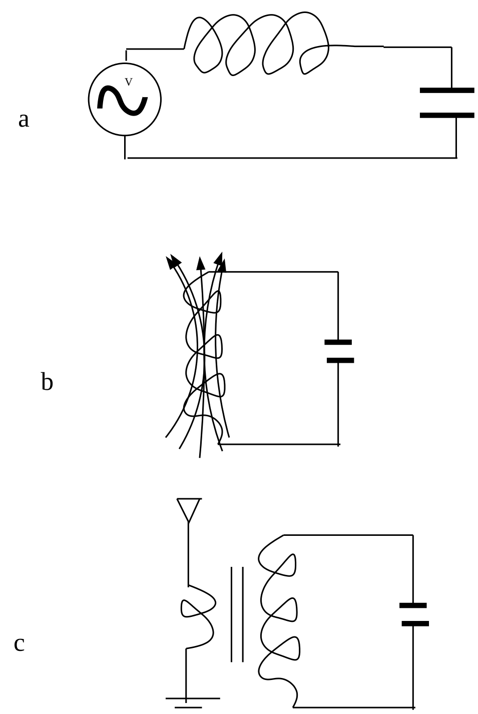



**Fig. 6.16** Fig. a shows a tank circuit. Fig. b shows a tank circuit in NMR detection. Fig. c shows a tank circuit in a radio receiver.

Fig. 6.16 shows a tank circuit to amplify small oscillating voltage $v_0 = A\cos\omega_0 t$ . Let $v_1$ be voltage across the capacitor. The current through the capacitor is $c\frac{dv_1}{dt}$ and hence the voltage across the inductor is $L\frac{d}{dt}(C\frac{dv_1}{dt})$ giving

$$v_0 - v_1 = LC\frac{d^2 v_1}{dt^2}, \tag{6.53}$$

with $\omega_0 = \frac{1}{\sqrt{LC}}$,

$$\omega_0^2 v_0 = \frac{d^2 v_1}{dt^2} + \omega_0^2 v_1. \tag{6.54}$$

Let $x = v_1$ and $y = \frac{1}{\omega_0}\frac{dv_1}{dt}$.

Then

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_0 v_0 \end{bmatrix}. \tag{6.55}$$

Using *variation of constant formula*,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \omega_0 A \int_0^t \cos\omega_0(t+\sigma)\begin{bmatrix} \sin\omega_0\sigma \\ \cos\omega_0\sigma \end{bmatrix} d\sigma \tag{6.56}$$

$$= \frac{A\omega_0 t}{2}\begin{bmatrix} -\sin\omega_0 t \\ \cos\omega_0 t \end{bmatrix}. \tag{6.57}$$

We find the voltage $v_1$ grows linearly, i.e, amplified.

### 6.4.1 NMR and radio receiver and tank circuit

Tank circuits are used in field of nuclear magnetic resonance (NMR) to detect nuclear spins. The magnetic moments of nuclear spins constitutes a flux passing through an inductor coil as shown in 6.16b. The precession of the spins in a magnetic field cause this flux to sinusoidally oscillate which induces a very small EMF in the coil. This can then be amplified using a tank circuit.

Tank circuits find use in radio receiver circuits as shown in 6.16c where antenna voltage is amplified using a tank circuit.

### 6.4.2 Resonant radio antenna and tank circuit

Shown in 6.17a is a Hertz dipole antenna. The applied voltage pulls charges towards it, making ends of the antenna charged. This charge opposes the pull. If we bend the ends as shown in 6.17b, it begins to look like a capacitor, where charges on capacitor plate oppose the pull. Therefore the antenna has a capacitance and of-course it has inductance as current flowing in antenna produces a magnetic field and change in this current or resulting magnetic field produces an EMF that opposes the change and finally the antenna has some resistance. Therefore antenna is like an RLC circuit

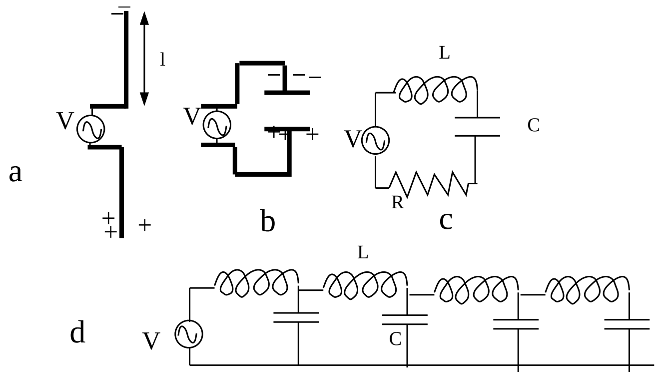



**Fig. 6.17** Fig. shows how resonant antenna acts a tank circuit and has gain inbuilt in it.

as shown in 6.17c. Neglecting resistance for a moment, the resonant frequency of this circuit is $\omega_0 = \frac{1}{\sqrt{LC}}$. If driving voltage $V$ is at this frequency, the impedance of the antenna vanishes and we see a large current flow in it and antenna efficiently transmits. Ofcourse in fig. 6.17a charge is not just concentrated at the ends but there is a charge density along the length. Hence the capacitance is distributed in the form of a transmission line as shown in 6.17d. If we choose the length of line (antenna) as $l$ such that $\beta l = \frac{\pi}{2}$, then the impedance of the antenna is $-jZ_0 \cot \beta l$ which becomes 0 at $\beta l = \frac{\pi}{2}$ and huge current flows (of course we are neglecting resistance) and antennna transmits efficiently.

We talked about antenna in the transmit mode. In receive mode, the incident electric field acts as a voltage source and the received voltage is the voltage across the capacitor as in 6.17c. As shown in our discussion of tank circuits, once the driving frequency is on resonance this voltage builds like crazy and is only limited by the resistance of the circuit. Once again, we can be more realistic and treat antenna as a transmission line. We excite it at one end with voltage $V_0$ and find the open circuit voltage at the other end. Since there is no current at the open end, $V_+ = V_-$ and voltage at the driving end is

$$V_0 = V_+ \exp(-j\beta l) + V_- \exp(j\beta l) = V_+ \exp(-j\beta l)(1 + \exp(j2\beta l)). \qquad (6.58)$$

When $2\beta l = \pi$, we get $V_+ \to \infty$. Thus transmission line amplifies the driving voltage $V_0$. Of course there is a resistance and infinity only means large gain as in a tank circuit.

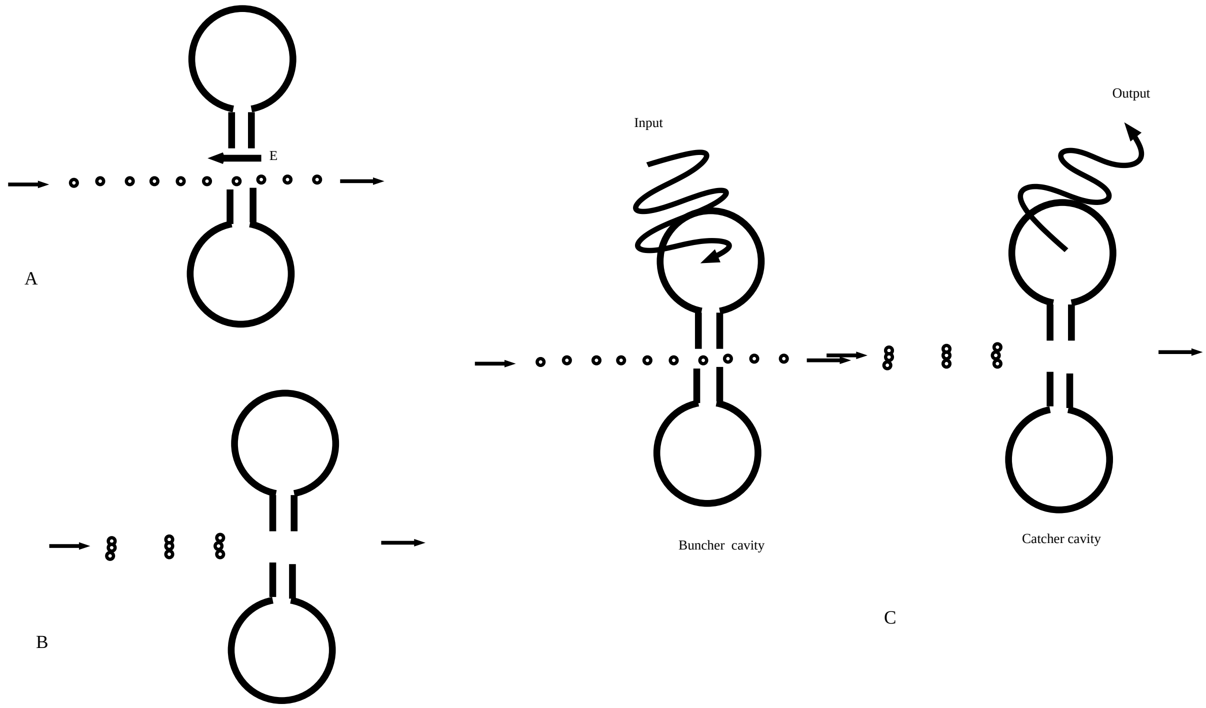



**Fig. 6.18** Fig. shows schematic of a microwave amplifier called Klystron.

## 6.5 Klystron microwave amplifier

Consider a microwave cavity as in fig. 6.18A. Suppose we pass electrons through it as shown. These will excite cavity mode as discussed in the Klystron oscillator. The electrons come one by one and trigger cavity excitation at different times. These modes are not in sync and cavity excitation doesnot build up. But suppose, we can bunch these electrons as in fig. 6.18B, so that excitation times are in resonance with the cavity, then the excitations due to electron will coherently add and we will see cavity fill up with microwaves. Now how do we bunch these electrons. We pass the electrons through a input cavity containing input microwaves that we want to amplify. The electric field due to these input signal will oscillate and oppose and accelerate the incoming electrons in a cycle. The result is in a cycle, it ends up bunching them. The bunching is more, if input signal is more, i.e, electric field is stronger. Then these bunched electrons go and fill the output cavity. When bunching is stronger output is stronger. Thus we get a linear amplifier called Klystron microwave amplifier.

## 6.6 Transistor Rf amplifiers

Rf signals can be amplified by a solid state device called transistor [7]. A transistor is made of semiconductor material like silicon. Silicon has 4 electrons in its outer shell. In a solid crystal, neighboring silicon atoms talk and these electrons of the silicon lattice organize themselves into waves states whose energies lie in a band. Wavestates are indexed by their wavenumber and each wavestate can carry two electrons. When all wavestates in a band are full as in silicon, we cannot conduct as conduction means increasing momentum or wavenumber which is not possible as the band is full. When we replace some of the silicon atoms with say phosphorus then we get extra electrons (phosphorus has 5 valence electrons instead of 4) which go into next available energy band called conduction band which is not full and can conduct. We say we have *n-doped* silicon. Similarly when you replace some of the silicon atoms with say aluminium (aluminium has 3 valence electrons instead of 4) then we get less electrons than native silicon which means some of wavestates of the filled band (also called valence band) become available and it can conduct. Valence band has lower energy than the conduction band. The energy gap is called the band gap. Fig. 6.19A shows bands for silicon, B shows bands for n-doped silicon and C shows bands for p-doped silicon.

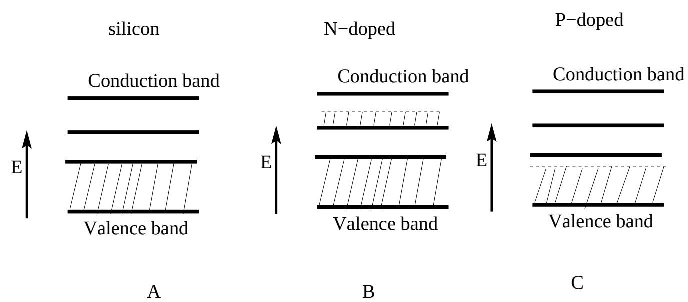



**Fig. 6.19**

When we bring a n-doped silicon next to a p-doped silicon, we form a junction called a pn junction. Some conduction electrons will flow from the n-region go the valence electrons in p-region to minimize energy. This will result in build up of $+$ and $-$ charge in the $n$ and $p$ region at the junction, which creates an electric field that eventually halts this flow. This interface filed corresponds to a voltage of .7 V. When we apply positive voltage to the p-region compared to n region, we pull valence electrons from p-region which are substituted by flow of conduction electrons from $n$ region and current flows. For this to happen the applied voltage should be larger to overcome the interface field which is roughly .7 V. We say we have forward biased the pn junction. if instead we apply positive pull on n electrons we cannot pull anything as the valence electrons in n-region are immobile and the conduction electrons can only be substituted by valence lectrons from p region but they have much lower energy. Hence current cannot flow and we say we have reverse biased the pn junction. pn junction only conducts in one direction.

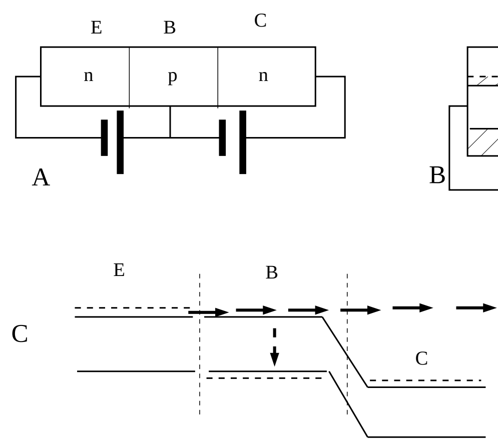



**Fig. 6.20** The fig A shows schematic ofa transistor. Fig. B shows a cartoon of how electrons and bands are in various regions without taking into account interface electric fields. Fig C shows the energy level diagram of a transistor.

Transistor has two junctions as shown in 6.20A. A $p$ region called base sandwitched between two $n$ regions (called emitter and collector) as shown in fig 6.20A. One pn junction is forward biased (emitter and base) and other reverse biased (base and collector). Forward biased pn junctions suck valence elctrons from p region and **hurls** electrons from the n region in the p-region as shown in 6.20B. These electron are immediately sucked by the reverse biased junction and on their fraction of them $\alpha$ recombine and contribute to the p junction current the remaining $1-\alpha$ fraction just fly by to other n region as shown in 6.20B. If $I$ is the base current $\frac{1-\alpha}{\alpha}I = \beta I$ is the collector current and $(1+\beta)I$ the emitter current. $\alpha$ is small so $\beta$ is large and is called the current gain of the transistor. Fig. 6.20C shows energy level diagram for the transistor.

### 6.6.1 Current Source and amplifier

Transistor acts a current source. Fig. 6.21A shows by use of a forward bias and resistance $R_1$, we establish a current in emitter, simply given by $I_e = \frac{V_1 - .7}{R_1}$. This gives a collector current $\frac{\beta}{\beta+1} I_e \sim I_e$ and the potential drop across resistance $R_2$ as $I_e R_2 = \frac{R_2(V_1 - .7)}{R_1}$. Thus the voltage $V_1$ is amplified to $\frac{R_2}{R_1} V_1$, where $\frac{R_2}{R_1}$ is the amplifier

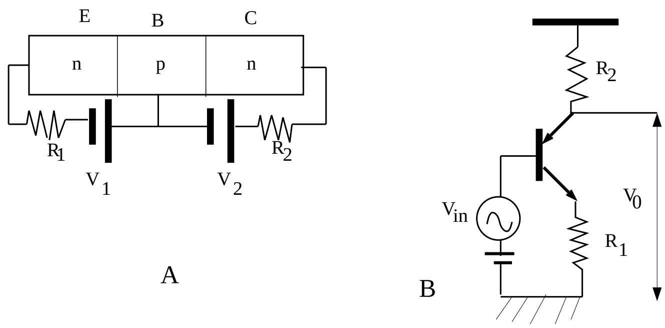



**Fig. 6.21** The fig A shows schematic of a transistor. Fig. B shows a cartoon of how electrons and bands are in various regions without taking into account interface electric fields. Fig C shows the energy level diagram of a transistor.

gain. $V_1$ can have oscillating parts as shown in figure 6.21B, which get amplified, the DC part of $V_1$ is used to bias the base emitter circuit as shown in figure 6.21B.

## 6.7 Rf and Microwave Filters

We talked about Rf and Microwave oscillators. Q of the oscillator limits the purity of resulting frequency source. Source signal may contain higher harmonics. We can improve signal purity by passing it through a band pass filter. We mention here cavity filters. We talked about cavity with transfer function

$$G(s) = \frac{1}{s - j\omega_0 + \alpha^2} \tag{6.59}$$

The factor $Q = \frac{\omega_0}{\alpha^2}$ measures the quality factor of the band-pass filter which can be high as $10^6$. The input port to the cavity couples the input signal. The signal at cavity resonance has large gain while other frequencies get attenuated. Filter also find natural application when a signal with noise needs to be amplified, we first filter out noise.

## 6.8 Cavity Coupling

The power between the cavity resonator and signal source or load can be coupled by means of a coaxial line, whose center conductor is extended inside the cavity in form of a probe 6.22A or loop 6.22B. The cavity may be coupled to a waveguide by means of an aperture or a slot in the main wall as in 6.22C and 6.22D [6].

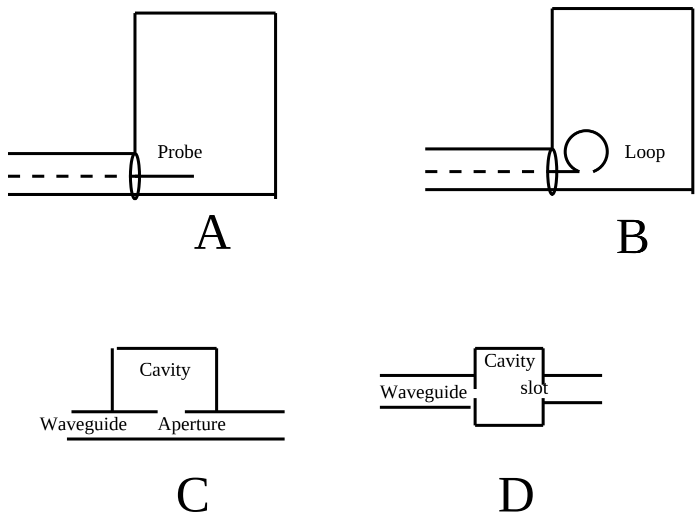



**Fig. 6.22** Fig. A shows a cavity coupled to a coax capacitively (probe coupling). Fig. B shows a cavity coupled to a coax inductively (loop coupling). Fig. C shows waveguide/cavity coupled through aperture. Fig. D shows waveguide/cavity coupled through slot

# Chapter 7
# Geometric Optics

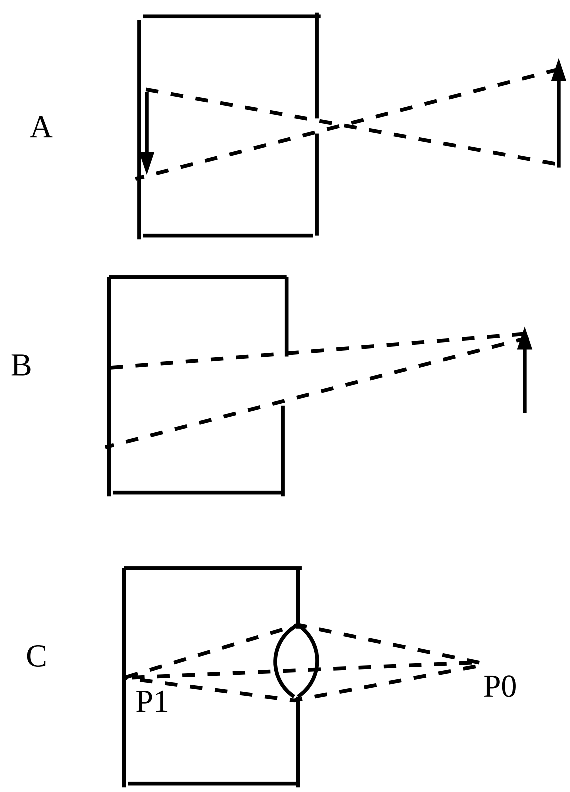



**Fig. 7.1** Fig. A shows a pin-hole camera. Fig. B shows a diffuse image if we increase the size of pin-hole. Fig. C shows how light can be focused with a lens.

## 7.1 Light Around Us

In past chapters, we talked about radio (3 kHz-3 GHz) and microwave (3-300 GHz) frequency EM waves. In this chapter, we focus on light or optical frequency (4-8 $\times$ $10^{14}$ Hz). This is all the world around us. Blue sky, green grass, light from the sun, all the world of beautiful colors. How do we see this world. Light reflects of objects and enter our eyes to form an image at the back (called retina) of the eye. Lets look at this image formation.

## 7.2 Pin Hole

We can model our eye as a pin hole camera as in 7.1A. The light rays from object after passing through pin-hole form an image on the screen as shown in 7.1A. This does not let very much light in. How can we gather all the light coming from the object. We can increase the size of the pin hole as shown in 7.1B. This produces a very diffuse image as light from a point is spread all over the screen as shown in 7.1B. How can we collect more light coming from the object and yet have a sharp image. We need a *lens* as shown in 7.1C [5]. The light emanating from each point on the object gets focused as shown in 7.1C. How does the lens work ?

## 7.3 Lens

Fig. 7.2 shows a dense spherical medium like glass. $r$ is the radius of the sphere. Light rays parallel to equator move into a rare medium like air. Consider a light ray making an angle $\alpha$ with radial line as shown in 7.2.

From Snell's laws of refraction

$$\frac{\sin\alpha+\beta}{\sin\alpha} = C. \tag{7.1}$$

$$r\sin\alpha = d\tan\beta, \tag{7.2}$$

for small $\alpha, \beta$, we have

$$\frac{\alpha+\beta}{\alpha} = C, \tag{7.3}$$

$$\frac{\alpha}{\beta} = (C-1)^{-1} = C'. \tag{7.4}$$

$$d = \frac{r}{C'}. \tag{7.5}$$

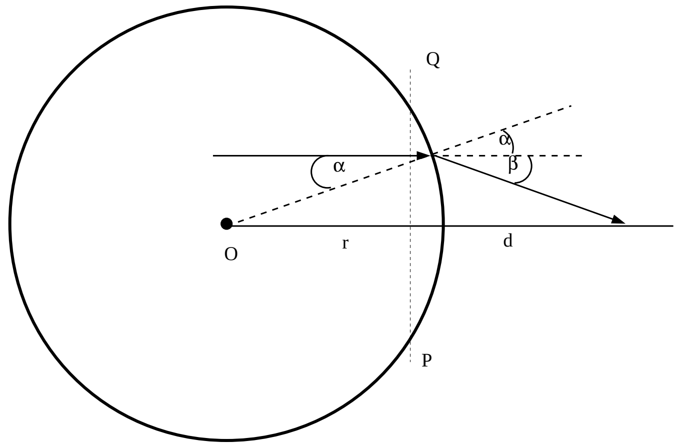



**Fig. 7.2** Fig. shows light moving from dense spherical medium to rare medium.

We find irrespective of $\alpha$, we focus at same $d$ as above. We call $\frac{d}{2} = f$, the focal point and $f$ focal length. We call $d$, 2F point. If we reverse light direction, we will start from the focal point and when light rays enter the glass medium, they become parallel. We assume all action is near equator, so that $\alpha, \beta$ are small. If we cut this sphere at the periphery, as shown by dotted line $QP$ in 7.2, we get a half lens and we join two such half lenses back to back to form a lens as shown in 7.3. Light rays emanating from 2F point become parallel inside the lens and focus on the 2F point on other side. The two lenses need not have same focal length. We can have two focal lengths $f_1$ and $f_2$. This is how the eye lens works. The focal length of the half lens that sees the world is changed to focus objects at different distances while the half lens facing retina focuses everything on retina and its focal length is not changed that much.

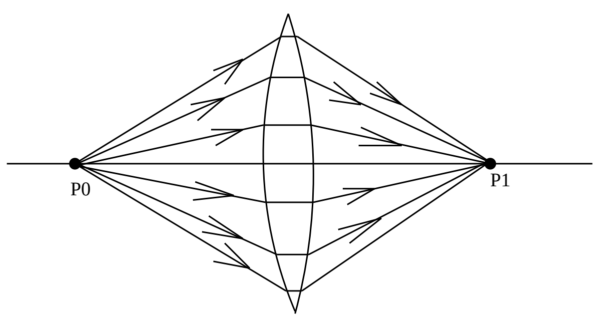



**Fig. 7.3** Fig. shows light rays emanating from 2F point become parallel inside the lens and focus on the 2F point on other side.

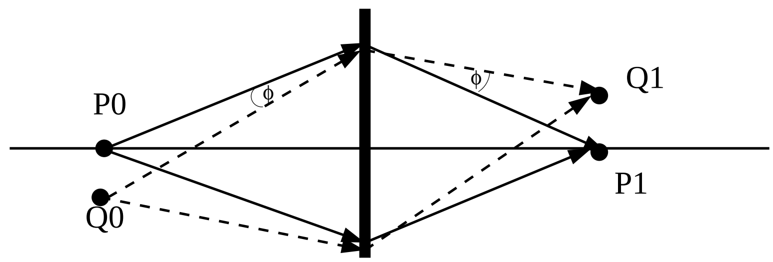



**Fig. 7.4** Fig. shows how light from point $Q_0$ shifted from 2F point converges to $Q_1$ on the other side and is shifted the other way.

We say how light rays emerging from a 2F point become parallel and focused again on the 2F point other side of lens. How about a point that is displaced vertically from the 2F point. The light rays tilt by an angle $\phi$ as shown in fig. 7.4. Fig. 7.4 shows that the point $Q_0$ converges to $Q_1$ on the other side that is shifted the other way. If focal lengths are different then if $Q_0$ is displaced by $x$ , $Q_1$ displaced by $y$ then for $\phi$ as shown in fig. 7.4,

$$\phi = \frac{x}{f_1} = \frac{y}{f_2}, \tag{7.6}$$

which gives $Q_1$ is displaced by $\frac{f_2 x}{f_1}$.

## 7.4 Microscopes

This forms the basis of what is called a microscope. If $f_2 \gg f_1$, the object gets magnified. This is shown in fig. 7.5.

## 7.5 Telescopes

Light coming from a distant object can be first focused with a lens (objective lens) and then magnified by another lens (eye piece) as in microscope, to form what is called a telescope that helps us see far. This is shown in fig. 7.6. With bare eyes, the image on retina is negligibly small. In telescope, the microscope action first enlarges the image $P1$, before it is seen.

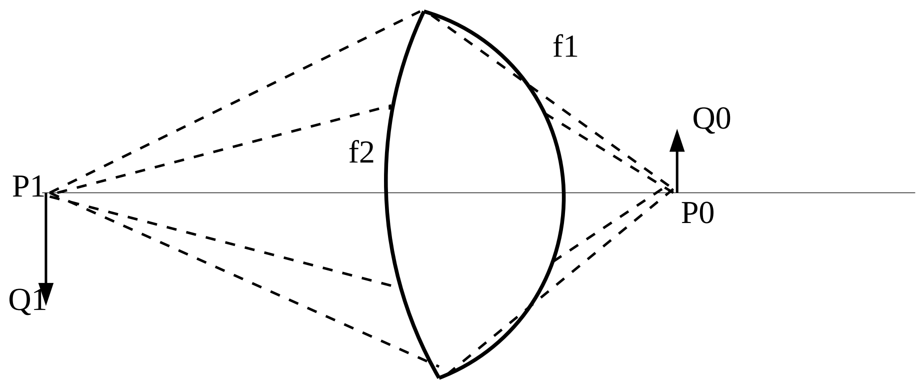



**Fig. 7.5** Fig. shows the magnification principle of a microscope with $P0Q0$ magnified to $P1Q1$.

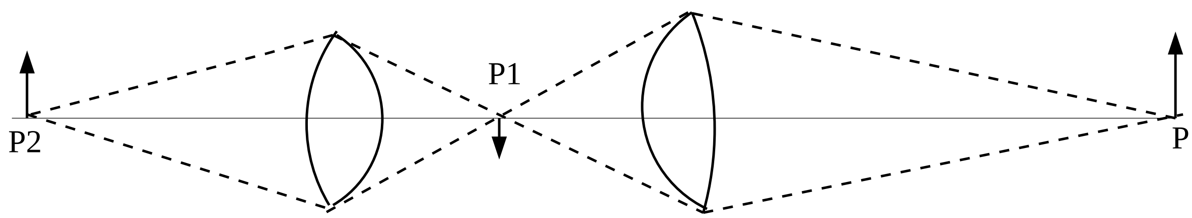



**Fig. 7.6** Fig. shows the magnification principle of a telescope, with $P$ magnified to $P2$.

## 7.6 Resolution and Diffraction

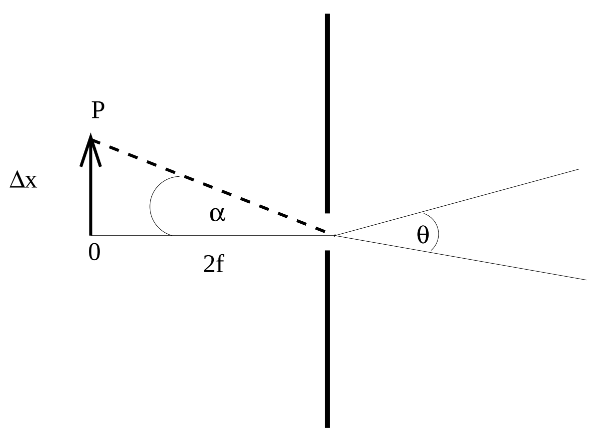



**Fig. 7.7** Fig. shows how diffraction limits resolution of magnification.

Microscopes can magnify. Is there any limit ? Can we magnify arbitrary small objects ? We find the small size is limited by the wavelength of light used. Consider a small slit in the wall in fig. 7.7 of aperture size $D$. Light coming from left passes through the slit and goes to the right but not in straight line. The slit acts as an antenna array and light spreads out, a phenomenon called diffraction. The spread $\theta$ of light in fig. 7.7 is as described in treatment of antenna arrays and given by

$$kD\theta = \pi, \tag{7.7}$$

where $k = \frac{2\pi}{\lambda}$ or $\theta = \frac{\lambda}{2D}$. To resolve $O$ and $P$, we have to make sure $\theta < \alpha$ or

$$\frac{\lambda}{2D} < \frac{x}{2f}, \tag{7.8}$$

$$x > \frac{f\lambda}{D}. \tag{7.9}$$

The limiting size is $x = \frac{f\lambda}{D}$, which becomes smaller if we use smaller wavelength light or larger $D$. We can now simply replace slit with a lens with $D$ as aperture of lens and above analysis say how small an object can be for it to be magnified.

## 7.7 Lens Equation

Until now we showed design of optical instruments with lenses whose half lenses were of different focal lengths. However it is possible to do everything with lenses with half lenses of same focal length. When we put object at $2f$ light rays converge other side at $2f$. They turn around by angle $2\phi$, where $\phi = \frac{x}{2f}$, where $x$ is vertical distance from lens center as shown in fig. 7.8A. If light rays started parallel when they turn around by $2\phi$, they will pass through the focal point as shown in fig. 7.8B. In general when they emanate from point $o$ as in fig. 7.8C, the angle $\phi_1 = \frac{x}{o}$ and they converge at point $i$ as in fig. 7.8C, the angle $\phi_2 = \frac{x}{i}$. They turn by $\phi + \phi_2 = 2\phi$, which gives the famous lens equation

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f}. \tag{7.10}$$

## 7.8 Revisiting Optical Instruments

Now we revisit design of optical instruments using lens equation. When we move object to be magnified close to $f$. The image point $i$ moves far and object size $x$ gets magnified to $\frac{ix}{o}$. This is shown in fig. 7.9. This forms the basis of a microscope.

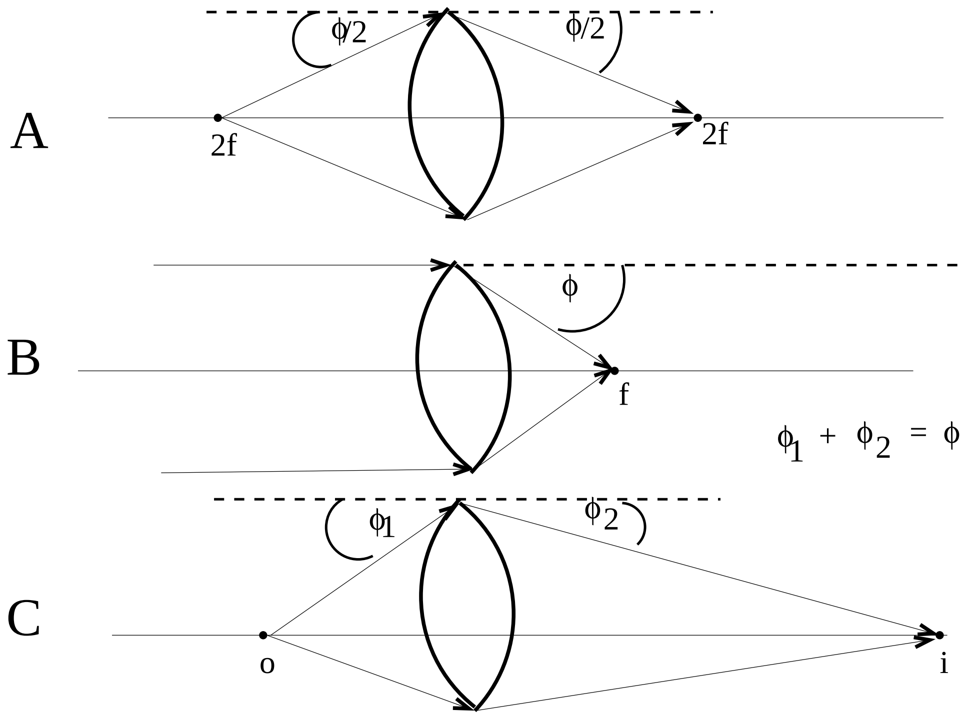



**Fig. 7.8** Fig. A shows for object at $2f$, light rays converge other side at $2f$. They turn around by angle $2\phi$. Fig. B shows parallel light rays pass through focal point. Fig. C shows object and image distances related by lens equation.

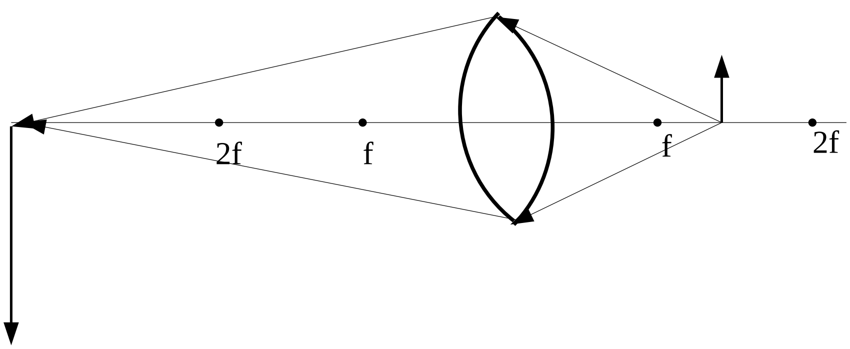



**Fig. 7.9** Fig. shows design of a microscope using lens equation.

Fig. 7.10 shows design of telescope using lens equation. The object is further than $2f$ for the objective lens whose image is formed between $f$ and $2f$ of the objective. This image is between $f$ and $2f$ of eye piece and is then magnified.

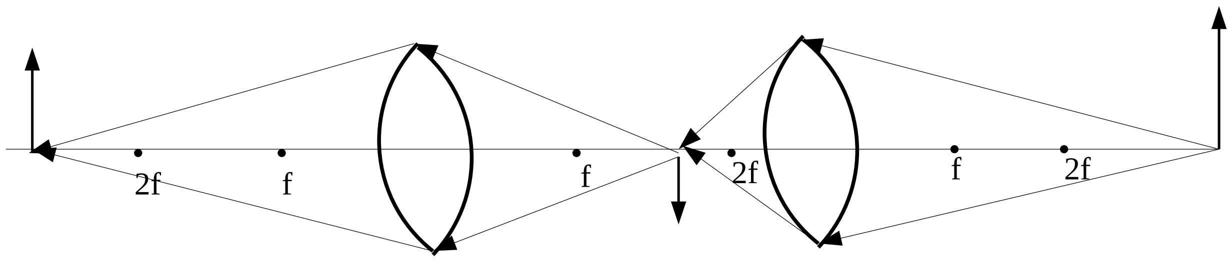



**Fig. 7.10** Fig. shows design of a telescope using lens equation.

## 7.9 Problems

1. Object distance to a lens of focal length 10 cm is 20 cm, what is the image distance.

2. In a microscope, object distance to objective lens of focal length 10 cm is 20 cm, if distance between objective and eye piece (focal length 10 cm) is 35 cm, what is the image magnification .

3. In a telescope, disyance between objective and eye piece is 22.5 cm, if both lenses have focal length 10 cm , what is magnification of telescope.

4. Size of a pin hole camera is 5 cm, what is the image size if the object is 5 m from the camera and 1 m tall.

5. For a light of 500 nm wavelength, what is the smallest size you can resolve, with an aperture of 5 cm.

# Chapter 8
# Optical resonance and colors

In this book, we talked about dielectric constant $\varepsilon$ of a media. We can write $\varepsilon = \varepsilon_0(1+\chi)$, where $\varepsilon_0$ is dielectric constant of vacuum and $\chi$ is called electric susceptibility of the medium. What is the source of $\chi$. $\chi$ results in different light velocity in a medium. Why do radio waves go through the wall as in mobile telephones while light does not. Radio waves are not scattered by the medium, light is. What is the source of all beautiful colors around us. In this chapter we will try to answer some of these questions.

## 8.1 Light absorbtion by an atom

Atom has electrons organized in orbitals. An electron from higher energy state (excited orbital) can jump to a lower energy state (ground state orbital) giving radiation with frequency $\omega_0$, where $\hbar\omega_0$ is the energy difference between the two states ($\hbar$ is the Planck's constant) . Similarly, when light is shone on an atom, with frequency $\omega_0$ it can absorb light, and jump from lower energy state to higher energy state.

Consider an EM wave incident on an atom with electron in ground state. We take the EM field as plane wave travelling along $x$ direction and electric field $E_0 \exp(j\omega_0 t)\hat{z}$. Lets us denote ground state orbital by $\phi_0$ and the excited state by $\phi_1$. Electron can be in the superposition $a\phi_0 + b\phi_1$. In this superposition state, it has a dipole moment along $x$ axis as

$$-e\langle a\phi_0 + b\phi_1|x|a\phi_0 + b\phi_1\rangle = d(ab^* + a^*b), \tag{8.1}$$

where $d = -e\langle\phi_1|x|\phi_0\rangle = -e\int \phi_1^*\phi_0 x$, which we take real.

When an electric field is applied say in the $x$ direction it produces a potential

$$V(x) = -eEx. \tag{8.2}$$

at the atomic site. This potential scatters $\phi_0$ to $\phi$ and we have dipole matrix element

$$d = -e\langle\phi_1|x|\phi_0\rangle. \tag{8.3}$$

Let state of the system be given by $\psi = a(t)\phi_0 + b(t)\phi_1$ which we write as $\psi(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$. Then if $h\omega_0$ is the difference in energies of $\phi_0$ and $\phi_1$ and $E = E_0\exp(j\omega_0 t)$ then $\psi(t)$ evolves as

$$\dot{\psi} = -\frac{i}{2}\begin{bmatrix} \omega_0 & \frac{d^*E_0}{\hbar}\exp(-j\omega_0 t) \\ \frac{dE_0}{\hbar}\exp(j\omega_0 t) & -\omega_0 \end{bmatrix}\psi. \tag{8.4}$$

Let $\frac{dE_0}{\hbar} = \omega_1$,

$$\Psi(t) = \exp(\frac{i}{2}\begin{bmatrix} \omega_0 & 0 \\ & -\omega_0 \end{bmatrix} t)\Psi(t), \tag{8.5}$$

then

$$\dot{\Psi} = -\frac{i}{2}\begin{bmatrix} 0 & \omega_1 \\ \omega_1 & 0 \end{bmatrix}\Psi, \tag{8.6}$$

then

$$\Psi(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos(\frac{\omega_1 t}{2}) \\ -j\sin(\frac{\omega_1 t}{2}) \end{bmatrix}. \tag{8.7}$$

$$\psi(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \exp(-j\frac{\omega_0 t}{2})\cos(\frac{\omega_1 t}{2}) \\ -j\exp(j\frac{\omega_0 t}{2})\sin(\frac{\omega_1 t}{2}) \end{bmatrix}. \tag{8.8}$$

At time $\omega_1 T = \pi$, we go to excited state. This is resonant absorbtion of light.

## 8.2 Light scattering by an atom

Input light need not be resonant to optical transition and can be far detuned, in which case the light photon cannot be absorbed as we donot conserve energy. The energy of the photon $\hbar\omega$ and ground atomic state is not equal to energy of the excited state. Instead what can happen is we go to intermediate excited state and then return to graound state and emit photon may be in a different direction. We say we have scattered. Scattering is understood as a three level atomic system so called $\Lambda$ system. In a $\Lambda$ system as shown in Fig. 8.1 we have two ground state levels $|1\rangle$ (ground state atom and input photon) and $|3\rangle$ (ground state atom and output photon) at energy $E_1$ and excited level (excited atom) $|2\rangle$ at energy $E_2$. The transition from $|1\rangle$ to $|2\rangle$ has strength $\Omega_1$ and transition from $|2\rangle$ to $|3\rangle$ has strength $\Omega_2$. In the interaction frame of natural Hamiltonian of the system, we get a second order term connecting level $|1\rangle$ to $|3\rangle$ with strength $\frac{\Omega_1\Omega_2}{(E_1-E_2)}$. This term creates an effective coupling between ground state levels and drives transition from $|1\rangle$ to $|3\rangle$. To see this observe,

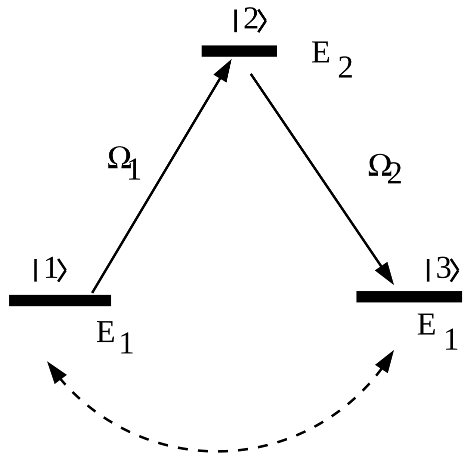



**Fig. 8.1** Above Fig. shows a three level $\Lambda$ system with two ground state levels $|1\rangle$ and $|3\rangle$ and an excited level $|2\rangle$.

The state of the three level system evolves according to the Schröedinger equation

$$\dot{\psi} = \frac{-i}{\hbar} \begin{bmatrix} E_1 & \Omega_1^* & 0 \\ \Omega_1 & E_2 & \Omega_2^* \\ 0 & \Omega_2 & E_1 \end{bmatrix} \psi. \tag{8.9}$$

We proceed into the interaction frame of the natural Hamiltonian (system energies) by transformation

$$\phi = \exp\left(\frac{i}{\hbar} \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_1 \end{bmatrix}\right) \psi. \tag{8.10}$$

This gives for $\Delta E = E_2 - E_1$,

$$\dot{\phi} = \underbrace{\frac{-i}{\hbar} \begin{bmatrix} 0 & \exp(-\frac{i}{\hbar}\Delta E\ t)\Omega_1^* & 0 \\ \exp(\frac{i}{\hbar}\Delta E\ t)\Omega_1 & 0 & \exp(\frac{i}{\hbar}\Delta E\ t)\Omega_2^* \\ 0 & \exp(-\frac{i}{\hbar}\Delta E\ t)\Omega_2 & 0 \end{bmatrix}}_{H(t)} \phi. \tag{8.11}$$

$H(t)$ is periodic with period $\Delta t = \frac{2\pi}{\Delta E}$. After $\Delta t$, the system evolution is

$$\phi(\Delta t) = (I + \int_0^{\Delta t} H(\sigma) d\sigma + \int_0^{\Delta t} \int_0^{\sigma_1} H(\sigma_1) H(\sigma_2) d\sigma_2 d\sigma_1 + \dots)\phi(0). \tag{8.12}$$

The first integral averages to zero, while the second integral

$$\int_0^{\Delta t}\int_0^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2 d\sigma_1 = \frac{1}{2}\int_0^{\Delta t}\int_0^{\sigma_1}[H(\sigma_1),H(\sigma_2)]d\sigma_2 d\sigma_1. \quad (8.13)$$

Evaluating it explicitly, we get for our system that second order integral is

$$\frac{-i\Delta t}{\hbar}\begin{bmatrix} 0 & & \frac{\Omega_1^*\Omega_2^*}{E_1-E_2} \\ 0 & 0 & 0 \\ \underbrace{\frac{\Omega_1\Omega_2}{E_1-E_2}}_{\mathscr{M}} & 0 & 0 \end{bmatrix}. \quad (8.14)$$

Thus we have created an effective Hamiltonian

$$\begin{bmatrix} 0 & & \frac{\Omega_1^*\Omega_2^*}{E_1-E_2} \\ 0 & 0 & 0 \\ \frac{\Omega_1\Omega_2}{E_1-E_2} & 0 & 0 \end{bmatrix}, \quad (8.15)$$

which couples level $|1\rangle$ and $|3\rangle$ and drives transition between them at rate

$$\mathscr{M} = \frac{\Omega_1\Omega_2}{(E_1-E_2)} \quad (8.16)$$

So in scattering, we scatter input photon, to a different direction output photon.

### 8.2.1 Electric Dipoles

Recall Maxwell equation in a medium

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad (8.17)$$
$$\nabla \times H = \frac{\partial D}{\partial t}. \quad (8.18)$$

$$\nabla \times E = -j\mu\omega H, \quad (8.19)$$
$$\nabla \times H = j\varepsilon\omega E. \quad (8.20)$$

In a material medium, we have

$$\varepsilon = \varepsilon_0(1+\chi(\omega)), \quad (8.21)$$
$$\chi(\omega) = \chi'(\omega) + j\chi''(\omega). \quad (8.22)$$

$\chi(\omega)$ is called optical susceptibility and depends on the frequency. It has a real part $\chi'(\omega)$ and a imaginary part $\chi''(\omega)$.

$$\sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{1+\chi'(\omega)+j\chi''(\omega)} = n + j\alpha, \tag{8.23}$$

$n$ is the index of refraction and $\alpha$ absorption coefficient. Let us understand the source of $\chi(\omega)$, the optical susceptibility.

### 8.2.2 Classical theory of optical susceptibility

Electrons are bound to atoms by positive charge of the nucleus. Electric field pulls the electrons away and we can describe the displacement as

$$m\ddot{x} + \gamma\dot{x} + kx = -eE. \tag{8.24}$$

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \omega_0^2 x = \frac{-eE}{m}, \tag{8.25}$$

where $\omega_0^2 = \frac{k}{m}$
with $E = E_0 \exp(j\omega t)$, we get

$$x(\omega) = \frac{-eE_0}{m}\frac{1}{(\omega_0^2-\omega^2)+j\omega\frac{\gamma}{m}} \tag{8.26}$$

Susceptibility is proportional to $x$. Fig. 8.2a and 8.2b gives the imaginary and real part of susceptibility as function of $\omega$. The imaginary part peaks as $\omega = \omega_0$.

### 8.2.3 Quantum theory of optical susceptibility

Lets $\phi_0$ and $\phi_1$ denote two electronic states (orbitals) of an atom. Let $\phi_0$ be ground state and $\phi_1$ excited state. When an electric field is applied say in the $x$ direction, it produces a potential

$$V(x) = -eEx. \tag{8.27}$$

at the atomic site. This potential scatters $\phi_0$ to $\phi_1$ and we have dipole matrix element

$$d = -e\langle\phi_1|x|\phi_0\rangle. \tag{8.28}$$

Let the state of the system be given by $\psi = a(t)\phi_0 + b(t)\phi_1$, which we write as $\psi(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$. Then if $h\omega_0$ is the difference in energies of $\phi_0$ and $\phi_1$ and $E = 2E_0\cos\omega t$, then $\psi(t)$ for $\omega_1 = \frac{2dE_0}{\hbar}$ evolves as

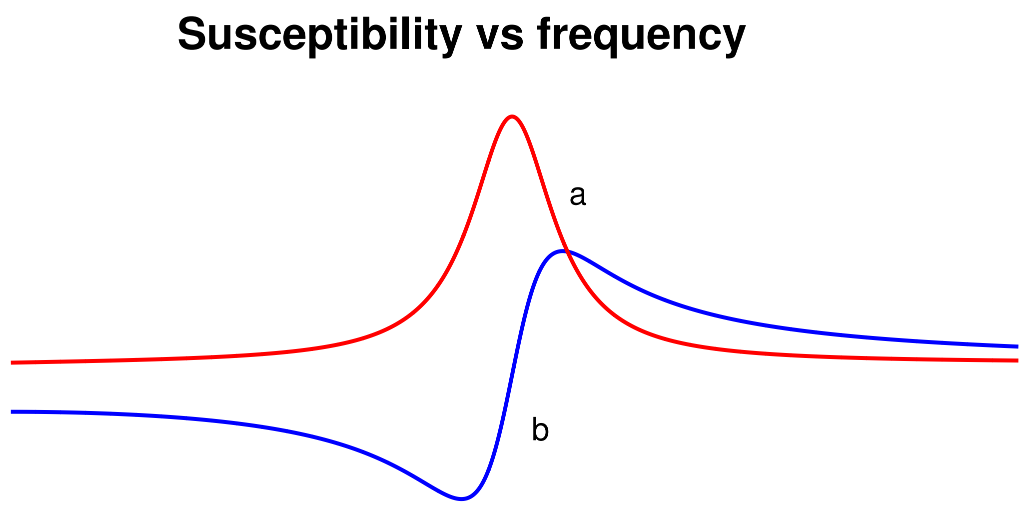



**Fig. 8.2**

$$\dot{\psi} = -\frac{i}{2}\begin{bmatrix} \omega_0 & \omega_1\cos(\omega t) \\ \omega_1\cos(\omega t) & -\omega_0 \end{bmatrix}\psi. \tag{8.29}$$

Let

$$\Psi(t) = \exp(\frac{i}{2}\begin{bmatrix} \omega & 0 \\ & -\omega \end{bmatrix} t)\psi(t), \tag{8.30}$$

then

$$\dot{\Psi} = -\frac{i}{2}\begin{bmatrix} (\omega_0-\omega) & \frac{\omega_1}{2}(1+\exp(2j\omega t)) \\ \frac{\omega_1}{2}((1+\exp(-2j\omega t)) & -(\omega_0-\omega) \end{bmatrix}\Psi. \tag{8.31}$$

We neglect the fast oscillating term $\exp(2j\omega t)$ to get

$$\dot{\Psi} = -\frac{i}{2}\begin{bmatrix} (\omega_0-\omega) & \omega_x \\ \omega_x & -(\omega_0-\omega) \end{bmatrix}\Psi, \tag{8.32}$$

where $\omega_x = \frac{dE_0}{\hbar}$. Then we can write

$$\rho = \Psi\Psi^\dagger = \frac{\mathbf{1}}{2} + s_x\sigma_x + s_y\sigma_y + s_z\sigma_z. \tag{8.33}$$

where,

$$\sigma_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\ \sigma_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix};\ \sigma_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{8.34}$$

$(s_x, s_y, s_x)$ have special meaning. $s_z$ corresponds to population in ground and excited state. When atom is in ground state $s_z = 1$ and in excited state $s_z = -1$. $s_x, s_y$ correspond to real and imaginary parts of susceptibility as shown below. Then observe the resulting dipole is

$$\langle a\phi_0 + b\phi_1 | x | a\phi_0 + b\phi_1\rangle, \tag{8.35}$$

$$= \langle \Psi_0 \exp(-\frac{j\omega_0 t}{2})\phi_0 + \Psi_1 \exp(\frac{j\omega_0 t}{2})\phi_1 | x | \Psi_0 \exp(-\frac{j\omega_0 t}{2})\phi_0 + \Psi_1 \exp(\frac{j\omega_0 t}{2}\phi_1)\rangle,$$

$$= (\Psi_1^*\Psi_0 + \Psi_1\Psi_0^*)\cos\omega t - i(\Psi_1^*\Psi_0 - \Psi_1\Psi_0^*)\sin\omega t, \tag{8.36}$$

$$= s_x \cos\omega t - s_y \sin\omega t. \tag{8.37}$$

From Eq. (8.32), we find that $(s_x, s_y, s_z)$ evolves as

$$\frac{d}{dt}\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} 0 & \omega - \omega_0 & 0 \\ \omega_0 - \omega & 0 & -\omega_x \\ \omega_x & 0 & 0 \end{bmatrix}\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}. \tag{8.38}$$

Due to dissipative processes like spontaneous emission (which we describe soon), the above equations have two decay constants $\Gamma_1$ and $\Gamma_2$, and the above equation takes the form

$$\frac{d}{dt}\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} -\Gamma_2 & \Delta\omega & 0 \\ -\Delta\omega & -\Gamma_2 & -\omega_x \\ 0 & \omega_x & -\Gamma_1 \end{bmatrix}\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \Gamma_1 \end{bmatrix}. \tag{8.39}$$

We can solve for the steady state of above equations to find [9],

$$(s_x, s_y, s_z)_\infty = \frac{1}{\Gamma_2^2 + \Delta\omega^2 + \omega_x \frac{\Gamma_2}{\Gamma_1}}(\Delta\omega\ \omega_x, \omega_x\Gamma_2, \Gamma_2^2 + \Delta\omega^2). \tag{8.40}$$

Here $s_x, s_y$ corresponds to real and the imaginary part of susceptibility and if we plot them as function of $\Delta\omega$, they take the form as in Fig. 8.2b and 8.2a respectively. Observe imaginary part decays as function of $\Delta\omega$. Also recall imaginary part is responsible for decay in a medium as in an conductor. Now optical resonance $\omega_0$ in in energy range $\hbar\omega_0 \sim eV$ or $\omega_0 \sim 10^{14} - 10^{15} Hz$. If we use radio waves, $\Delta\omega$ is very large, and hence $\chi''(\omega)$ is negligible, loss is negligible and radio waves go through walls. Light doesn't, because $\Delta\omega$ is small, hence $\chi''(\omega)$ is not negligible and we have attenuation or loss. Furthermore $\chi''(\omega)$ decays much faster as function of $\Delta\omega$ than $\chi'(\omega)$. hence even for radio or microwaves we have non-zero $\chi'(\omega)$ a medium which explains dielectric response $\varepsilon$ as different from $\varepsilon_0$.

## 8.3 Spontaneous Emission

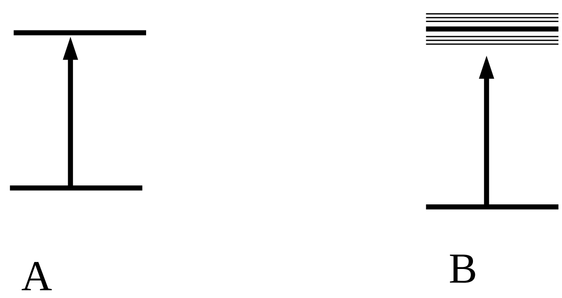



**Fig. 8.3** Fig. A shows transition from ground to excited state. Fig. B shows excited state level broadened.

Consider two level system as shown in 8.3A. Let $\Omega$ be the rate of on resonance excitation from ground state to excited state. The dynamics of the system with $\psi = \begin{bmatrix} \psi_1 \\ \psi_0 \end{bmatrix}$ is

$$\dot{\psi} = -\frac{i}{2}\begin{bmatrix} \omega_0 & \Omega\exp(-j\omega_0 t) \\ \Omega\exp(j\omega_0 t) & -\omega_0 \end{bmatrix}\psi. \tag{8.41}$$

In the rotating frame $\psi \to \exp(\frac{i}{2}\begin{bmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{bmatrix} t)\psi$ we have

$$\dot{\psi} = -\frac{i}{2}\begin{bmatrix} 0 & \Omega \\ \Omega & 0 \end{bmatrix}\psi. \tag{8.42}$$

Now suppose the excited level is detuned by energy $\Delta\omega$, then

$$\dot{\psi} = -\frac{i}{2}\begin{bmatrix} \omega_0 + 2\Delta\omega & \Omega\exp(-j\omega_0 t) \\ \Omega\exp(j\omega_0 t) & -\omega_0 \end{bmatrix}\psi, \tag{8.43}$$

then in the rotating frame $\psi \to \exp(\frac{i}{2}\begin{bmatrix} \omega_0 + 2\Delta\omega & 0 \\ 0 & -\omega_0 \end{bmatrix} t)\psi$, we have

$$\dot{\psi} = -\frac{i}{2}\begin{bmatrix} 0 & \Omega\exp(j\Delta\omega t) \\ \Omega\exp(-j\Delta\omega t) & 0 \end{bmatrix}\psi. \tag{8.44}$$

Then starting from $\psi_0(0) = 1$, we have $\psi_1(\Delta t) = \int_0^{\Delta t}\Omega\exp(j\Delta\omega\tau)d\tau$. Now consider many excited levels as in 8.3B, with level $\psi_n$ detuned by $n\Delta\omega$. Then $\psi_n(\Delta t) = \int_0^{\Delta t}\Omega\exp(jn\Delta\omega\tau)d\tau$. Then

$$\psi_k^*(\Delta t)\psi_k(\Delta t) = \int_0^{\Delta t}\int_0^{\Delta t}\Omega^2\exp(jk\Delta\omega(t-\tau))d\tau dt. \tag{8.45}$$

Then $\Phi = \sum_{k=-n}^{n}\psi_k^*(\Delta t)\psi_k(\Delta t) =$

$$\frac{\Omega^2}{\Delta\omega}\int_0^{\Delta t}\int_0^{\Delta t}(\int_{-B}^{B}\exp(j\omega(t-\tau))d\omega)\ d\tau dt, \tag{8.46}$$

$$= \frac{\Omega^2}{\Delta\omega}\int_0^{\Delta t}\int_0^{\Delta t}\frac{\sin(B(t-\tau)}{t-\tau}\ d\tau dt. \tag{8.47}$$

$$= \frac{\Omega^2}{\Delta\omega}\Delta t. \tag{8.48}$$

where we use the approximation $B \gg \Omega$. There is time $\Delta t$ such that $\Omega\Delta t \ll 1$ and $B\Delta t \gg 1$. Then

$$\Gamma = \frac{\Phi}{\Delta t} = \frac{\Omega^2}{\Delta\omega}, \tag{8.49}$$

gives the transition rate out of ground state.

With Eq. 8.49 also known as *Fermi Golden Rule*, we can study the spontaneous emission rate of an atom. It is the rate at which atom decays into ground sate. Let $\hbar\omega_0$ be the energy difference between the excited and ground state. When atom decays from excited to ground sate it radiates EM wave mode. These modes are plane waves with defined $k=(k_x,k_y,k_z)$. We give the mode some width $(\Delta k_x,\Delta k_y,\Delta k_z)$, which gives the mode a finite volume $V^{-1}=\frac{\Delta k_x\Delta k_y\Delta k_z}{(2\pi)^3}$. Fig. 8.4 depicts this width for a mode with $k$ along $x$ axis. The energy of this mode is $\hbar\omega_0$ and hence it has electric field $\varepsilon_0 E^2 V = \hbar\omega_0$ and hence the transition rate due to this electric field $\Omega = \frac{Ed}{\hbar}$. Infact, transition rate is $\cos\theta\Omega$, where $\theta$ is the vector k vector makes with $z$ axis as transition is induced by electric field in the equatorial plane. The mode has width $\Delta k$ along radial direction and hence frequency width of $\Delta\omega = c\Delta k$. The modes are spread radially with energy spacing of $\Delta\omega$. Therefore by Fermi golden rule the decay rate $\Gamma$ is $\Gamma = \frac{\Omega^2\mathscr{M}}{\Delta\omega}$, where $\mathscr{M}$ is the number of modes on the surface of sphere of radius $k_0$. This is

$$\mathscr{M} = 2\pi k_0^2\Delta k\int\cos^2\theta\sin\theta\ d\theta,$$

where $\cos^2\theta$ comes from the factor $\cos\theta$ in transition rate. When we put everything together we get [4]

$$\Gamma = \frac{d^2\omega_0^3}{6\pi^2c^3\hbar\varepsilon_0} = \frac{d^2\omega_0^3}{3\pi c^3 h\varepsilon_0}. \tag{8.50}$$

### 8.3.1 Spontaneous Emission in a Resonant Cavity and Purcell effect

Now consider a resonant cavity, with volume $V$. Let its natural frequency be $\omega_0$. Let $Q$ be the quality factor of the cavity and consider a two level atom in the cavity with

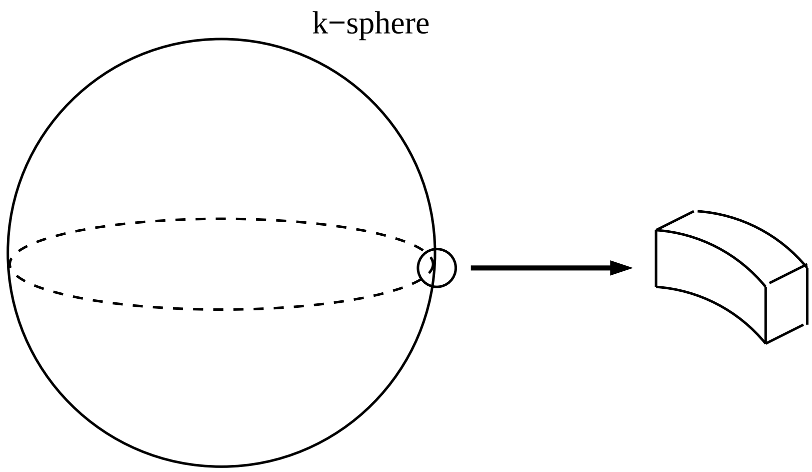



**Fig. 8.4** Fig. shows blowup of a mode depicting it has some width in k-space.

natural frequency $\omega_0$. We are interested in the spontaneous emission rate of the atom in the cavity. This rate is the same rate as exciting the atom from the ground state with electric field worth $\hbar\omega_0$ worth of energy, i.e., $\varepsilon_0 E_0^2 V = \hbar\omega_0$. The corresponding transition frequency is $\Omega = \frac{E_0 d}{\hbar}$. The electric field decays as $E_0 \exp(-\alpha t)$, with $\frac{\alpha}{\omega_0} = \xi$, $Q = 1 - \xi$. The transition with decaying field $E$ can be simulated by a broadened transition with $n$ excited levels, with frequency spacing $\Delta\omega$, such that $n\Delta\omega = \alpha$, and transition frequency $\frac{\Omega}{\sqrt{n}}$. Then by the Fermi Golden rule we have

$$\Gamma = \frac{\Omega^2}{n\Delta\omega} = \frac{Q}{V}\frac{d^2}{\varepsilon\hbar}. \tag{8.51}$$

Observing that $\omega_0^3 \propto \frac{c^3}{V}$, we get

$$\Gamma = Q\frac{d^2\omega_0^3}{c^3\varepsilon\hbar}. \tag{8.52}$$

Thus there is modification of spontaneous emission rate over vacuum, this modification $Q$ is called *Purcell effect*.

## 8.4 World of Colors

We are surrounded by beautiful colors. Really beautiful. Our clothes have beautiful colors on them. Plants are green, blood is red. What is the source of this color. Clothes have dyes containing pigments, like paints have pigments. These pigments are primarily inorganic in the sense, they have a transition metal element in them. Transition metal elements are the one found in the center of the periodic table that have electrons in their d-orbitals. These include, for example, Cobalt (Co) , Cadmium (Cd), Chromium (Cr), Manganese (Mn) etc. For example, Cobalt (atomic number 27), has electronic configuration $1s^22s^22p^63s^23p^63d^74s^2$. The d-orbitals are five fold degenerate. These orbitals are $d_{z^2}, d_{x^2-y^2}, d_{xy}, d_{yz}, d_{xz}$. However in a transition metal compound, binding with other atoms called ligands, this degeneracy gets broken. We have orbitals $d_{z^2}, d_{x^2-y^2}$ called $e_g$ manifold at higher energy than the orbitals $d_{xy}, d_{yz}, d_{xz}$ called $t_{2g}$ manifold as shown below

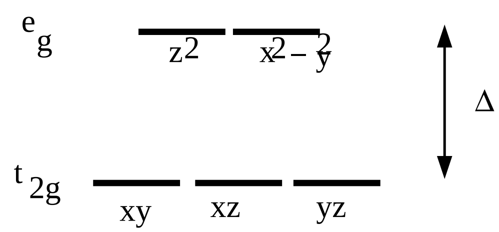



**Fig. 8.5** Fig. shows splitting of energy of d-orbitals in $e_g$ manifold and $t_{2g}$ manifold.

The energy difference $\Delta = \hbar\omega_0$ is sub-eV and corresponds to visible wavelength. When we shine light the right color is absorbed, rest scattered back, which we see. We see complimentary colors.

## 8.5 Scattering and Blue Sky

Why is sky Blue? From 8.16, blue light is more detuned and hence scatters less and reaches us (from sun above us) , red light gets back scattered as less detuned. At eveening time we mainly see scattered light (sun on the side) hence red light.

## 8.6 Problems

1. For dipole transition $d = 1$ A°, and $E = 1000V/m$ calculate the atomic absorption rate (rabi frequency) on resonance.

2. For dipole transition $d = 1$ A$^\circ$, and $E = 1000V/m$ calculate the scattering rate (rabi frequency) for 1 GHz detuning.

3. Calculate the spontaneous emission rate for $d = 1$ A$^\circ$ and transition frequency $\omega_0 = 10^{15}$ Hz.

4. Calculate the Purcell spontaneous emission rate for $d = 1$ A$^\circ$ and $\omega_0 = 10^{15}$ Hz with $Q = .1$.

5. calculate the optical susceptibility with $d = 1$ A$^\circ$, and $E = 1000V/m$ , 1 GHz detuning and 1 ns decay rate.

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