Navin Khaneja

Scattering

– Monograph –

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Dedicated to my Parents

Preface

The book grew out of author's research in Nuclear Magnetic Resonance (NMR). Modern NMR spectrometers use a superconducting magnet to create a very large and stable magnetic field. The superconducting coil in magnet carries current that never dissipates. One just has to cool the coil with liquid Helium and it operates round the clock. What an amazing physical phenomenon, called superconductivity. In superconductivity electrons at Fermi surface scatter of each other by exchange of phonons to form bound cooper pairs. Superconductivity arises out of phonon mediated scattering. Scattering is a phenomenon of emission and absorption of a Boson or vice-versa. It is ubiquitous and heart of this book.

Most mundane scattering is blue sky. The sky is blue, at night it is red, because red light from sun is scattered (photon direction changes when it meets an atom in atmosphere), and we get blue light and at night of-course sun is on side and we see red scattered light. This is called Rayleigh scattering. Rayleigh scattering is elastic scattering only direction of light changes, we can also have scattering where wavelength also changes, called in-elastic scattering as in Raman scattering. Light when scatters of molecules changes wavelength and lost energy excites vibrational modes of a molecule. Rayleigh and Raman scattering happens when photon talks to bound electron of an atom. We also have high energy X-ray photons scattering of the free electrons in a solid, something called Compton scattering.

Raman scattering is enhanced if the light is emitted into a laser with the same frequency called stimulated Raman. In Rayleigh scattering, we have atom absorbing and re-emitting photon. But we can have more interesting scenarios. The atom may absorb a photon and emit two photons of half the frequency, or absorb two photons and emit a photon of twice the frequency. In nonlinear optics [48], light of frequency ω gets converted to 2ω, called second harmonic generation (SHG). One photon of 2ω get converted to two one photons of 2ω. More generally photons at ω_1 and ω_2 get converted to $\omega_3 = \omega_1 + \omega_2$. Also called sum frequency generation (SFG).

Electron can exchange momentum with a atom raising it to higher energy state as in Frank Hertz experiment and sodium, neon, mercury vapor lamps, or ionization of atom as in bubble and cloud chamber experiments, or just production of X-rays as in Bremsstraulung, where scattered electron emits radiation. Elastic scattering

of electrons happens in electron diffraction and electron microscopes. Scattering of electrons in the condensed matter range from scattering of electrons of periodic potential, to give Bloch waves, scattering of electrons of phonons and impurities to give resistance, scattering of electrons of lattice to give cooper pairs and superconductivity. Beautiful scattering happens in electron-proton scattering both elastic and inelastic, as in deep inelastic scattering experiments and understand the independence of inelastic cross-section of with respect to transferred momentum. We see, why we can just say that there are three quarks in proton from elastic cross-section.

Two electrons can exchange photon (one emit other absorb) and hence exchange momentum. The net energy of a two electron system gets modified by a second order energy correction, which is the beautiful Coulomb potential. The exchanged photon energy can get modified when it interacts with electron vacuum creating electron-positron pairs and annihilating then. Modification is a second order calculation that is heart of High energy physics. This modification, modifies he coulomb potential a phenomenon termed vacuum polarization. A electron can directly scatter a photon changing its and photon momentum a phenomenon termed Compton scattering observed a change in wavelength of a scattered X-ray light. Coulomb potential between moving electrons is different, that can be thought of a part of moving frame and due to Einstein relativity have enhanced interaction which is manifested as a magnetic field which gives spinor its energy $\mu \cdot B$, but μ gets modified (due to vacuum coupling) to what we call anomalous magnetic moment of electron, a second order effect we can calculate in QED. Coupling to vacuum also modifies the orbital energies around nucleus of an atom a phenomenon that manifests itself as Lamb shift in hydrogen, a second order effect that we can calculate in QED. Electron-Positron when collide can of-course scatter but can also annihilate and then create another particle-antiparticle pair, which may be heavier (like muons) as long as their is enough kinetic energy in colliding beam.

Photons are of-course excitation of Electromagnetic (EM) vacuum. This is not the only vacuum, we have W-Z vacuum (weak interaction vacuum), whose excitations are heavy bosons $W+, W-$ and Z. Electrons can change momentum by emitting heavy boson, and changing to a neutrino, similarly quarks can change flavor by emitting, absorbing heavy bosons. Neutrinos are of-course without charge but can scatter of electrons, positrons by exchange of heavy bosons. Heavy particle can change to light particle (neutron to proton) by emitting W Boson which can create a electron neutrino pair a process called β decay. Similarly pions can decay to muons and muons to electrons. These are all weak interactions, because interacting boson is very heavy.

But of-course there are strong force or color interactions that binds quarks into hadrons, mesons (pions and kaons) and Baryons (protons and neutrons). The interaction is color interaction where by quarks have three possible colors red, green, blue and can exchange momentum and color with interaction strength much higher. Color interaction is mediated by photons we call gluons, of eight kinds. Protons collide, exchange momentum with gluons, lose energy and create pions and kaons etc. Quarks are confined to protons, protons to nuclei with nuclear force. Protons are color neutral cannot mediate interactions with exchange of gluons but can produce

pions whose exchange leads to nuclear force, which is very short range as pions are massive. Gluons can interact, scatter of each other, the exchanged gluon can change energy due to this second order effect which can weaken the interaction between gluons, a effect very pronounced at close distances called asymptotic freedom.

Collisions do not have to be between leptons (electron-positron, electron-neutrino), or between hadrons (proton-proton), we can collide electrons with protons. The interaction if of-course EM photon. At low energies, the electron just elastically scatters of the proton and scattering cross-section dependence with exchanged momentum says how many quarks we have, three in this case. If electron is very energetic it can excite internal modes (quark orbitals) and can create a heavy proton, a process called inelastic scattering. The exchanged photon may generate pions or other mesons a signature of inelastic scattering.

I will like to thank the wonderful colleagues and academic environment of SYSCON at IIT Bombay that provided ample opportunity for self development. Finally I like to acknowledge the support of my family which made this effort possible.

IIT Bombay, *Navin Khaneja*
March 2024

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Part I

Chapter 1
Introduction to Scattering

1.1 Introduction

This is a text on scattering, the beautiful subject of scattering. The sky is blue, at night it is red, because red light from sun is scattered (photon direction changes when it meets an atom in atmosphere), and we get blue light and at night of-course sun is on side and we see red scattered light. This is called Rayleigh scattering.

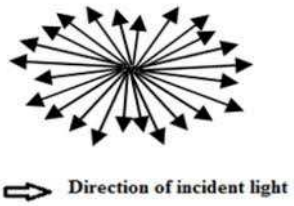


Fig. 1.1 Fig. shows Rayleigh scattering

Rayleigh scattering is elastic scattering only direction of light changes, we can also have scattering where wavelength also changes, called in-elastic scattering as in Raman scattering. Light when scatters of molecules changes wavelength and lost energy excites vibrational modes of a molecule.

Rayleigh and Raman scattering happens when photon talks to bound electron of an atom. We also have high energy X-ray photons scattering of the free electrons in a solid, something called Compton scattering.

X-rays can of-course elastically scatter from a group of atoms and scattering amplitudes coherently add to give what is called Bragg-scattering where we get a strong reflection when wavelength of X-rays match lattice spacing.

Scattering happens when photon collides with an electron. Photon is absorbed and can be re-emitted a process called scattering. Electron can also scatter of electrons by emitting a photon which is absorbed by the second electron. Electrons of right

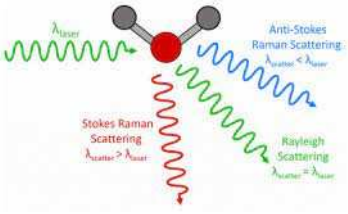


Fig. 1.2 Fig. shows Raman scattering

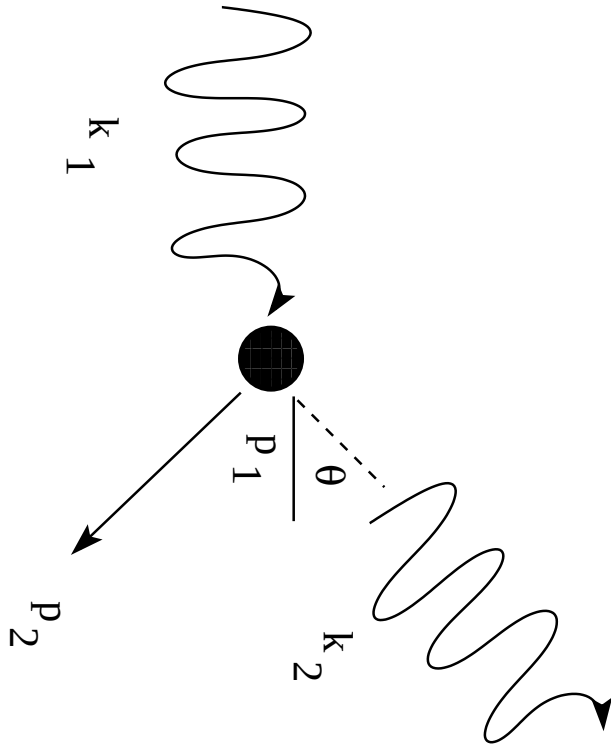


Fig. 1.3 Fig. shows Compton scattering

energy and wavelength can also Bragg scatter of the lattice, a phenomenon called X-ray diffraction, first absorbed in nickel crystal by Davisson and Germer in 1927, that reaffirmed wave nature of electrons. Neutrons have much smaller wavelength and can Bragg scatter of lattice at much lower energy.

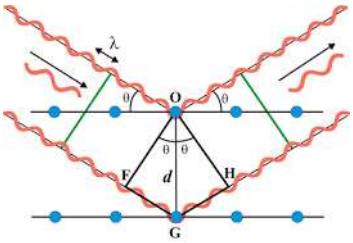


Fig. 1.4 Fig. shows Bragg scattering

1.2 Variants of Raman Scattering

Many optical phenomenon can be understood as variants of Raman scattering. These include Resonance Raman, Stimulated Raman and CARS (coherent anti-stokes raman scattering). When the scattering molecule has excited state energy close to incoming photon energy then the amplitude of the Raman scattering is greatly enhanced and we call it Resonance Raman. when scattering is done in presence of second external radiation whose frequency is the energy of outgoing scattered light then again scattering amplitude is enhanced and it is called stimulated Raman.

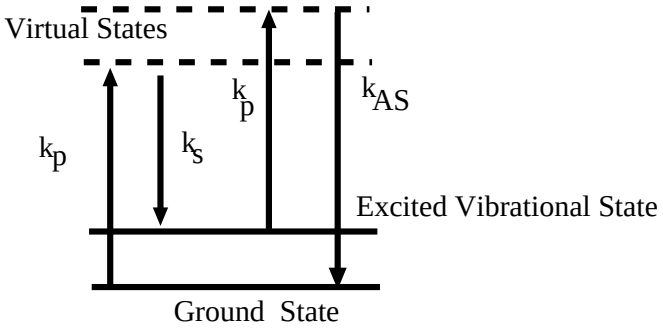


Fig. 1.5 Figure shows the Coherent Raman Anti-stokes process. Starting from ground state, pump laser k_p is absorbed and stokes k_s emitted and we transit to excited vibrational level, then, k_p is absorbed and anti-stokes k_{AS} emitted, returning to ground state.

CARS [45] is like Raman, except two Raman back to back. In Fig. 3.9 starting from ground state, pump laser k_p is absorbed and stokes k_s emitted and we transit to excited vibrational level, then, k_p is absorbed and anti-stokes k_{AS} emitted, returning to ground state. k_p, k_s form a Raman pair with amplitude $\mathcal{M}_1$ and k_p, k_{AS} form another Raman pair with amplitude $\mathcal{M}_2$.

1.3 Nonlinear Optical scattering

In Rayleigh scattering, we have atom absorbing and re-emitting photon. But we can have more interesting scenarios. The atom may absorb a photon and emit two photons of half the frequency, or absorb two photons and emit a photon of twice the frequency.

In nonlinear optics [48], light of frequency ω gets converted to 2ω, called second harmonic generation (SHG). One photon of 2ω get converted to two one photons of 2ω. More generally photons at ω_1 and ω_2 get converted to $\omega_3 = \omega_1 + \omega_2$. Also called sum frequency generation (SFG).

In Down conversion light of frequency 2ω gets converted to ω. Also called Difference Frequency Generation (DFG), one photon of ω_3 get converted to photons of ω_1 and ω_2 such that $\omega_3 = \omega_1 + \omega_2$.

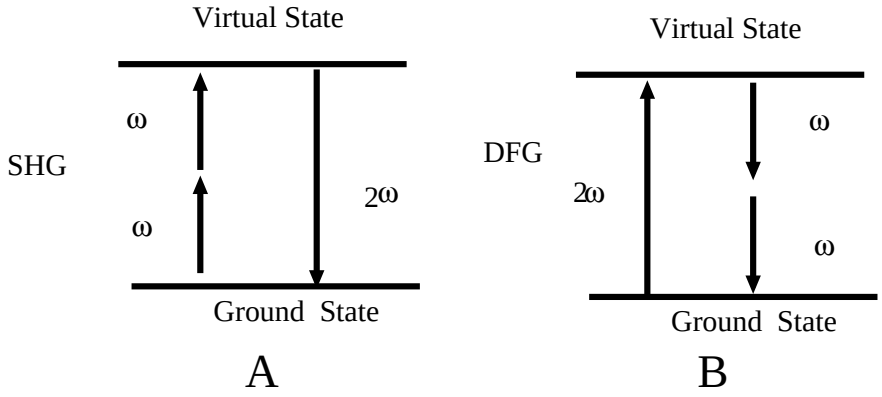


Fig. 1.6 Fig. A shows second harmonic generation (SHG) and Fig. B shows down conversion.

Fig. 3.12 A shows second harmonic generation (SHG) and Fig. 3.12 B shows down conversion. The 4 level system analogy is as shown in 3.11.

All these phenomenon become transparent, when we study electron-photon interaction in Dirac equation subsequently.

1.4 Absorption vs Scattering

Scattering is emission and absorption of photon or vice-versa. But in many optical processes we can simply absorb a photon and transit to higher energy atomic orbital. This is the simple processing Laser spectroscopy where laser light of right frequency promotes an electron to higher atomic orbital.

1.4.1 Laser cooling

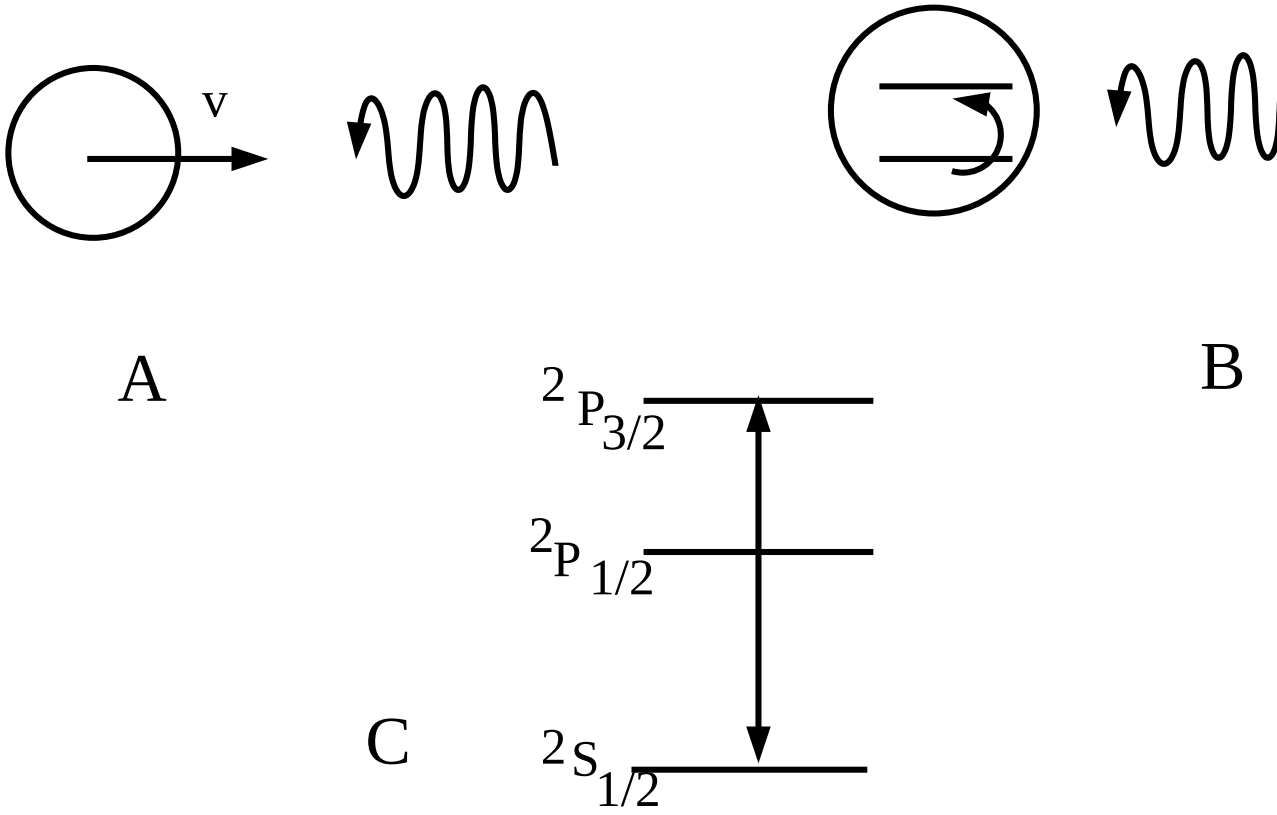


Fig. 1.7 Figure A shows how an atom is hit with light with momentum $\hbar k$ and slows down. Figure B shows atom will absorb light if frequency $\hbar\omega$ matches the difference of energy between the internal energy levels. Figure C shows these energy levels for sodium. These are the electronic states with $n = 3$.

Many experiments in physics require slow atoms [38]. At room temperatures, we have atoms moving at say 300 m/s. We like to slow them down to say 10 m/s. Slow atoms are used to make Bose Einstein condensates (velocity as low as cm/s). How do we slow atoms. We can do it by hitting them with light of the right frequency. Atoms have internal energy levels. When the frequency of the laser light matches this, the light is absorbed (optical transition). But light carries momentum $\hbar k$ and therefore on absorption atom gets a kick which slows it down. The absorbed photon is spontaneously emitted but in a random direction. When we average over many such absorption and emission, the kick is always in the same direction while the recoil due to spontaneous emission is random and averages to zero and in the end atom slows down. If frequency of the laser light is ω an atom moving towards the

light source will see the frequency shifted to $\omega(1+\frac{v}{c})$. Then if detune the laser frequency to be slightly less than the internal energy level, due to this Doppler shift the atom will see just the right frequency and will absorb it. Atoms moving slowly will have negligible Doppler shift and will not absorb light. Hence we will only cool fast atoms and not slow ones. This way we will bring all to same velocity. This is the basic idea of laser cooling. It has been successful in cooling atoms to very low velocities where they form a Bose Einstein condensate. Figure C shows these energy levels for sodium. These are the electronic states with principal quantum number $n = 3$. The transition shown is called the D_2 line.

1.4.2 Two photon microscopy

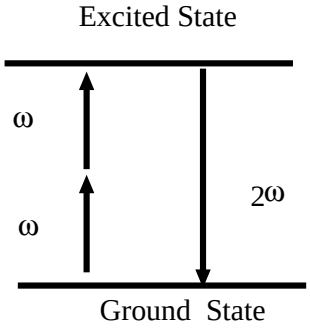


Fig. 1.8 Figure shows a two photon absorption process. There is atomic transition at 2ω frequency. A laser a frequency ω is absorbed to a virtual level. Then another photon is absorbed making transition to excited atomic state. This together constitutes a Raman process. This is followed by spontaneous emission to ground state.

Figure, 3.10 shows a two photon absorption process [43, 44] . There is atomic transition at 2ω frequency. A laser a frequency ω is absorbed to a virtual level. Then another photon is absorbed making transition to excited atomic state. This together constitutes a Raman process. This is followed by spontaneous emission to ground state.

1.4.3 World of Colors

We are surrounded by beautiful colors. Really beautiful. Our clothes have beautiful colors on them. Plants are green, blood is red. What is the source of this color. Clothes have dyes containing pigments, like paints have pigments. These pigments are primarily inorganic in the sense, they have a transition metal element in them.

Transition metal elements are the one found in the center of the periodic table that have electrons in their d-orbitals. These include, for example, Cobalt (Co) , Cadmium (Cd), Chromium (Cr), Manganese (Mn) etc. For example, Cobalt (atomic number 27), has electronic configuration $1s^2 2s^2 2p^6 3s^2 3p^6 3d^7 4s^2$. The d-orbitals are five fold degenerate. These orbitals are $d_{z^2}, d_{x^2-y^2}, d_{xy}, d_{yz}, d_{xz}$. However in a transition metal compound, binding with other atoms called ligands, this degeneracy gets broken. We have orbitals $d_{z^2}, d_{x^2-y^2}$ called e_g manifold at higher energy than the orbitals d_{xy}, d_{yz}, d_{xz} called t_{2g} manifold as shown below

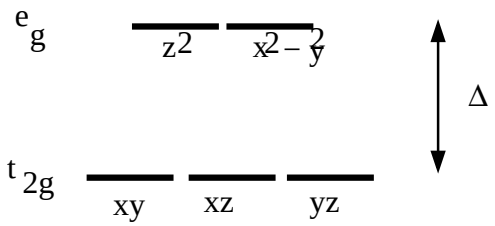


Fig. 1.9 Fig. shows splitting of energy of d-orbitals in e_g manifold and t_{2g} manifold.

The energy difference $\Delta = \hbar\omega_0$ is sub-eV and corresponds to visible wavelength. When we shine light the right color is absorbed, rest scattered back, which we see. We see complimentary colors.

1.5 Electron Scattering

In this book we study electron scattering. A electron of mass m, when electrically scatters of nucleus, of mass M, transfers momentum q to the nucleus. The energy lost by electron is more than the energy gained by the nucleus. The resulting energy goes in exciting the atom to a higher energy state as in Frank Hertz experiment and sodium, neon, mercury vapor lamps, or ionization of atom as in bubble and cloud chamber experiments, or just production of X-rays as in Bremsstraulung. In this book, we study these phenomenon. These experiments are inelastic scattering experiments. We remark, why neutrinos donot scatter and can penetrate earth, why muons travel further than electrons in materials and why a material like lead plate can slow down electrons and positrons efficiently. We look at the elastic scattering of electrons as in electron diffraction and electron microscopes. We look at scattering of electrons in the condensed matter, these phenomenon range from scattering of electrons of periodic potential, to give Bloch waves, scattering of electrons of phonons and impurities to give resistance, scattering of electrons of lattice to give cooper pairs

and superconductivity. We study electron scattering from exchange potential as in Fermi liquid theory and resulting T^2 resistance at low temperatures. Electron scattering of exchange potential resulting in chemical reactions. We turn our attention to electron-proton scattering both elastic and inelastic, as in deep inelastic scattering experiments and understand the independence of inelastic cross-section of with respect to transferred momentum. We see, why we can just say that there are three quarks in proton from elastic cross-section.

1.6 Scattering in High energy physics

1.6.1 Scattering in quantum electrodynamics (QED)

QED is the science of electron photon scattering , where by two electrons can scatter by exchanging a photon, which gives rise to Coulomb potential between electron pairs. Photon can scatter of a electron exchanging momentum and hence changing color as in Compton scattering. Electron-Positrons can collide and annihilate to give photons which can create muon-antimuon pairs.

Quantum electrodynamics (QED) is one of the most successful theories of modern physics era [9, **?**, 63, 13]. In QED, electrons interact by electromagnetic coupling to vacuum. Electron emits photon which is absorbed by the second electron leading to momentum exchange between electrons which we call electric force. The emission and absorption changes the energy of the two electrons by what we call the electric potential energy. In calculating this energy, which is a second order calculation, we make use of the energy of photon $E_k = \hbar c k$ where k is its momentum. But this emitted photon can further interact with the vacuum by creating electron positron pairs, which annihilate to give the photon back. This again has its own energy which modifies the energy of the photon E_k to E'_k. we can calculate this modification or correction and we find this will change the electromagnetic potential between two electrons. We may think of this as simply changing ϵ_0 the vacuum permittivity and this is called vacuum polarization, very much like light propagating in a medium polarizes it and changes ϵ_0 and slows down. On another note, an electron can emit and absorb a photon and the process modifies the rest energy of the electron mc^2 to $m'c^2$ a process we call mass correction.

1.6.2 Scattering in Weak Interactions

Photons are of-course excitation of Electromagnetic (EM) vacuum. This is not the only vacuum, we have W-Z vacuum (weak interaction vacuum), whose excitations are heavy bosons $W+, W-$ and Z. Electrons can change momentum by emitting heavy boson, and changing to a neutrino, similarly quarks can change flavor by emitting, absorbing heavy bosons. Neutrinos are of-course without charge but can

scatter of electrons, positrons by exchange of heavy bosons. Heavy particle can change to light particle (neutron to proton) by emitting W Boson which can create a electron neutrino pair a process called β decay. Similarly pions can decay to muons and muons to electrons. These are all weak interactions, because interacting boson is very heavy. This is scattering at its best, we emit and absorb, heavy photons

1.6.3 Scattering in Quantum Chromodynamics

But of-course there are strong force or color interactions that binds quarks into hadrons, mesons (pions and kaons) and Baryons (protons and neutrons). The interaction is color interaction where by quarks have three possible colors red, green, blue and can exchange momentum and color with interaction strength much higher. Color interaction is mediated by photons we call gluons, of eight kinds. Protons collide, exchange momentum with gluons, lose energy and create pions and kaons etc. Quarks are confined to protons, protons to nuclei with nuclear force. Protons are color neutral cannot mediate interactions with exchange of gluons but can produce pions whose exchange leads to nuclear force, which is very short range as pions are massive. Gluons can interact, scatter of each other, the exchanged gluon can change energy due to this second order effect which can weaken the interaction between gluons, a effect very pronounced at close distances called asymptotic freedom.

1.6.4 Collisions

Collisions do not have to be between leptons (electron-positron, electron-neutrino), or between hadrons (proton-proton), we can collide electrons with protons. The interaction if of-course EM photon. At low energies, the electron just elastically scatters of the proton and scattering cross-section dependence with exchanged momentum says how many quarks we have, three in this case. If electron is very energetic it can excite internal modes (quark orbitals) and can create a heavy proton, a process called inelastic scattering. The exchanged photon may generate pions or other mesons a signature of inelastic scattering.

Chapter 2
Relativity, electrons and photons

2.1 notation

Three vectors are denoted by boldface type.

$$x^\mu = (x^0, \mathbf{x}) \tag{2.1}$$
$$x_\mu = (x^0, -\mathbf{x}) \tag{2.2}$$
$$\partial_\mu = (\frac{\partial}{\partial x^0}, \nabla) \tag{2.3}$$
$$\partial^\mu = (\frac{\partial}{\partial x^0}, -\nabla) \tag{2.4}$$

$$x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

and

$$\partial_\mu \partial^\mu = (\frac{\partial}{\partial x^0})^2 - (\frac{\partial}{\partial x^1})^2 - (\frac{\partial}{\partial x^2})^2 - (\frac{\partial}{\partial x^3})^2$$

2.2 Relativity

Consider lab frame O and a frame O' , moving with respect to lab frame with velocity v in the x-direction as shown in Fig. 2.1.

Then the space time increment $(\Delta x, \Delta t)$ in O, corresponds to $(\Delta x', \Delta t')$ in O'. The phase increment of the light wave in both frames is the same. The velocity of light is same in both frames, which is the central tenet of theory of relativity.

Then

$$k\Delta x - \omega\Delta t = k'\Delta x' - \omega'\Delta t' \tag{2.5}$$

$$k(\Delta x - c\Delta t) = k'(\Delta x' - c\Delta t'). \tag{2.6}$$

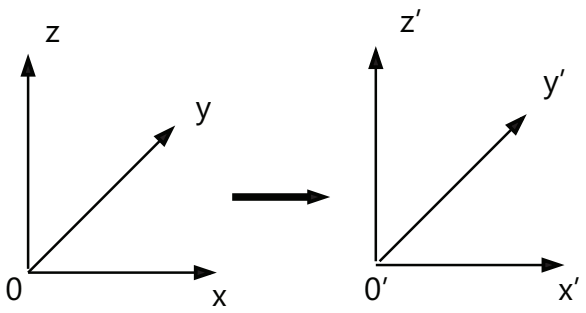


Fig. 2.1 Fig. shows frames O' moving relative to O at velocity v.

For light travelling in opposite direction

$$k'(\Delta x + c\Delta t) = k(\Delta x' + c\Delta t'). \tag{2.7}$$

The two relations give

$$(c\Delta t)^2 - \Delta x^2 = (c\Delta t')^2 - \Delta x'^2. \tag{2.8}$$

For $\Delta x' = 0$, we have, $\Delta x = v\Delta t$ and this gives

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.9}$$

This is called time dilation. Furthermore

$$\frac{k'}{k} = \frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.10}$$

Then combining Eq. (2.6. 2.7, 2.10), we get

$$\begin{bmatrix} \Delta x \\ c\Delta t \end{bmatrix} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} \Delta x' \\ c\Delta t' \end{bmatrix} \tag{2.11}$$

For a rod of length l' in O' we have $(\Delta x', \Delta t') = (l', 0)$, the $l = \Delta x - v\Delta t = l'\sqrt{1 - \frac{v^2}{c^2}}$. This is called length contraction.

For an object moving at velocity in the frame O' at velocity u, for time $\Delta t'$, we have $(\Delta x', \Delta t') = (u\Delta t', \Delta t')$. Then from (Eq. 2.11), the relative velocity

$$v = \frac{\Delta x}{\Delta t} = \frac{u + v}{1 + \frac{uv}{c^2}} \tag{2.12}$$

Of-course world is three dimensional, with $X = (x, y, z, ct)$, we have for O moving with v along x direction to O', we have for $\gamma = \sqrt{1 - \frac{v^2}{c^2}}$,

$$\Delta X = \begin{pmatrix} \frac{1}{\gamma} & \frac{v}{c\gamma} & 0 & 0 \\ \frac{v}{c\gamma} & \frac{1}{\gamma} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Delta X' \tag{2.13}$$

Now with $u = (u_x, u_y, u_z)$, we have

$$\upsilon = (\frac{u_x + v}{1 + \frac{u_x v}{c^2}}, \frac{u_y \gamma}{1 + \frac{u_x v}{c^2}}, \frac{u_z \gamma}{1 + \frac{u_x v}{c^2}}) \tag{2.14}$$

For conservation of momentum to hold in relativistic frame transformation we have to define momentum as

$$\mathbf{p} = \frac{m}{\gamma}(v_x, v_y, v_z) \tag{2.15}$$

Kinetic energy can be computed by finding work done in accelerating m_0 from rest to v, this is

$$W = \int d(mv)v = \int d(mv^2) - mvdv = mv^2 + \int d(m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} = mc^2 - m_0 c^2 \tag{2.16}$$

But we have rest mass energy $m_0 c^2$, giving total energy mc^2. To see rest mass energy, Let the energy of the rest mass m_0 in O' be U. Then its energy in O is $U + (m - m_0)c^2$. let this mass disintegrate giving two photons in forward and backward direction of energy $\hbar\omega_0 = \frac{U}{2}$ each. Then in frame 0, the energies of photons are $\hbar\omega_1$ and $\hbar\omega_2$. Then we get

$$\hbar\omega_1 + \hbar\omega_2 = 2\hbar\omega_0(\frac{\omega_1}{2\omega_0} + \frac{\omega_2}{2\omega_0}) = \frac{2\hbar\omega_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This gives $U = m_0 c^2$ and $E = mc^2$.

We can define $p = c(mc, p_x, p_y, p_z)$ and then a direct verification gives the frame transformation,

$$p = \begin{pmatrix} \frac{1}{\gamma} & \frac{v}{c\gamma} & 0 & 0 \\ \frac{v}{c\gamma} & \frac{1}{\gamma} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} p' \tag{2.17}$$

Consider now mass m_0 moving with velocity $u = (u_x, u_y, u_z)$, its energy $E = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}}$ increments slightly if we go in the frame moving with velocity v such that the new velocity is given by Eq. (2.14), and

$$\frac{dE}{dv_x}|v_x = 0 = mu_x,$$

, this is a new interpretation of x momentum, it is differential energy change as we move in a frame in x direction. Similarly y and z momentum can be defined. Now let us use this new interpretation of momentum.

Consider a electron matter wave with frequency, wave-vector (ω, k) and (ω', k') respectively. Then

The phase increment of the matter wave in both frames is the same.

Then

$$k\Delta x - \omega\Delta t = k'\Delta x' - \omega''\Delta t' \tag{2.18}$$

$$\left[\, k \; -\frac{\omega}{c} \,\right] \begin{bmatrix} \Delta x \\ c\Delta t \end{bmatrix} = \left[\, k \; -\frac{\omega}{c} \,\right] \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} \Delta x' \\ c\Delta t' \end{bmatrix} = \left[\, k' \; -\frac{\omega'}{c} \,\right] \begin{bmatrix} \Delta x' \\ c\Delta t' \end{bmatrix} \tag{2.19}$$

This gives

$$\left[\, k \; -\frac{\omega}{c} \,\right] \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{bmatrix} = \left[\, k' \; -\frac{\omega'}{c} \,\right] \tag{2.20}$$

Rewriting this equation we get

$$\begin{bmatrix} k \\ \frac{\omega}{c} \end{bmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} k' \\ \frac{\omega'}{c} \end{bmatrix} \tag{2.21}$$

Once again we use our interpretation of momentum and ask what is $\frac{dE(v)}{dv}|_0 = \hbar k'$. Therefore momentum of our complex wave ω', k' is simply

$$\hbar k'.$$

Thus we have two basic results in quantum mechanics the energy is $\hbar\omega$ and momentum $\hbar k$.

Also note that directly be definition $E = \sqrt{(pc)^2 + (m_0 c^2)^2}$.

2.3 Photon

Consider Maxwell equations in free space in coordinate system O.

$$\nabla \cdot E = 0, \tag{2.22}$$
$$\nabla \cdot B = 0, \tag{2.23}$$
$$\nabla \times B = \mu_0 \frac{\partial D}{\partial t}, \tag{2.24}$$
$$\nabla \times E = -\frac{\partial B}{\partial t}. \tag{2.25}$$

In coordinate system O', the E, B transform to E', B' such that Maxwell equations stay same, i.e., the new E' and B' fields should also satisfy Maxwell equations. So what should be the transformation rule. Recall, we call write from 2.23 that

$$B = \nabla \times \mathbf{A}. \tag{2.26}$$

($\mathbf{A} = (A_x, A_y, A_z)$) and substituting in 2.25, we get

$$E = -\frac{\partial \mathbf{A}}{\partial t} - \nabla A_0. \tag{2.27}$$

With $A^\mu = (A_0, \mathbf{A})$, observe the gauge Transformation

$$A_\mu \rightarrow A_\mu - \partial_\mu \chi, \tag{2.28}$$

does not change E and B so we choose Lorentz gauge

$$\partial_\mu A^\mu = 0. \tag{2.29}$$

Substituting for E, B in terms of A in Eq. 2.22 and 2.24, we find

$$\partial^\mu \partial_u A^\nu = 0. \tag{2.30}$$

Now define

$$A'(x'(x)) = \Lambda\ A(x). \tag{2.31}$$

Then we can check that

$$\partial_\mu A'^\mu = 0, \tag{2.32}$$

and

$$\partial^\mu \partial_u A'^\nu = 0. \tag{2.33}$$

Now we can define E' and B' in terms of A' as in 2.26 and 2.27, this insures that B' and E' satisfy Maxwell equation 2.23 and 2.25. Then using 2.32 and 2.33, we get

E' and B' also satisfy 2.22 and 2.24. Therefore the new E' and B' fields also satisfy Maxwell equations. Remember, transformation rule is 2.31.

Electromagnetic field is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{2.34}$$

We know give a variational interpretation to the equation of A field 2.33. We equip A with a dynamics by defining Lagrangian as density

$$L = -\frac{\epsilon_0}{4} F_{\mu\nu} F^{\mu\nu} = \frac{\epsilon_0}{2}(E^2 - B^2). \tag{2.35}$$

The corresponding energy density is

$$H = \epsilon_0 (F_{0\mu} F^{0\mu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}) = \frac{\epsilon_0}{2}(E^2 + B^2). \tag{2.36}$$

Once we have the Lagrangian, we can write the Euler Lagrange equations that give us Eq. 2.33. Lets make a small detour on how Euler Lagrange equations arise from Lagrangian.

2.3.1 Euler Lagrange Equations

Recall given a mechanical system with Lagrangian $L(x, \dot{x})$, we want to find the trajectory connecting two fixed points that minimize

$$S = \int L(x, \dot{x})\, dt. \tag{2.37}$$

$$\delta S = \int \delta L(x, \dot{x})\, dt, \tag{2.38}$$

$$\delta S = \int \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x}. \tag{2.39}$$

Integrating by parts with δx as 0 at the endpoints/boundary, we have

$$\delta S = \int (\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}) \delta x. \tag{2.40}$$

For above to be true for arbitrary δ we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}. \tag{2.41}$$

As an example consider a spring mass system with mass m and spring constant k, then

$$L(x,\dot{x}) = \frac{1}{2}(m\dot{x}^2 - kx^2). \tag{2.42}$$

Then Euler Lagrange equations read

$$m\ddot{x} = -kx. \tag{2.43}$$

2.3.2 Klein Gordon Field

Consider a scalar field with Lagrangian density

$$L = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m\phi^2), \tag{2.44}$$

Given the action

$$S = \int L\ d^3x \tag{2.45}$$

Then

$$\delta S = \int \partial_\mu\phi\partial^\mu\delta\phi - m\phi\delta\phi\ d^3x \tag{2.46}$$

$$\delta S = \int (-\partial_\mu\partial^\mu\phi - m\phi)\delta\phi\ d^3x \tag{2.47}$$

where we integrate by parts with variation zero at boundary and we get for arbitrary $\delta\phi$, it should be true that

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0, \tag{2.48}$$

or

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = 0. \tag{2.49}$$

2.3.3 Variation of A field

We equip A with a dynamics by defining Lagrangian as density

$$L = -\frac{\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.50}$$

$$S = \int L\ d^3x \tag{2.51}$$

Then

$$\delta S = \int F_{\mu\nu}(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu)\, d^3x, \tag{2.52}$$

$$\delta S = \int \partial^\mu F_{\mu\nu} \delta A^\nu\, d^3x, \tag{2.53}$$

where we used integration by parts and zero variation at the boundary to get

$$\partial^\mu F_{\mu\nu} = 0. \tag{2.54}$$

These are four beautiful Maxwell's equations. If we transform $A = \Lambda A'$, then we see that Maxwell's equations are satisfied in the transformed frame, that says it must indeed be the way that A transforms. This transformation gives $F^{\mu\nu} = \Lambda F'^{\mu\nu}\Lambda^T$.

Invoking the Lorentz gauge we get

$$\partial^\mu \partial_\mu A_\nu = 0, \tag{2.55}$$

i.e.,

$$(\frac{\partial^2}{c^2 \partial t^2} - \nabla^2) A_\nu = 0. \tag{2.56}$$

The solution is for $\varepsilon^\mu = (\varepsilon^0, \varepsilon)$, we have

$$A = \varepsilon \cos(\mathbf{k}\cdot\mathbf{r} - \omega t), \tag{2.57}$$

is a wave propagating in $\mathbf{k} = (k_x, k_y, k_z)$ direction with $\omega = c|\mathbf{k}|$. with $k^\mu = (\frac{\omega}{c}, \mathbf{k})$. The Lorentz gauge condition then becomes

$$k_\mu \varepsilon^\mu = 0. \tag{2.58}$$

For example,

$$A = \varepsilon \cos(k\cdot z - \omega t), \tag{2.59}$$

is a wave propagating in z direction with $\omega = c|k|$. From 2.36, the energy of this A field is $(\varepsilon_2^2 + \varepsilon_3^2)\frac{\epsilon_0 \omega^2}{2c^2} V$. Therefore for $\varepsilon_2^2 + \varepsilon_3^2 = 1$, we have,

$$A = c\sqrt{\frac{2\hbar}{V\epsilon_0\omega}}\varepsilon\cos(k\cdot z - \omega t) = c\sqrt{\frac{\hbar}{2\epsilon_0\omega V}}\varepsilon\ (\ \exp\ i(k\cdot z - \omega t) + \exp\ -i(k\cdot z - \omega t)\ ) \tag{2.60}$$

has energy $\hbar\omega$, and this elementary excitation is termed **Photon**. More generally, when ε is complex

$$A = c\sqrt{\frac{\hbar}{2\epsilon_0\omega V}}\ (\varepsilon\ \exp\ i(k\cdot z - \omega t) + \varepsilon^*\ \exp\ -i(k\cdot z - \omega t)\ ) \tag{2.61}$$

For instance if $\varepsilon = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\i\\0\end{bmatrix}$, we have

$$A = c\sqrt{\frac{\hbar}{\epsilon_0 \omega V}}\begin{bmatrix}0\\ \cos(k\cdot z-\omega t)\\ \sin(k\cdot z-\omega t)\\ 0\end{bmatrix} \tag{2.62}$$

constitutes circularly polarized light. if we move into a frame that rotates around z axis with angular velocity $\Delta\omega$, the find the A transforms to

$$A' = c\sqrt{\frac{\hbar}{\epsilon_0 \omega V}}\begin{bmatrix}0\\ \cos(k\cdot z-(\omega+\Delta\omega)t)\\ \sin(k\cdot z-(\omega+\Delta\omega)t)\\ 0\end{bmatrix} \tag{2.63}$$

If we calculate the energy of A', we get two contributions, one due to z dependence of $\frac{\hbar\omega}{2}$ and t dependence, which is $\frac{\hbar(\omega+\Delta\omega)^2}{2\omega}$. Going from A to A' the energy changes by $\Delta E = \hbar\Delta\omega$ and the angular momentum is just $\frac{\Delta E}{\Delta\omega} = \hbar$. Thus circularly polarized light carries angular momentum of $\hbar$.

2.4 Electron

We now come to relativistic equation of an electron. Recall Pauli matrices

$$\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix};\ \ \sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix};\ \ \sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix};\ \ \mathbf{1} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \tag{2.64}$$

For an electron wave $\exp(i(k\cdot r-\omega t)$, we want relativistic dispersion relation

$$\hbar\omega = \sqrt{(m_o c^2)^2 + (c\hbar k_x)^2 + (c\hbar k_y)^2 + (c\hbar k_z)^2}.$$

For this we make the wave a four vector $\psi = \exp(i(k\cdot r-\omega t)u$, where u is $+1$ eigenvector of $\cos\theta\alpha_\mu + \sin\theta\beta$, where $\sin\theta = \frac{m_0c^2}{E}$ and $\beta = \sigma_x\otimes\mathbf{1}$ and $\alpha_u = -\sigma_z\otimes\sigma_\mu$.

ψ satisfies,

$$i\hbar\frac{\partial\psi}{\partial t} = (mc^2\beta - ic\hbar\alpha_j\partial_j)\psi.$$

Using $\hbar = c = 1$ and multiplying both sides with β gives

$$(i\gamma^\mu\partial_\mu - m)\psi = 0,$$

where $\gamma^0 = \sigma_x\otimes\mathbf{1}$ and $\gamma^\mu = i\sigma_y\otimes\sigma_\mu$. Let us diagonalize the matrix

$$C = m\beta + p_j\alpha_j = E(-\cos\theta\sigma_z \otimes \sigma_\alpha + \sin\theta\sigma_x \otimes \mathbf{1}) \tag{2.65}$$

$$= E\exp(i\frac{\theta}{2}\sigma_y \otimes \sigma_\alpha)(-\sigma_z \otimes \sigma_\alpha)\exp(-i\frac{\theta}{2}\sigma_y \otimes \sigma_\alpha) \tag{2.66}$$

Let $\xi_\alpha^\pm$ be eigenvectors of σ_α with eigenvalues ± 1. Then $\begin{bmatrix}1\\0\end{bmatrix} \otimes \xi_\alpha^\pm$ are eigenevectors of $-\sigma_z \otimes \sigma_\alpha$ with eigenvalues ± 1 and $\begin{bmatrix}0\\1\end{bmatrix} \otimes \xi_\alpha^\pm$ are eigenevectors of with eigenvalues ∓ 1. Then

$$\exp(i\frac{\theta}{2}\sigma_y \otimes \sigma_\alpha)\begin{bmatrix}1\\0\end{bmatrix} \otimes \xi_\alpha^\pm = \begin{bmatrix}\cos\frac{\theta}{2}\\ \pm\sin\frac{\theta}{2}\end{bmatrix} \otimes \xi_\alpha^\pm \tag{2.67}$$

are eigenvectors of C with eigenvalues $\pm E$.

$$\exp(i\frac{\theta}{2}\sigma_y \otimes \sigma_\alpha)\begin{bmatrix}0\\1\end{bmatrix} \otimes \xi_\alpha^\pm = \begin{bmatrix}\mp\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{bmatrix} \otimes \xi_\alpha^\pm \tag{2.68}$$

are eigenvectors of C with eigenvalues $\mp E$. let u and v be these eigenvectors with $\pm$ eigenvalues respectively. Then let

$$u_1(p) = \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} \otimes \xi_\alpha^+; \qquad u_2(p) = \begin{bmatrix}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{bmatrix} \otimes \xi_\alpha^-; \tag{2.69}$$

$$v_1(p) = \begin{bmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{bmatrix} \otimes \xi_\alpha^+; \qquad v_2(p) = \begin{bmatrix}\cos\frac{\theta}{2}\\ -\sin\frac{\theta}{2}\end{bmatrix} \otimes \xi_\alpha^- \tag{2.70}$$

where $1, 2$ represent positive and negative helicity respectively.

2.4.1 Completeness Relation

Then using

$$\sum_{s=1,2} u_s(p)u_s(p)^\dagger + v_s(p)v_s(p)^\dagger = \mathbf{1} \tag{2.71}$$

and

$$E_p \sum_{s=1,2} u_s(p)u_s(p)^\dagger - v_s(p)v_s(p)^\dagger = (mc^2\beta + c\ \alpha_i p_i) \tag{2.72}$$

This gives for $\mathbf{m} = mc^2$,

$$\sum_{s=1,2} u_s(p)\bar{u}_s(p) = \frac{\not{p} + \mathbf{m}}{2E_p} \tag{2.73}$$

$$\sum_{s=1,2} v_s(p)\bar{v}_s(p) = \frac{\not{p} - \mathbf{m}}{2E_p}. \tag{2.74}$$

where $\not{p} = p_\mu \gamma^\mu$.

Coming back to Dirac equation

$$(\gamma^\mu p_\mu + m)u(p) = 0. \tag{2.75}$$

Let momentum p' is related to p by a Lorentz transformation. The Lorentz transformation that takes p to p' can be written as boost from p to rest and arbitrary rotation and then boost from rest to p'. We can represent this as

$$\Lambda = B_{p'}(\zeta')\exp(\theta_1 \Omega_\alpha) B_p(-\zeta), \tag{2.76}$$

where B_x is boost along x direction. On spinor, it takes the form,

$$u(p') = \Sigma u(p), \tag{2.77}$$

where

$$\Sigma = \exp(-\frac{\zeta'}{2}\sigma_z \otimes \sigma_{p'})\exp(i\frac{\theta_1}{2}\mathbf{1} \otimes \sigma_\alpha)\exp(\frac{\zeta}{2}\sigma_z \otimes \sigma_p). \tag{2.78}$$

Then it can be verified that

$$(\gamma^\mu p'_\mu + m)u(p') = 0. \tag{2.79}$$

It should be noted that Eq. (2.77) does not preserve norm. However what is true is that if we normalize the spinors such that $\mathbf{u}(p) = \sqrt{E_p}u(p)$ then under Lorentz transformation,

$$\Sigma \mathbf{u}(p) = \mathbf{u}(p'). \tag{2.80}$$

To see this, let

$$u(p) = \begin{bmatrix} \cos\frac{\theta(p)}{2} \\ \sin\frac{\theta(p)}{2} \end{bmatrix} \otimes \xi_p. \tag{2.81}$$

Consider a spinor $\mathbf{u}(0) = \sqrt{\frac{m}{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \xi_p$ at rest. If we boost it to momentum p, we can write $u(0) \to \mathbf{w}$.

$$\mathbf{w} = \exp(-\frac{\zeta}{2}\sigma_z \otimes \sigma_p)u(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} \exp(-\frac{\zeta}{2}) \\ \exp(\frac{\zeta}{2}) \end{bmatrix} \times \xi_p. \tag{2.82}$$

Since $\mathbf{w} \propto u(p)$, we have

$$\tan\frac{\theta(p)}{2} = \exp(\zeta). \tag{2.83}$$

Then

$$|\mathbf{w}|^2 = m\cosh(\zeta) = \frac{m}{\sin\theta(p)} = E_p, \tag{2.84}$$

but this says that

$$\mathbf{u}(p) = \mathbf{w}. \tag{2.85}$$

Then from 2.78,

$$\Sigma\,\mathbf{u}(p) = \exp(-\frac{\zeta'}{2}\sigma_z \otimes \sigma_{p'}) \exp(i\frac{\theta_1}{2}\mathbf{1} \otimes \sigma_\alpha)\,\mathbf{u}(0), \tag{2.86}$$

$$= \sqrt{m}\,\exp(-\frac{\zeta'}{2}\sigma_z \otimes \sigma_{p'})(a\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \xi^+_{p'} + b\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \xi^-_{p'}), \tag{2.87}$$

where $a^2 + b^2 = 1$. Then

$$\Sigma\,\mathbf{u}(p) = \sqrt{m}\,(a\frac{1}{\sqrt{2}}\begin{bmatrix}\exp(-\frac{\zeta'}{2})\\ \exp(\frac{\zeta'}{2})\end{bmatrix} \otimes \xi^+_{p'} + b\frac{1}{\sqrt{2}}\begin{bmatrix}\exp(\frac{\zeta'}{2})\\ \exp(\frac{-\zeta'}{2})\end{bmatrix} \otimes \xi^-_{p'}), \tag{2.88}$$

$$\propto (a\begin{bmatrix}\cos\frac{\theta(p')}{2}\\ \sin\frac{\theta(p')}{2}\end{bmatrix} \otimes \xi^+_{p'} + b\begin{bmatrix}\sin\frac{\theta(p')}{2}\\ \cos\frac{\theta(p')}{2}\end{bmatrix} \otimes \xi^-_{p'}). \tag{2.89}$$

Then equating coefficients of $\xi_{p'}$ we get $\propto$ in last equation is $\sqrt{E_{p'}}$ and

$$\Sigma\mathbf{u}(p) = \mathbf{u}(p'). \tag{2.90}$$

2.5 Electron-Photon Interaction

2.5.1 Electric-Magnetic Field Lagrangian and Hamiltonian

A charged particle with mass m and charge q in electric-magnetic field has Lagrangian

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - qA_0, \tag{2.91}$$

where $\mathbf{A}$ is vector potential and $B = \nabla \times \mathbf{A}$, i.e.,

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \tag{2.92}$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \tag{2.93}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \tag{2.94}$$

$$E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial A_0}{\partial x}, \tag{2.95}$$

$$E_y = -\frac{\partial A_y}{\partial t} - \frac{\partial A_0}{\partial y}, \tag{2.96}$$

$$E_z = -\frac{\partial A_z}{\partial t} - \frac{\partial A_0}{\partial z}. \tag{2.97}$$

We have Euler Lagrange equations

$$m\ddot{x} + q\dot{A_x} = q(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}) - q\frac{\partial A_0}{\partial x}. \tag{2.98}$$

Writing

$$\dot{A_x} = \frac{\partial A_x}{\partial t} + \dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{y}\frac{\partial A_x}{\partial y}. \tag{2.99}$$

Substituting in 2.98 we get ,

$$m\ddot{x} = q(\dot{y}B_z - \dot{z}B_y) - q(\frac{\partial A_x}{\partial t} + \frac{\partial A_0}{\partial x}), \tag{2.100}$$

and similarly for y, z gives in all that

$$m\dot{v} = q(v \times B + E). \tag{2.101}$$

The *Lorentz* force law. The momentum $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x$ and similarly for y, z gives the Hamiltonian or energy

$$H = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m} + \frac{(p_z - qA_z)^2}{2m} + qA_0. \tag{2.102}$$

The energy E is then

$$E - qA_0 = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m} + \frac{(p_z - qA_z)^2}{2m}. \tag{2.103}$$

The relativistic generalization is

$$E - qA_0 = c\sqrt{(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2 + (mc)^2}. \tag{2.104}$$

We use this energy to define Dirac equation in the electric-magnetic field.

2.5.2 Gauge Coupling and Transitions

Recall how the Dirac equation reads with $p^\mu = (E, \mathbf{p}c)$, we have

$$(\gamma^\mu p_\mu - m)\psi = 0 \tag{2.105}$$

From 2.104, we have the Dirac equation in presence of electromagnetic field as $p_\mu \rightarrow p_\mu - qA_\mu$. If we identify p_μ with $i\partial_\mu$, the Dirac equation reads

$$(i\gamma^\mu \partial_\mu - mc - qA_\mu \gamma^\mu)\psi = 0 \tag{2.106}$$

or in terms of matrices α_j, β,

$$i\partial_t \psi = (-c\alpha_j \, i\partial_j + mc^2\beta - qA_j\alpha_j + qA_0)\psi \tag{2.107}$$

without A, we have the free Dirac equation

$$i\partial_t \psi = (-c\alpha_j \, i\partial_j + mc^2\beta)\psi \tag{2.108}$$

The term

$$T = -qA_j\alpha_j + qA_0, \tag{2.109}$$

is **transition term**. In absence of this if electron is in eigenstate of the Dirac equation $\psi_1 = u_1(p)\exp(ip\cdot r)$, it will stay in this state. In presence of the transition term it makes a transition. Lets imagine a photon is present, then A is as in Eq. 2.110

$$A = c\sqrt{\frac{2\hbar}{\epsilon_0 \omega V}}\varepsilon \cos(k\cdot r - \omega t) = c\sqrt{\frac{\hbar}{2\epsilon_0\omega}}\varepsilon \; (\; \exp\; i(k\cdot r - \omega t) + \exp\; -i(k\cdot r - \omega t)\;) \tag{2.110}$$

Then T acting on ψ_1 induces change $u_1(p)\exp(ip\cdot r) \rightarrow u_1(p+k)\exp(i(p+k)\cdot r)$. Lets call the state $u_1(p+k)\exp(i(p+k)\cdot r)$ as ψ_2. Let x_1 and x_2 denote coefficients of ψ_1 and ψ_2, Then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-i}{\hbar}\begin{bmatrix} E_p & \Omega^\dagger \exp(i\omega t) \\ \Omega\exp(-i\omega t) & E_{p+k} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.111}$$

where E_P and E_{p+k} are energies of the initial and final electron states and

$$\Omega = qc\sqrt{\frac{\hbar}{2V\epsilon_0\omega}}\; u_1^\dagger(p+k)(-\epsilon_j\alpha_j + \epsilon_0)u_1(p) = \frac{C}{\sqrt{2E_k}}\,\bar{u}_1(p+k)(\epsilon_\mu\gamma^\mu)u_1(p), \tag{2.112}$$

where $\bar{u} = u^\dagger\beta$, $E_k = \hbar\omega$ the energy of the photon and

$$C = \frac{qc\hbar}{\sqrt{V\epsilon_0}}. \tag{2.113}$$

If we denote $\tilde{x}_1 = \exp(-i\omega t)x_1$, then

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_1\\ x_2\end{bmatrix} = \frac{-i}{\hbar}\begin{bmatrix}E_p + E_k & \Omega^*\\ \Omega & E_{p+k}\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ x_2\end{bmatrix} \tag{2.114}$$

where $E_k = \hbar\omega_k$ the energy of the photon. Observe $\tilde{x}_1$ is coefficient of state evolving with energy $E_p + E_k$, the joint state of electron and photon. d then represent the transition out of this state. The initial state has electron and photon, while the final state only has electron. We denote this as

$$|p, k\rangle \rightarrow |p + k\rangle \tag{2.115}$$

Drawn as a energy level diagram it looks like Fig. 2.2.

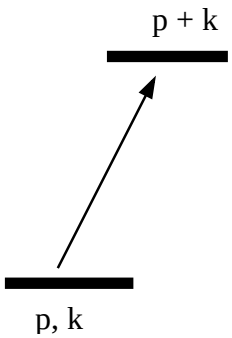


Fig. 2.2 Fig. shows transitions between electron, photon states.

We showed transition to state $u_1(p + k)$, similarly we have transition to state $u_2(p + k)$, etc, where $1, 2$ represent helicity.

2.5.3 Feynman Diagrams

The above level diagram is also represented by so called a Feynman diagram as shown in 2.3. An electron and photon of momentum p and k respectively react to form an electron of momentum $p + k$.

This reaction is very unfavourable for large energy difference between input and out states of the two levels as shown in fig. 2.2. However the outgoing electron can immediately dissociate into an electron and photon of momentum p' and k such that $p' + k' = p + k$. Furthermore $E_P + E_k = E_{p'} + E_{k'}$. so that the initial and final

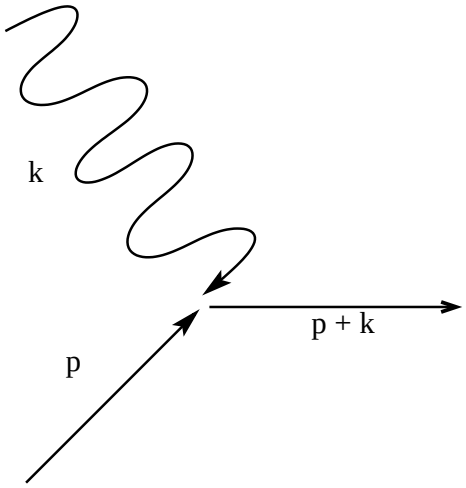


Fig. 2.3 Fig. shows a Feynamn diagram of electron and photon of momentum p and k respectively react to form an electron of momentum $p+k$

states have same energy-momentum. Then the overall reaction which is represented by a level diagram in 2.4A and Feynman diagram 2.4B becomes favorable. We can compute the rate of this reaction, which we do in the next chapter on Quantum Electrodynamics a subject that calculates such reaction rates.

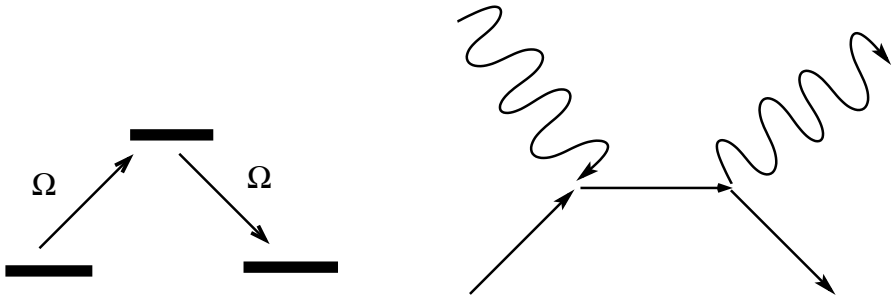


Fig. 2.4 Fig. A shows a three level diagram and Fig. B shows a Feynman diagram for electron-photon scattering.

2.6 Lorentz gauge vs $E \cdot x$ gauge

For a plane wave along z direction, with electric field $E_x \sin(kz - \omega t)$, the Lorentz gauge is

$$(A_0, A_x, A_y, A_z) = \frac{E_x}{\omega} \cos(kz - \omega t)(0, 1, 0, 0)$$

. But this gauge is not suited for calculating optical transitions, because we don't recover the Rabi frequency $qE_x d$ (d electric dipole moment). What we find is something orders of magnitude smaller. Nor is it suitable for calculating electron electron scattering as described in next chapter on quantum electrodynamics (QED), because we don't recover Coulomb potential. What we find is something orders of magnitude smaller. Instead, we work with $E \cdot x$ gauge

$$(A_0, A_x, A_y, A_z) = \frac{E_x}{2}(-x\ \sin(kz - \omega t), \frac{\cos(kz - \omega t)}{\omega}, 0, -\frac{x}{c}\sin(kz - \omega t)) \quad (2.116)$$

(c light velocity) to find everything correct. What we get is new propagator that describes amplitude of electron electron scattering which gives us Coulomb potential.

A more general gauge is

$$(A_0, A_x, A_y, A_z) = E_x(-\cos^2\theta x\ \sin(kz - \omega t), \sin^2\theta \frac{\cos(kz - \omega t)}{\omega}, 0, -\cos^2\theta \frac{x}{c}\sin(kz - \omega t)), \quad (2.117)$$

we will find that we use this general gauge for successful renormalization in the next chapter, when photon wavelength is very small compared to Compton wavelength.

Lets see why in Optical transitions, Rabi frequency goes bad, if we donot use right gauge. This equation is not very tractable, because it is nonlinear in A, lets write a linear equation, which is the Dirac equation, which takes the form

$$i\frac{\partial \phi}{\partial t} = \left(\sum_{j=x,y,z} c(-i\,\hbar \frac{\partial}{\partial x_j} - qA_j)\alpha_j + \beta mc^2 + qA_0 \right) \phi. \quad (2.118)$$

where $\alpha_j = \sigma_z \otimes \sigma_j$ and $\beta = \sigma_x \otimes \mathbf{1}$ are Dirac matrices, where σ_j are the Pauli matrices, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ϕ is electron spinor, for a electron wave with momentum k, takes the form $\phi = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \otimes \uparrow$, where $\uparrow$ is spin up, $\cos\theta = \frac{\hbar k}{mc} = \frac{v}{c}$, where $v = \frac{\hbar k}{m}$, is electron wave group velocity. Electron Orbitals are of size $\sim A^\circ$, their $k \sim 10^{10} m$, then $v \sim 10^6 m/s$ and $\cos\theta \sim \frac{10^6}{3\times 10^8} \sim 10^{-3}$. Electron is non-relativistic, $\cos\theta = \frac{v}{c} \sim 0$, $\theta \sim \frac{\pi}{2}$, $\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \uparrow$.

To fix ideas, take incoming EM wave, along z direction, with electric field $E_x \sin(kz - \omega t)$, the Lorentz gauge is $(A_0, A_x, A_y, A_z) = \frac{E_x}{\omega}\cos(kz - \omega t)(0, 1, 0, 0)$. Electron wave with momentum q absorbs the photon with momentum k, and transits to momentum $q + k$. The transition is driven by Dirac matrix α_x, with transition amplitude

$$\mathcal{M} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \otimes \uparrow \underbrace{(\sigma_z \otimes \sigma_x)}_{\alpha_x} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \otimes \downarrow = qcA_x \frac{\upsilon}{c} = qE_x \frac{\upsilon}{\omega} \tag{2.119}$$

If we have electron orbital ϕ_0 then $k' = \frac{M}{M+m}k$ of photon momentum goes to electron-nuclear relative coordinate, while $k'' = k$ momentum to CM (center of mass), where M is nucleus mass. The process drives the transition

$$\phi_0\uparrow \longrightarrow \exp(ik'z)\phi_0\downarrow,$$

with amplitude $\mathcal{M} = qE_x\frac{\upsilon}{\omega}$.

When orbital ϕ_1 is different from ϕ_0 we go to

$$\phi_0\uparrow \longrightarrow \exp(ik'z)\phi_0\downarrow,$$

with amplitude $\mathcal{M} = qcA_x\frac{\upsilon}{c}$ whose overlap with ϕ_1 is

$$\mathcal{M}_1 = qcA_x\frac{\upsilon}{c}ik'\underbrace{\langle\phi_1|z|\phi_0\rangle}_{d_z} = iqE_x d_z\ \frac{\upsilon}{c},$$

where $ck' \sim \omega$.

But this is not suited for study of optical transitions, because we don't recover the Rabi frequency $qE_x d$. What we find is orders of magnitude smaller (down by $\frac{\upsilon}{c}$). Instead we work with gauge

$$(A_0, A_x, A_y, A_z) = \frac{-E_x}{2}(x\ \sin(kz-\omega t), -\frac{\cos(kz-\omega t)}{\omega}, 0, \frac{x}{c}\sin(kz-\omega t)).$$

Now we have process driven by x term. For the $\mathcal{E}$ process, the amplitude of $\phi_0 \to \phi_0$ is just 0, as $\langle\phi_0|x|\phi_0\rangle = 0$ and the amplitude of $\phi_0 \to \phi_1$ is simply

$$\mathcal{M}_1' = qE_x\underbrace{\langle\phi_1|x|\phi_0\rangle}_{d_x} = qE_x d_x,$$

Dipole elements d_z, d_x are approx, Bohr radius $\sim A^\circ$. Due to the factor $\frac{\upsilon}{c} \sim 10^{-3}$, $\mathcal{M}_1 \ll \mathcal{M}_1'$. Therefore transition between different atomic orbitals are largely driven by the x term.

2.7 Schröedinger Equation

In classical mechanics, we talk about a particle say an electron with a position x and velocity v. In quantum mechanics, particle state is represented by complex waves $\exp(ikx)$ or sum of such waves $\sum_j \exp(ik_j x)$. In complex wave $\exp(ikx)$, k is the wavenumber of the particle. The wave evolves in time as $\exp(i(kx-\omega(k)t))$, $\omega(k)$

is the frequency of the wave and depends on wavenumber k. The dependence $\omega(k)$ is called the dispersion relation of the wave. First postulate of quantum mechanics is that the energy of the wave is $E = \hbar\omega(k)$, where $\hbar$ is a fundamental constant called Planck's constant. Its units are angular momentum and in SI units its value is 6.6×10^{-34}.

The momentum of our complex wave ω, k is simply $\hbar k$.

Now from classical mechanics $E = \frac{p^2}{2m}$. Then we get $\hbar\omega = \frac{\hbar^2 k^2}{2m}$ or $\omega = \frac{\hbar k^2}{2m}$. Thus my complex wave $\psi(x,t) = \exp(i(kx - \omega t))$ satisfies

$$i\hbar\frac{\partial\psi}{\partial t} = (-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}). \tag{2.120}$$

This equation (2.120) is called *Schröedinger* equation. It is still true if we have

$$\psi(x,t) = \sum_j \alpha_j \exp(i(k_j x - \omega(k_j)t)).$$

as individual exponential satisfy these equation.

$\psi(x,t)$ is called a wavefunction of electron, it is superposition of plane waves. This is a feature of quantum mechanics, we can be in superposition of states. It satisfies the Schröedinger equation. All we are saying is that if we start with initial state $\psi(x) = \sum_j \alpha_j \exp(ik_j x)$, these ways will evolve by their characteristic energies as $\psi(x,t) = \sum_j \alpha_j \exp(i(k_j x - \omega(k_j)t))$ and $\psi(x,t)$ satisfies the Schröedinger equation.

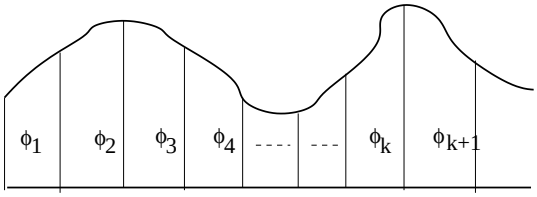


Fig. 2.5 Figure shows how $V(x)$ is decomposed as piece-wise constant potential.

Now how does my wavefunction evolve if I have a potential V. Then from classical mechanics $E - V = \frac{p^2}{2m}$, implying $\hbar\omega - V = \frac{\hbar^2 k^2}{2m}$ or my wave satisfies

$$i\hbar\frac{\partial\psi}{\partial t} = (-\hbar\frac{\partial^2}{\partial x^2} + V)\psi. \tag{2.121}$$

and again same is true if we have superposition of plane waves.

Now how does the evolution of $\psi(x)$ take place when we have $V(x)$. Then we can break $\psi(x)$ into small pieces ϕ_i over which $V(x)$ is constant as V_i. See fig 2.5. Then each ϕ_i sees a potential V_i. Its evolution will be same if V_i was globally true. Then we can break ϕ into exponentials and conclude it satisfies the equation

$$i\hbar\frac{\partial\phi_i}{\partial t} = (-\hbar\frac{\partial^2}{\partial x^2} + V_i)\phi_i. \tag{2.122}$$

Then adding them all we get

$$i\hbar\frac{\partial\psi}{\partial t} = (-\hbar\frac{\partial^2}{\partial x^2} + V(x))\psi. \tag{2.123}$$

Thus we have derived a fundamental equation of quantum mechanics. Wavefunction $\psi(x)$ has a probabilistic interpretation. $\int_a^b |\psi(x)|^2 dx$ gives the probability of finding the particles in the interval $[a, b]$

2.7.1 Hydrogen Atom

In polar coordinates $r = \sqrt{x^2+y^2}$ and $\phi = \tan^{-1}(\frac{y}{x})$.

Then

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}.$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2}.$$

$$\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2}.$$

Using $R = \sqrt{z^2+r^2}$ and $\theta = \tan^{-1}(\frac{r}{z})$.

$$\begin{aligned}\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial}{\partial \theta^2} + \frac{\cot\theta}{R^2}\frac{\partial}{\partial \theta} + \frac{1}{R^2\sin^2\theta}\frac{\partial}{\partial \phi^2} \\ &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial}{\partial R}) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{R^2\sin^2\theta}\frac{\partial}{\partial \phi^2}.\end{aligned}$$

To Schroedinger Eigenvalue Eq. reals

$$\{\frac{\hbar^2}{2m}(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial}{\partial R}) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{R^2\sin^2\theta}\frac{\partial}{\partial \phi^2}) + (E - V(R))\}\psi = 0.$$

$$\{(\frac{\partial}{\partial R}(R^2\frac{\partial}{\partial R}) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{\sin^2\theta}\frac{\partial}{\partial \phi^2}) + \frac{2mR^2}{\hbar^2}(E - V(R))\}\psi = 0.$$

We write the solution $\psi = f(R)Y(\theta, \phi)$.

$$(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}+\underbrace{l(l+1)}_{E_1})Y(\theta,\phi)=0. \tag{2.124}$$

Writing $Y(\theta,\phi)=\Theta(\theta)e^{im\phi}$, we get

$$(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})-\frac{1}{\sin^2\theta}m^2+\underbrace{l(l+1)}_{E_1})\Theta(\theta)=0.$$

For $x=\cos\theta$, the above equation reads

$$(1-x^2)\Theta''-2x\Theta'+(l(l+1)-\frac{m^2}{1-x^2})\Theta=0.$$

The solution Θ_l^m exits for integer l,m satisfying $0\le|m|\le l$. For $m\ge 0$

$$\Theta_l^m(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

with

$$\Theta_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}\Theta_l^m(x).$$

Then the equation for R gives

$$\frac{\partial}{\partial R}(R^2\frac{\partial f}{\partial R})=(l(l+1)+\frac{2mR^2}{\hbar^2}(V(R)-E))f.$$

Let $u=Rf$, then

$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial R^2}+(V+\frac{\hbar^2}{2mR^2}l(l+1))u=Eu,$$

where $V=\frac{-e^2}{4\pi\epsilon_0 r}$. This is one-dimensional Schroedinger equation. Guess a solution of the form $u(r)=R^{l+1}e^{-\frac{R}{a_0}}$. Then twice differentiating R^{l+1} cancels the centrifugal part. Differentiating R^{l+1} and $e^{-\frac{R}{a_0}}$, cancels V, when $\frac{\hbar^2}{m}\frac{l+1}{a_0}=\frac{e^2}{4\pi\epsilon_0}$, i.e,

$$a_0=\frac{(l+1)\hbar^2 4\pi\epsilon_0}{me^2},\quad E=\frac{\hbar^2}{2ma_0^2}.$$

However, we donot have to cancel V immediately. We can add another term

$$u(r)=R^{l+1}e^{-\frac{R}{a_0}}+c_1R^{l+2}e^{-\frac{R}{a_0}}.$$

Then centrifugal part of second term c_1 can cancel the part of first term obtained by differentiating R^{l+1} and $e^{-\frac{R}{a_0}}$. For this c_1 has to be chosen correct. Now we cancel V by differentiating R^{l+2} and $e^{-\frac{R}{a_0}}$.

Then in general

$$u(r) = R^{l+1} e^{-\frac{R}{a_0}} (1 + \sum_{j=1}^{d} c_j R^j),$$

with $n = l + d + 1$, the principle quantum number. Then

$$\frac{\hbar^2}{m} \frac{n}{a_0} = \frac{e^2}{4\pi\epsilon_0}. \quad a_0 \propto n$$

and

$$\frac{c_j}{c_{j-1}} = \frac{2(l+j-n)a_0^{-1}}{j(2l+j+1)}.$$

This gives a_0 and finally

$$E = \frac{\hbar^2}{2ma_0^2}, \quad E \propto \frac{1}{n^2}.$$

2.7.2 Angular Momentum

$$L = r \times p.$$

$$L_x = ypZ - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x.$$

Using $[p_x, x] = -i\hbar$, etc, we have

$$L^2 = L_x^2 + L_y^2 + L_z^2 = R^2(p_x^2 + p_y^2 + p_z^2) - (xp_x + yp_y + zp_z - i\hbar)^2 + \hbar^2.$$

A quick calculation shows

$$xp_x + yp_y + zp_z = -i\hbar R \frac{\partial}{\partial R}.$$

Now substituting for

$$p_x^2 + p_y^2 + p_z^2 = -\hbar^2 \left(\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial}{\partial R}\right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial}{\partial \phi^2}\right) \quad (2.125)$$

$$L^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial}{\partial \phi^2}\right). \quad (2.126)$$

Then from Eq. 2.124,

$$L^2Y(\theta,\phi) = \hbar^2 l(l+1).$$

and

$$L_zY(\theta,\phi) = -i\hbar\frac{\partial}{\partial\phi}Y(\theta,\phi) = \hbar mY(\theta,\phi).$$

We denote the eigenfunction as Y_{lm}
Observe easily verifiable commutation relations

$$[L_x,L_y] = i\hbar L_z,\ \ [L_y,L_z] = i\hbar L_x,\ \ [L_z,L_x] = i\hbar L_y. \tag{2.127}$$

Define $L^- = L_x - iL_y$ and $L^+ = L_x + iL_y$. L^- is called lowering operator and L^+ is called raising operator.

$$[L^2,L^\pm] = 0,\ \ [L_z,L^\pm] = \pm\hbar L^\pm. \tag{2.128}$$

Then note $[L^2,L^\pm]Y_{lm} = 0$ implies $L^2L^\pm Y_{lm} = \hbar^2 l(l+1)L^\pm Y_{lm}$ hence $L^\pm Y_{lm}$ is a linear combination of Y_{lm} for different m. Now $[L_z,L^+]Y_{lm} = L^+Y_{lm}$ implying $L_zL^+Y_{lm} = \hbar(m+1)L^+Y_{lm}$ implying $L^+Y_{lm} = a_mY_{l,m+1}$. Similarly $L^-Y_{lm} = b_mY_{l,m-1}$. Then observe $L^+Y_{ll} = 0$ and $L^-Y_{l,-l} = 0$. Furthermore

$$[L^+,L^-] = 2\hbar L_z. \tag{2.129}$$

Furthermore we get

$$L^+L^- + L^-L^+ = 2(L^2 - L_z^2). \tag{2.130}$$

Then we get

$$L^+L^- = L^2 - L_z^2 + \hbar L_z \tag{2.131}$$

$$L^-L^+ = L^2 - L_z^2 - \hbar L_z. \tag{2.132}$$

For normalized Y_{lm} we get

$$b_{lm} = \hbar\sqrt{l(l+1) - m(m-1)} \tag{2.133}$$

$$a_{lm} = \hbar\sqrt{l(l+1) - m(m+1)} \tag{2.134}$$

We talked about orbitals with principle quantum number n and integer angular momentum number l and z angular momentum l, with $|m| \le l \le n-1$. Here l was integer. In principle it can be half integer and is ascribed to an intrinsic angular momentum called spin. We use the quantum number s instead of l. In particular $s = \frac{1}{2}$ is called spin $\frac{1}{2}$ a property of electron. We then have two values of $s_z = \pm\frac{1}{2}$. Then an electron as two set of quantum numbers l,m and s,s_z.

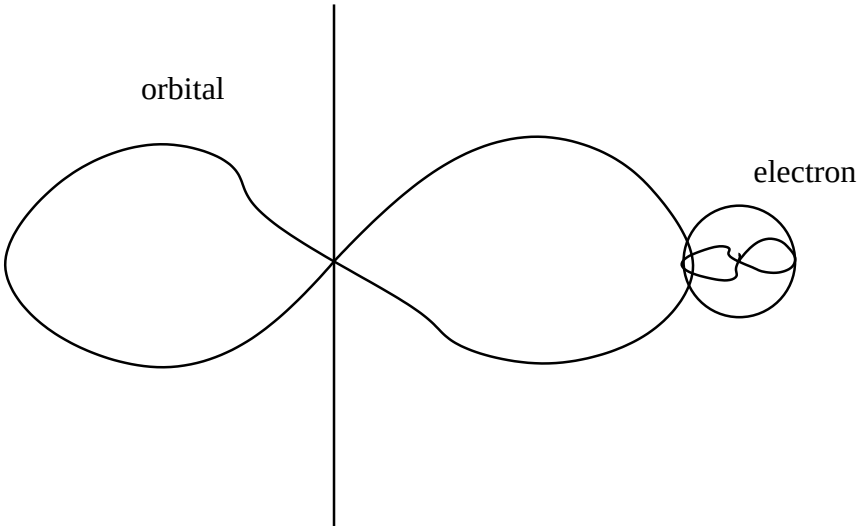


Fig. 2.6 Fig. shows an atomic orbital and an electron with an inner orbital that constitutes its spin angular momentum

2.8 Fine Structure and Spin orbital coupling

We talked about spin. Lets try to understand the physics of it. You are familiar with earth spinning on its axis. This gives earth a angular momentum. Now imagine our earth was charged. Then spinning will give earth a magnetic moment. Imagine a loop of wire carrying current (circulating charge), then it has a magnetic moment $M = I.A$, where I is the current and A area of the loop, from your basic physics. Now imagine a charge q going around in a loop of radius r, with angular velocity ω. Then it makes $\frac{\omega}{2\pi}$ rotations per sec. The current is then $\frac{q\omega}{2\pi}$ and its magnetic moment is $\mu_S = \frac{q\omega\pi r^2}{2\pi} = \frac{q}{2m}(mvr)$ where $l = mvr$ is the angular momentum. Then $\mu_S = \frac{q}{2m}L$, the ratio $\gamma = \frac{q}{2m}$ is called the gyromagnetic ratio, it relates angular momentum to magnetic moment. For reasons coming from relativity we infact have $\gamma = \frac{q}{m}$.

There is coupling between electron spin and orbital angular momentum. There is coupling Hamiltonian of the form

$$H_{so} = \beta L \cdot S. \tag{2.135}$$

Let us see how this coupling arises. When electron is at a certain point on its orbital it has a velocity v and momentum p. From perspective of the electron the nucleus is moving in the opposite direction with same magnitude of velocity. Then from Biot Savart law the moving nucleus produces a magnetic field on the site of electron given by

$$B = \frac{e\mu_0}{4\pi}\frac{p \times r}{mr^3} = \frac{e\mu_0}{4\pi}\frac{L}{mr^3}. \tag{2.136}$$

The energy of the electron in this field is

$$B \cdot \mu_S = \gamma B \cdot S = \frac{e^2 \mu_0}{4\pi m^2 r^3} L \cdot S = \frac{e^2}{4\pi \epsilon_0 c^2 m^2 r^3} L \cdot S. \tag{2.137}$$

Thus $\beta = \frac{e^2}{4\pi \epsilon_0 c^2 m^2 r^3}$. Due to phenomenon called Thomas precession, β is called by another factor of 2 and $\beta = \frac{e^2}{8\pi \epsilon_0 c^2 m^2 r^3}$.

In presence of this Hamiltonian our orbitals will change. let us compute how the orbitals change and what are the new energies.

$$L \cdot S = L_z S_z + L_x S_x + L_y S_y = L_z S_z + \frac{L^+ S^- + L^- S^+}{2}. \tag{2.138}$$

For this define a new operator

$$J^2 = (L+S)^2 = L^2 + S^2 + 2L \cdot S. \tag{2.139}$$

$$J_z = L_z + S_z, \quad J^\pm = L^\pm + S^\pm. \tag{2.140}$$

Given l and s, we start with the state $l_z = l$ and $s_z = s$. Denote this state by (l, s). This state is an eigenstate of the operator $L \cdot S$ with eigenvalue l, s and hence it is an eigenstate of J^2 with eigenvalue $j(j+1)$ with $j = l+s$. Now as before we can apply lowering operator. From last section $J^-(j, j_z) = b(j, j_{z-1})$ with $b = \hbar\sqrt{j(j+1) - j_z(j_z-1)}$, so by applications of J^- we decrease j_z until it is $-j$. Hence we have constructed $2j$ or $2j+1$ orbitals depending on if j is integer or half integer.

Observe $J^-(l, s) = (l-1, s) + (l, s-1)$. There is another orthogonal state $e_1 = (l-1, s) - (l, s-1)$ which is eigenfunction of J_z with eigenvalue $l+s-1$ and hence must be an eigenfunction of J^2. We eigenvalue of J^2 cannot be $j(j+1)$ as we have exhausted all these vectors as $J^+ e_1 = 0$. Only possible value of J^2 is $(j-1)j$, we gain apply lowering operators and go from $j_z = j-1, \ldots, -(j-1)$.

Now we consider $J^{-2}(l, s) = (l-2, s) + (l-1, s-1) + (l, s-2)$, which has $J_z = l+s-2$. We have constructed two eigenvectors $J^2 = j(j+1)$ and $J^2 = (j-1)j$. We can form a third eigenvector, we can show it has J^2 value $(j-1)(j-2)$, we can again apply lowering operators and construct eigenvectors with J^2. Instead of writing J^2 we say this J which in this case has value $j-2$.

We start with one term (l, s). Then $J^-(l, s)$ has two terms, $J^{-2}(l, s)$ has three terms. This process continues till smaller of l, s say s becomes $-s$. Then lowering doesn't increase number of terms. Then starting with $j = l+s$ we go until $j = l-s$. Thus all states can be indexed by $j = l+s, \ldots, l-s$ and for a given j we have $j_z = j, \ldots, -j$. Thus starting with state $|l, l_z\rangle|s, s-z\rangle$ we have formed state

$$|j, j_z\rangle = \sum_{l_z, s_z} c_{l_z, s_z} |l, l_z\rangle |s, s_z\rangle, \tag{2.141}$$

where as just told, $j = l+s, \ldots, l-s$ and for a given j we have $j_z = j, \ldots, -j$.

In the basis $|j, j_z\rangle$, we have $L \cdot S$ is diagonal with eigenvalue $\frac{j(j+1)-l(l+1)-s(s+1)}{2}$. The coefficients c_{l_z,s_z} are called Clebsch Gordon coefficients. Fig. (2.7) shows how $n = 2$, p orbital gets split due to fine structure.

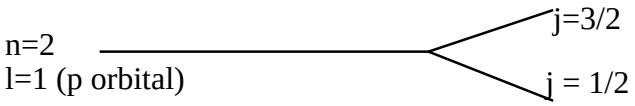


Fig. 2.7 Fig. shows how $n = 2$, p orbital gets split due to fine structure.

As we can see in the figure. A energy level $n = 1, l = 1$ in presence of $L \cdot S$ coupling gets split into two set of orbitals $j = \frac{3}{2}$ with $(j_z = \frac{3}{2}, \ldots, -\frac{3}{2})$ and $j = \frac{1}{2}$ with $(j_z = \frac{1}{2}, \ldots, -\frac{1}{2})$ with different energies. This is called *fine-structure*. If we estimate how big this is it is $\beta = \frac{\hbar^2 e^2}{4\pi c^2 \epsilon_0 m_e^2 r^3} \sim 10^2 eV \sim 10^3 GHz$. It arises because the angular momentum of the orbital and the spin of the electron talk to each other. Evaluating spin orbit coupling,

$$\langle \frac{1}{r^3} \rangle = \frac{1}{n^3 l(l+\frac{1}{2})(l+1)a^3} \tag{2.142}$$

$$E_{so} = \alpha^4 mc^2 \frac{j(j+1) - l(l+1) - \frac{3}{4}}{4n^3 l(l+\frac{1}{2})(l+1)} \tag{2.143}$$

Electron has a spin, so does the nucleus of the atom. It is called nuclear spin. We denote nuclear spin with I like we denote electron spin with S. We assume that we again have an interaction between nuclear spin and electron orbital and spin angular momentum as

$$I \cdot (L+S) = I \cdot J. \tag{2.144}$$

What was between L and S is between I and J so we can define the total angular momentum

$$F = I + J. \tag{2.145}$$

Given i and j the coupling gives f taking values between $i+j, \ldots, |i-j|$. Thus a j orbital gets split into f orbitals. This is called hyperfine splitting. The eigenvalues of $I \cdot J$ takes one values $\frac{f(f+1)-j(j+1)-i(i+1)}{2}$. Thus if we estimate how much this is, it is $\beta = \frac{\hbar^2 e^2 \mu_0}{4\pi m_e m_p r^3} \sim 1 GHz$, where m_p is proton mass which is 10^3 heavier than electron mass.

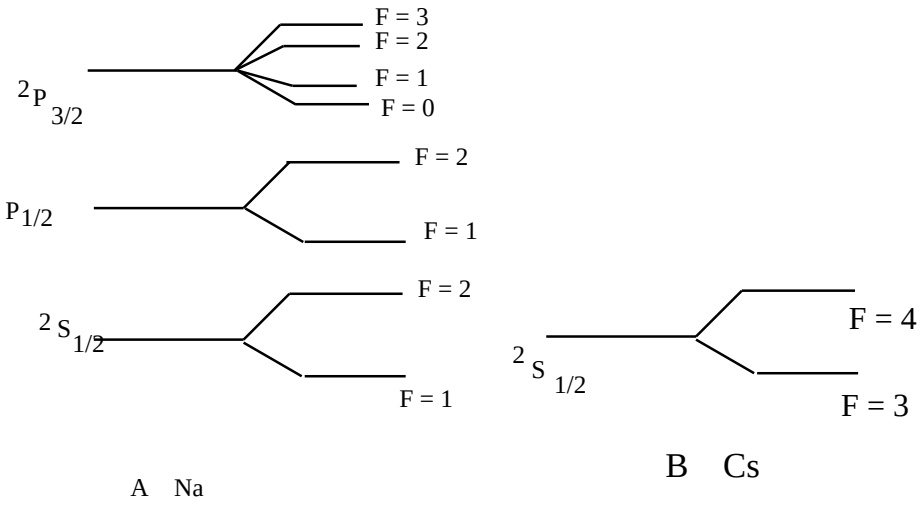


Fig. 2.8 Fig. A shows hyperfine levels for sodium. Fig. B shows hyperfine levels for Cesium

2.9 Relativistic Correction

In Schröedinger equation we used kinetic energy as $\frac{p^2}{2m}$. If we use relativistic formula of

$$E = \sqrt{(pc)^2 + (mc^2)^2} \sim mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}. \tag{2.146}$$

Then we find that

We have correction to energy

$$\Delta E_{rel} = -\frac{p^4}{8m^3c^2}. \tag{2.147}$$

Using the fact $\frac{p^2}{2m} = E - V$, on an orbital we can calculate

$$\langle \Delta E_{rel} \rangle = -\frac{1}{2mc^2} \; E_n^2 - 2E_n\langle V \rangle + \langle V^2 \rangle. \tag{2.148}$$

$$E_n = -\frac{\alpha^2 mc^2}{2n^2}; \;\; \alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}. \tag{2.149}$$

Using $V = \frac{e^2}{4\pi\epsilon_0 r}$,

$$\langle \frac{1}{r} \rangle = \frac{1}{n^2 a}, \;\; \langle \frac{1}{r^2} \rangle = \frac{1}{n^3 (l + \frac{1}{2}) a^2} \tag{2.150}$$

where a is Bohr radius. Putting everything together we find

$$\langle \Delta E_{rel} \rangle = -\frac{\alpha^4 mc^2}{4n^4}\left(\frac{2n}{l+\frac{1}{2}} - \frac{3}{2}\right). \tag{2.151}$$

Adding Eq. (2.152) and (2.143) we get what is called fine structure

$$\langle \Delta E_{fs} \rangle = -\frac{\alpha^4 mc^2}{4n^4}\left(\frac{2n}{j+\frac{1}{2}} - \frac{3}{2}\right). \tag{2.152}$$

2.10 Lamb Shift

2.11 Positronium

Positronium (Ps) is a system consisting of an electron and its anti-particle, a positron, bound together into an exotic atom, specifically an onium. The system is unstable: the two particles annihilate each other to predominantly produce two or three gamma-rays, depending on the relative spin states. The orbit and energy levels of the two particles are similar to that of the hydrogen atom (which is a bound state of a proton and an electron). However, because of the reduced mass, the frequencies of the spectral lines are less than half of the corresponding hydrogen lines.

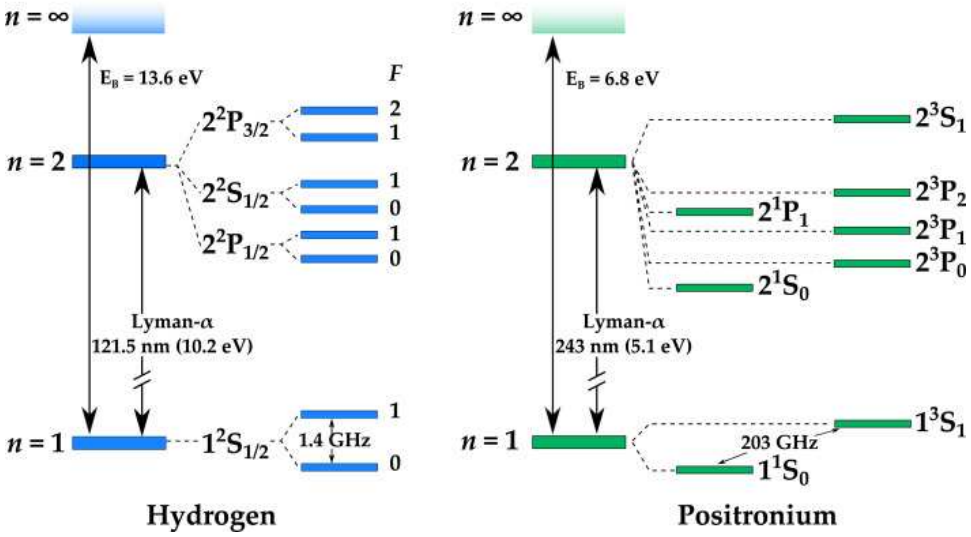


Fig. 2.9 Fig. shows energy levels for hydrogen atom and positronium.

2.12 Problems

1. Frame O' moves with respect to frame O with velocity u along x axis. If velocity of a particle in frame O' is $v' = (v'_x, v'_y, v'_z)$, find its velocity in frame O.

2. The rest mass of a proton is 938 Mev/c^2. If it moves with velocity .9 c, where c is velocity of light, find its energy.

3. In above problem if the proton moves such that its momentum is 1000 Mev/c, find its energy.

4. Muon at rest, decay in time 2.2 μsec. How much time will it take them to decay if they are moving at velocity .99 c.

5. Lagrangian of electromagnetic field is

$$L = \frac{-\epsilon_0}{4} \int F_{\mu\nu} F^{\mu\nu} d^3x$$

   By taking variation of L show that we get Maxwell equations $\partial^\mu F_{\mu\nu} = 0$.

Chapter 3
Optical Scattering

In this chapter, we develop a subtle nuance in our understanding of the optical transitions [33, 34]. The starting point is the electron Hamiltonian, H, in a electromagnetic wave. $H = \sum_{j=x,y,z} \frac{(p_j - qA_j)^2}{2m} + A_0$, where q is the electron charge, m the electron mass, p is the electron momentum and A the vector potential. For a plane wave along z direction, with electric field $E_x \sin(kz - \omega t)$, the Lorentz gauge is $(A_0, A_x, A_y, A_z) = \frac{E_x}{\omega} \cos(kz - \omega t)(0, 1, 0, 0)$. But this gauge is not suited for calculating optical transitions, because we donot recover the Rabi frequency $qE_x d$ (d electric dipole moment). What we find is something orders of magnitude smaller. Instead, we work with gauge $(A_0, A_x, A_y, A_z) = \frac{-E_x}{2}(x \sin(kz - \omega t), -\frac{\cos(kz-\omega t)}{\omega}, 0, \frac{x}{c}\sin(kz - \omega t))$ (c light velocity) to find everything correct. We can classify optical transitions into two types, one that are not mediated by x term in Hamiltonian ($\mathcal{A}$ process), and other by the x term in Hamiltonian ($\mathcal{E}$ process). Equipped with this, we study many optical processes. These include atomic transitions, elastic and inelastic scattering processes like Rayleigh scattering, Raman scattering, Two photon absorption, stimulated Raman, second harmonic generation, parametric down conversion, photoelectric effect, optical processes in semiconductors, etc. We show that the $\mathcal{E}$ process is stronger than the $\mathcal{A}$ process, when we transit between atomic levels. The $\mathcal{A}$ process happens, when we have virtual levels. In processes like Raman scattering, two photon absorption , we have both the $\mathcal{A}$ and the $\mathcal{E}$ processes. The $\mathcal{A}$ process flips the spin, while the $\mathcal{E}$ process doesn't. The $\mathcal{A}$ processes are therefore always circularly polarized. Transition amplitude of the $\mathcal{E}$ process is straightforward and is everyday calculation of the Rabi frequency using dipole moment. The transition amplitude of the $\mathcal{A}$ process is done through Dirac equation, the relativistic cousin of Schrödinger equation.

3.1 Introduction

Take a classical electron, with coordinates $(x, y, z) = (x_1, x_2, x_3)$. Its Lagrangian in the electromagnetic field is

$$L = \frac{m}{2}\sum_i \dot{x_i}^2 + q\sum_i A_i\dot{x_i} - qA_0, \tag{3.1}$$

where q amd m are electron charge and mass. $\mathcal{A}$ and V are vector and scalar potentials.

The Euler Lagrange equations are the familiar Lorentz force law $m\dot{v} = q(E + v \times B)$, where v is the velocity vector, $E_i = -\frac{\partial A_i}{\partial t} - \frac{\partial V}{\partial x_i}$, $B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}$, the electric and magnetic fields.

The momentum $p_i = \frac{\partial L}{\partial x_i}$ and the Hamiltonian of the system $H = p_i \frac{\partial}{\partial \dot{x}_i} - L$ is

$$H = \sum_{j=x,y,z} \frac{(p_j - qA_j)^2}{2m} + qA_0. \tag{3.2}$$

3.1.1 Dirac and Schrödinger Equation

The Electron Schrödinger Equation [36] is

$$i\frac{\partial \psi}{\partial t} = \left(\sum_{j=x,y,z} \frac{(-i\hbar\frac{\partial}{\partial x_j} - qA_j)^2}{2m} + qA_0\right)\psi, \tag{3.3}$$

where ψ is electron wavefunction. This equation is not very tractable, because it is nonlinear in $\mathcal{A}$, lets write a linear equation, which is the Dirac equation [63], which takes the form

$$i\frac{\partial \phi}{\partial t} = \left(\sum_{j=x,y,z} c(-i\,\hbar\frac{\partial}{\partial x_j} - qA_j)\alpha_j + \beta mc^2 + qA_0\right)\phi. \tag{3.4}$$

where $\alpha_j = \sigma_z \otimes \sigma_j$ and $\beta = \sigma_x \otimes \mathbf{1}$ are Dirac matrices, where σ_j are the Pauli matrices, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ϕ is electron spinor, for a electron wave with momentum k, takes the form $\phi = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \otimes \uparrow$, where $\uparrow$ is spin up, $\cos\theta = \frac{\hbar k}{mc} = \frac{\upsilon}{c}$, where $\upsilon = \frac{\hbar k}{m}$, is electron wave group velocity. Electron Orbitals are of size $\sim A^\circ$, their $k \sim 10^{10}m$, then $\upsilon \sim 10^6 m/s$ and $\cos\theta \sim \frac{10^6}{3\times 10^8} \sim 10^{-3}$. Electron is non-relativistic, $\cos\theta = \frac{\upsilon}{c} \sim 0$, $\theta \sim \frac{\pi}{2}$, $\phi = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \uparrow$.

To fix ideas, take incoming EM wave, along z direction, with electric field $E_x \sin(kz - \omega t)$, the Lorentz gauge is $(A_0, A_x, A_y, A_z) = \frac{E_x}{\omega}\cos(kz - \omega t)(0, 1, 0, 0)$. Electron wave with momentum q absorbs the photon with momentum k, and transits to momentum $q + k$. The $\mathcal{A}$ process transition is driven by Dirac matrix α_x, with transition amplitude

$$\mathcal{M} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \otimes \uparrow \underbrace{(\sigma_z \otimes \sigma_x)}_{\alpha_x} \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \otimes \downarrow = qcA_x \frac{\upsilon}{c} = qE_x \frac{\upsilon}{\omega} \tag{3.5}$$

If we have electron orbital ϕ_0 then $k' = \frac{M}{M+m}k$ of photon momentum goes to electron-nuclear relative coordinate, while $k'' = k$ momentum to CM (center of mass), where M is nucleus mass. The $\mathcal{A}$ process drives the transition

$$\phi_0\uparrow \longrightarrow \exp(ik'z)\phi_0\downarrow,$$

with amplitude $\mathcal{M} = qE_x\frac{\upsilon}{\omega}$.

When orbital ϕ_1 is different from ϕ_0 we go to

$$\phi_0\uparrow \longrightarrow \exp(ik'z)\phi_0\downarrow,$$

with amplitude $\mathcal{M} = qcA_x\frac{\upsilon}{c}$ whoose overlap with ϕ_1 is

$$\mathcal{M}_1 = qcA_x\frac{\upsilon}{c}ik'\underbrace{\langle\phi_1|z|\phi_0\rangle}_{d_z} = iqE_x d_z\ \frac{\upsilon}{c},$$

where $ck' \sim \omega$.

But this is not suited for study of optical transitions, because we donot recover the Rabi frequency $qE_x d$. What we find is orders of magnitude smaller (down by $\frac{\upsilon}{c}$). Instead we work with gauge

$$(A_0, A_x, A_y, A_z) = \frac{-E_x}{2}(x\ \sin(kz-\omega t), -\frac{\cos(kz-\omega t)}{\omega}, 0, \frac{x}{c}\sin(kz-\omega t)).$$

Now we have an $\mathcal{E}$ process driven by x term. For the $\mathcal{E}$ process, the amplitude of $\phi_0 \to \phi_0$ is just 0, as $\langle\phi_0|x|\phi_0\rangle = 0$ amd the amplitude of $\phi_0 \to \phi_1$ is simply

$$\mathcal{M}_1' = qE_x\underbrace{\langle\phi_1|x|\phi_0\rangle}_{d_x} = qE_x d_x,$$

Dipole elements d_z, d_x are approx, Bohr radius $\sim A^\circ$. Due to the factor $\frac{\upsilon}{c} \sim 10^{-3}$, $\mathcal{M}_1 \ll \mathcal{M}_1'$. Therefore transition between different atomic orbitals are largely driven by the $\mathcal{E}$ process.

$\mathcal{A}$ process flips electron spin, $\mathcal{E}$ doesn't. When ϕ_1 and ϕ_0 have different angular $\mathcal{A}$ process cannot drive the transition, but $\mathcal{E}$ process can, with a circularly polarized light and

$$\mathcal{M}_1' = qE\underbrace{\langle\phi_1|r|\phi_0\rangle}_{d} = qEd,$$

Since $\mathcal{A}$ process always flips spin, it is necessarily done with circularly polarized light.

3.1.2 Bremsstraulung and $\mathcal{E}\cdot x$ gauge

The $E\cdot x$ gauge explains the phenomenon of Bremsstraulung [37], where when an electron breaks (accelerates) its emit radiation. If only $\mathcal{A}$ process was around, this radiation will be in direction of momentum change of electron. In practice, in syncrotons this radiation is tangential to circling electron and hence perpendicular to the radial acceleration which can only be accounted by $\mathcal{E}\cdot x$ gauge.

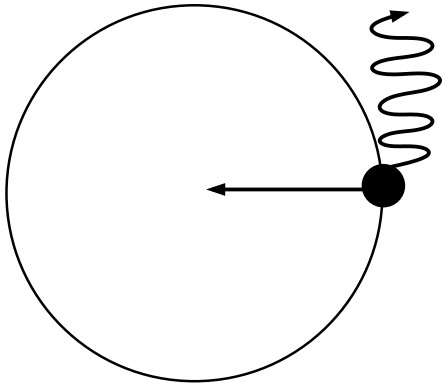


Fig. 3.1 Fig. depicts how in Bremsstraulung radiation is tangential to circling electron and hence perpendicular to the radial acceleration.

3.1.3 General gauge, qed and coulomb potential

We can have a more general gauge,

$$(A_0,A_x,A_y,A_z)=-E_x(\alpha x\,\sin(kz-\omega t),-(1-\alpha)\frac{\cos(kz-\omega t)}{\omega},0,\alpha\frac{x}{c}\sin(kz-\omega t)). \tag{3.6}$$

for $0\leq\alpha\leq 1$. If we use this gauge, to calculate electron electron scattering by exchange of photon in qed and calculate coulomb potential, we find $\alpha\sim\frac{1}{2}$ for a correct potential.

Under a frame transformation (ω,k) and (E,B), transform under ususal Lorentz transformation to give a vector potential in $\mathcal{E}\cdot x$ gauge.

3.2 Atomic transitions

The first place to look at Optical transitions in Atomic transitions [38]. Lets look at D1 and D2 line of Sodium. D1 line Transition from $3\ ^2S_{\frac{1}{2}}$ ($J = \frac{1}{2}$) to $3\ ^2P_{\frac{1}{2}}$ ($J = \frac{1}{2}$). D1 line Transition from $3\ ^2S_{\frac{1}{2}}$ ($J = \frac{1}{2}$) to $3\ ^2P_{\frac{3}{2}}$ ($J = \frac{3}{2}$). Fig. 3.2 depicts J states of Sodium $n = 3$ (valence electron).

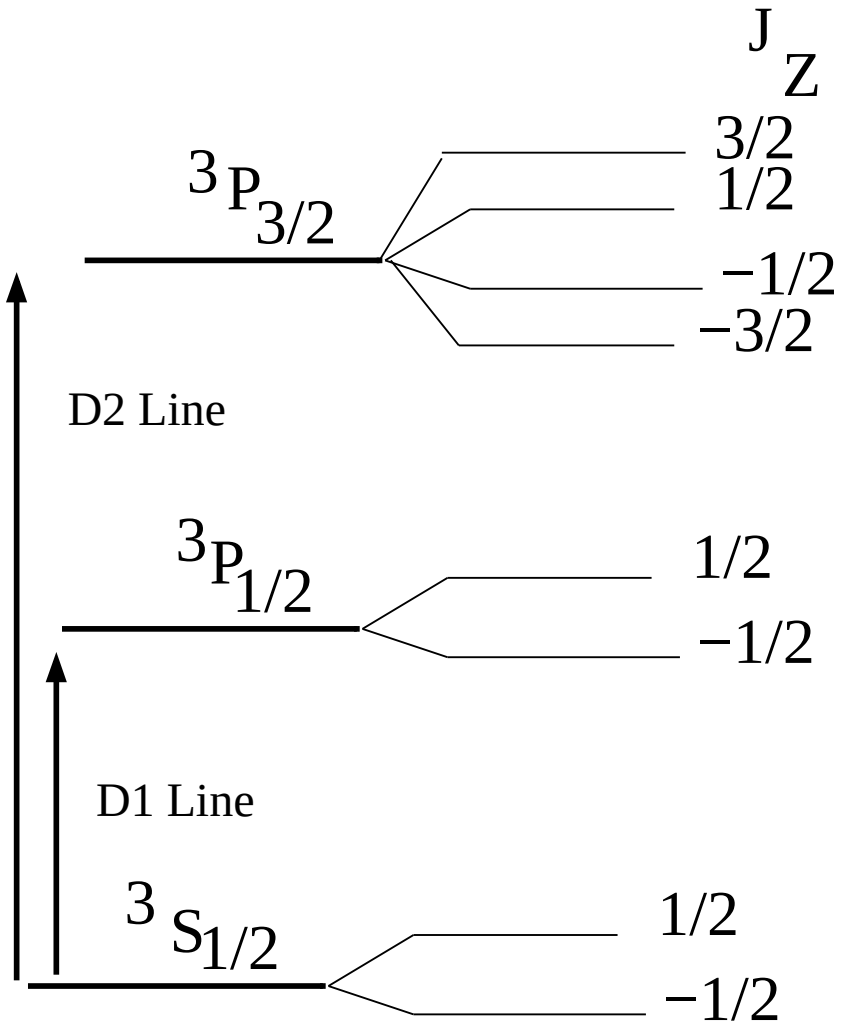


Fig. 3.2 Fig. depicts J states of Sodium principal quantum number n=3.

J states are made of orbital and electron spin states. When we make a transition $\Delta J_z = 1$, either orbital angular momentum increments or spin angular momentum increments. When orbital angular momentum increments, transition is make by an $\mathcal{E}$ process. When spin angular momentum increments, transition is make by an $\mathcal{A}$ process. Either case circular polarized light is involved.

Sodium nucleus is spin $\frac{3}{2}$, when J angular momentum is coupled to nuclear spin angular momentum, we get hyperfine F states.

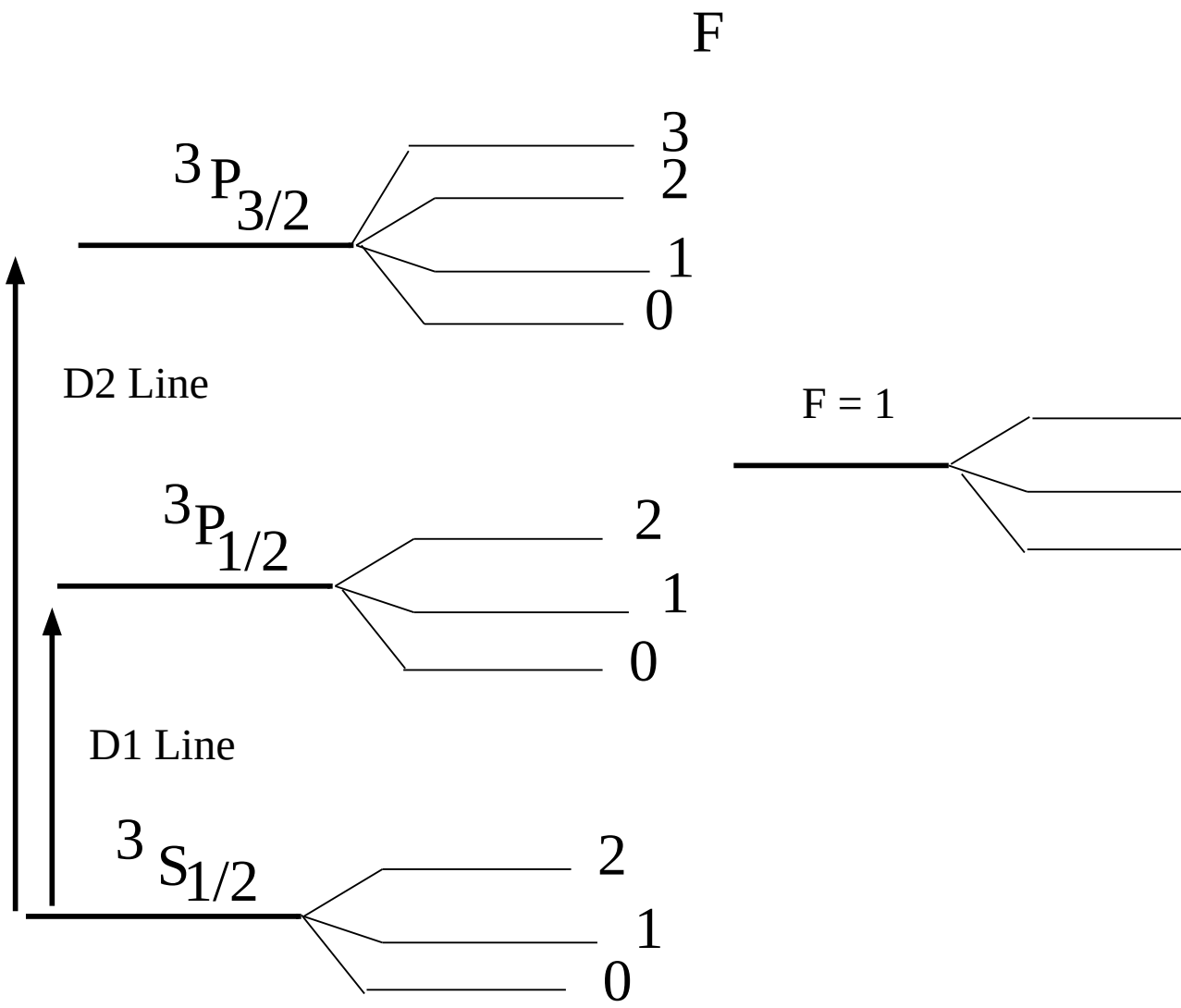


Fig. 3.3 Fig. depicts F states of Sodium with principal quantum number n=3.

F states are made of orbital and electron spin states and nuclear spin states. When we make a transition $\Delta F_z = 1$, either orbital angular momentum increments or electron spin angular momentum increments or nuclear spin angular momentum increments. When orbital angular momentum increments, transition is make by an $\mathcal{E}$ process. When electron or nuclear spin angular momentum increments, transition is make by an $\mathcal{A}$ process. Either case circular polarized light is involved.

When $\mathcal{A}$ process flips nuclear spin, $k' = \frac{m}{M+m}k$ of photon momentum goes to electron-nuclear relative coordinate. The transition amplitude of this process is

$$\mathcal{M}_2 = qcA_x \frac{\upsilon_n}{c} ik' \underbrace{\langle \phi_1 | z | \phi_0 \rangle}_{d_z} = iqE_x d_z \, (\frac{m}{M})^2 \frac{\upsilon}{c},$$

where $\upsilon_n \sim \frac{m}{M}\upsilon = 10^3 m/s$ is velocity of nuclear coordinate. This amplitude $\mathcal{M}_2$ is small.

Fig. 3.3 depicts F states of Sodium $n = 3$ (valence electron).

3.3 Rayleigh and Raman Scattering

Atomic transitions are first order, we are on resonance to energy difference between energy levels. Now we discuss some second order processes like Rayleigh [39] and Raman Scattering [40, 41]. Rayleigh is elastic scattering of light, Raman inelastic. In Rayleigh scattering, light of wavevector k is scattered by atom to light of wavevector k_1 such that $|k| = |k_1|$, only direction of light changes. In Raman scattering, light of wavevector k is scattered by atom to light of wavevector k_1 such that $|k| \neq |k_1|$. The difference of the photon energy $\Delta\omega = \hbar c(|k| - |k_1|)$, results in atomic transition with $\Delta E = \hbar\omega$. When $\Delta E > 0$ its Stokes process. When $\Delta E < 0$ its anti-Stokes process.

Calculation of transition amplitude for these processes, involves a second order calculation manifested in a three level system. This is as shown in Figure 5.9. There are three levels $|1\rangle$, $|2\rangle$ and $|3\rangle$ with energies E_1, E_2, E_3, with $E_1 = E_3$, levels 1 and 3 are degenerate. Ω_1 and Ω_2 are transition amplitudes between level 1 and 2 and level 2 and 3 respectiviely.

Then gives for $|E_1 - E_2| \gg \Omega_i$, there is transition amplitude of going from $|1\rangle$ to $|3\rangle$, given by,

$$\mathcal{M} = \frac{\Omega_1 \Omega_2}{E_1 - E_2} \tag{3.7}$$

Fig 5.3A depicts Rayleigh scattering, incoming photon at wavevector k scatters to wavevector k_1. Fig B, depicts a three level 3 level Rayleigh process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to ϕ

In 3 level system analogy, State $|1\rangle$ is electron-nuclear, photon state ϕ, k and state $|2\rangle$ is electron-nuclear state Φ and State $|3\rangle$ is electron-nuclear, photon state ϕ, k'. $|k| = |k'|$, so that $E_1 = E_3$. Ω_1 is a $\mathcal{A}$ process with value

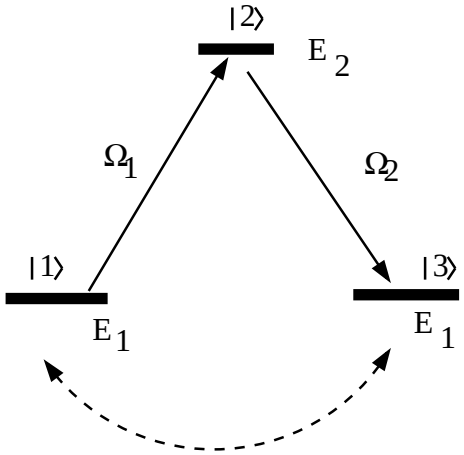


Fig. 3.4 Fig. depicts a 3 level system with transition amplitude Ω_1 and Ω_2 between level 1 and 2 and level 2 and 3 respectiviely.

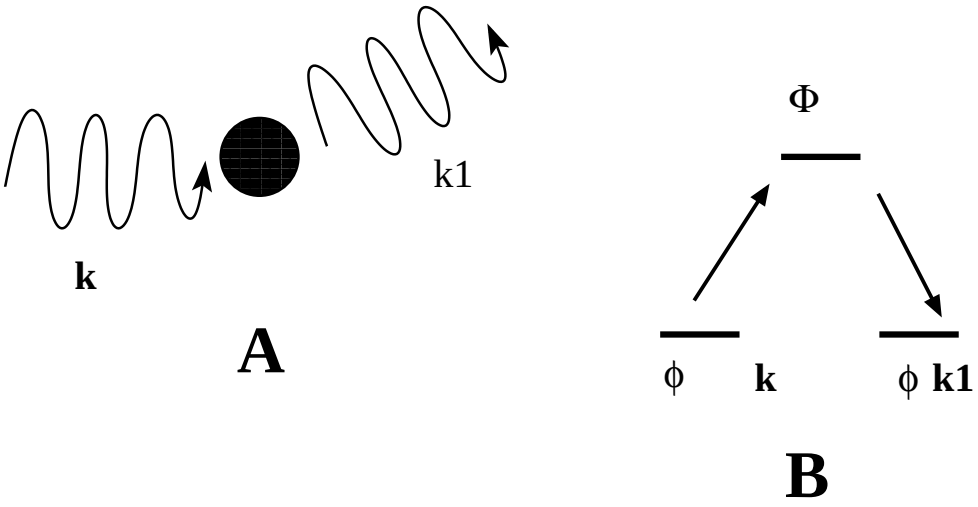


Fig. 3.5 Fig. A, depicts Rayleigh scattering, incoming photon at wavevector k scatters to wavevector k_1. Fig B, depicts a three level 3 level Rayleigh process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to ϕ_0

$$\Omega_1 = qE_x \frac{\upsilon}{\omega}, \tag{3.8}$$

where E_x is electric field of incoming EM wave.

$$\Omega_2 = qE_o \frac{\upsilon}{\omega}, \tag{3.9}$$

where E_0 is the electric field of spontaneously emitted photon at k_1. How much is E_0.

Spontaneous emission time Δt, in atomic systems is $\Delta t \sim 10's$ ns, corresponding to a Rabi frequency of $\Omega = 100$ MHz, which is $10^{-26}J$, in which photon travels $\Delta l = c\Delta t \sim 1m$ (leaves the atom). The volume of the photon $\sim 1m^3$. Bandwidth of the photon $\Delta B = \sim \frac{2\pi}{\Delta t}$, how many photons are there in all the directions? Then wavevector sphere of radius $k_0 = \frac{2\pi}{\lambda_0}$ (say 300 nm photon, $\lambda_0 = 300nm$). Number of direction $n = 4\pi k_0^2/(\Delta B)^2 \sim 10^{12}$.

Rabi frequency $\Omega = q\sqrt{n}E_0 d = 10^{-26}$, giving $E_0 = mV/m$. Ofcourse, energy of the photon $\hbar\omega = \frac{\epsilon_0}{2}E_0^2 V$ with $V \sim 1m^3$, we have for $\omega \sim 10^{15}$, $E_0 \sim mV/m$ (all self consistent).

Coming back to Rayleigh scattering, from Eq. 5.27, 5.10, 5.11, we have

$$\mathcal{M} = \frac{q^2 E_x E_0}{\hbar}\frac{\upsilon^2}{\omega^3}. \tag{3.10}$$

or Rayleigh Rabi frequency,

$$\Omega_{Rayleigh} = \frac{q^2 E_x E_0}{\hbar^2}\frac{\upsilon^2}{\omega^3}. \tag{3.11}$$

for $E_x \sim 10^3 V/m$, $\omega = 10^{15}$, we get $\Omega_{Rayleigh} \sim 10^{-4}$ Hz.

Finally Ω_1 and Ω_2 both $\mathcal{A}$ process.

3.3.1 Raman Scattering

Raman scattering is an inelastic scattering process. Photon as wavenumber k is absorbed by atom and k_1 emitted. Energy of absorbed and emitted photon is not same $k \neq k_1$. Deficit goes in exciting atomic transition.

Fig 5.4A depicts Raman scattering, incoming photon at wavevector k scatters to wavevector k_1. Atom is excited from initial atomic level ϕ to ϕ_1. Fig B, depicts a three level Raman process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to new atomic level ϕ_1.

In 3 level system analogy, State $|1\rangle$ is electron-nuclear, photon state ϕ, k and state $|2\rangle$ is electron-nuclear state Φ and State $|3\rangle$ is electron-nuclear, photon state ϕ_1, k_1. $|k| \neq |k_1|$, but $E_1 = E_3$. Ω_1 is a $\mathcal{A}$ process with value

$$\Omega_1 = qE_x\frac{\upsilon}{\omega}, \tag{3.12}$$

where E_x is electric field of incoming EM wave. Ω_2 is a $\mathcal{E}$ process (atomic levels change) with value

$$\Omega_2 = qE_o d, \tag{3.13}$$

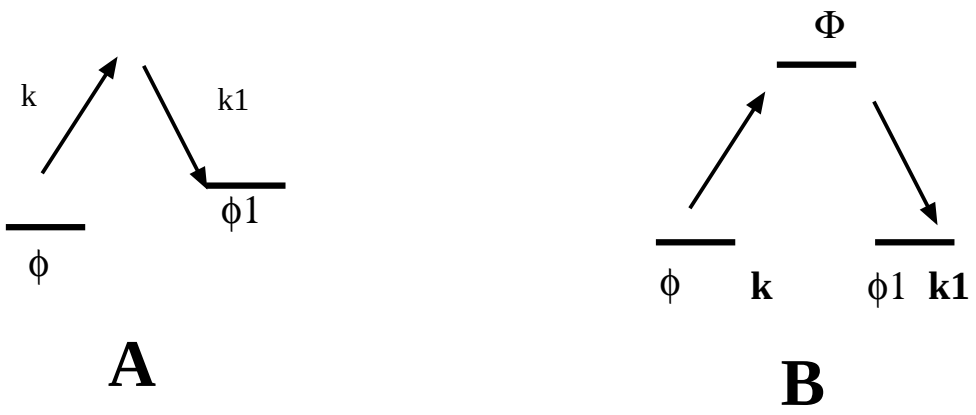


Fig. 3.6 Fig. A, depicts Raman scattering, incoming photon at wavevector k scatters to wavevector k_1. Fig B, depicts a three level 3 level Rayleigh process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to ϕ_0

where E_0 is the electric field of spontaneously emitted photon at k_1.

Coming back to Rayleigh scattering, from Eq. 5.27, 5.14, 5.15, we have

$$\mathcal{M} = \frac{q^2 E_x E_0 d}{\hbar} \frac{\upsilon}{\omega^2}. \tag{3.14}$$

or Raman Rabi frequency,

$$\Omega_{Raman} = \frac{q^2 E_x E_0 d}{\hbar^2} \frac{\upsilon}{\omega^2}. \tag{3.15}$$

for $E_x \sim 10^3 V/m$, $\omega = 10^{15}$, we get $\Omega_{Raman} \sim 10^{-5}$ Hz.

Raman is a $\mathcal{A}$ and $\mathcal{E}$ process, Ω_1 is $\mathcal{A}$ and Ω_2 is $\mathcal{E}$. Ω_2 can be $\mathcal{A}$ but the amplitude is much smaller than.

We have been talking of ϕ and ϕ_1 as atomic levels, which is just fine in principle. In practice, these are vibrational levels of two nuclie making a molecule. Then these are energy levels of their (two nuclie) relative coordinate.

3.3.2 Resonance Raman

Resonance Raman [42] is Raman scattering, an inelastic scattering, where there is atomic transition close to incoming photon energy. Photon at wavenumber k is absorbed by atom and k_1 emitted. Energy of absorbed and emitted photon is not same $k \neq k_1$.

Fig 3.7A depicts Resonance Raman scattering, incoming photon at wavevector k scatters to wavevector k_1. Atom is excited from initial atomic level ϕ_1 to ϕ_2. Fig B,

depicts a three level Raman process, where initial atomic level ϕ_1 absorbs photon k and moves to ϕ_2. k_1 is emitted and we return to new atomic level ϕ_3.

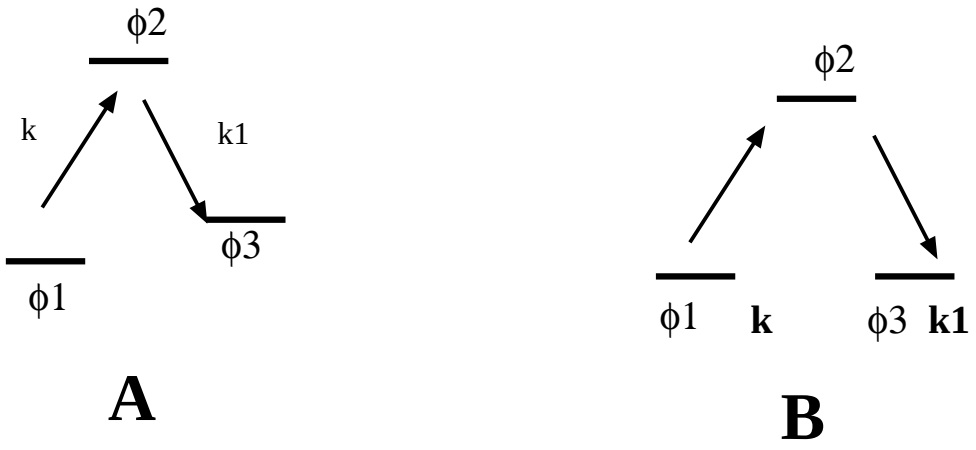


Fig. 3.7 Fig. A, depicts Resonance Raman, energy difference between ϕ_1 and ϕ_2 is close to incoming photon energy. Fig B, depicts a three level 3 level Resonance Raman process, where initial atomic level ϕ_1 absorbs photon k and moves to ϕ_2. k_1 is emitted and we return to ϕ_3

In 3 level system analogy, State $|1\rangle$ is atom, photon state ϕ_1, k and state $|2\rangle$ is atom state ϕ_2 and State $|3\rangle$ is atom, photon state ϕ_3, k_1. $|k| \neq |k_1|$, but $E_1 = E_3$. Ω_1 is a $\mathcal{E}$ process (atomic levels change) with value

$$\Omega_1 = qE_x d_{12}, \tag{3.16}$$

where E_x is electric field of incoming EM wave, $d_{12} = \langle \phi_2 | x | \phi_1 \rangle$ transition element . Ω_2 is also a $\mathcal{E}$ process (atomic levels change) with value

$$\Omega_2 = qE_o d_{23}, \tag{3.17}$$

where E_0 is the electric field of spontaneously emitted photon at k_1 and $d_{23} = \langle \phi_3 | x | \phi_2 \rangle$.

For ω frequency of incoming photon,

$$\Delta\omega = \frac{(E_{\phi_2} - E_{\phi_1})}{\hbar} - \omega,$$

from Eq. 5.27, 3.16, 3.17, we have

$$\mathcal{M} = \frac{q^2 E_x E_0 d_{12} d_{23}}{\hbar \Delta\omega}. \tag{3.18}$$

or Resonance Raman Rabi frequency,

$$\Omega_{Resonance-Raman} = \frac{q^2 E_x E_0 d_{12} d_{23}}{\hbar^2 \Delta\omega}. \tag{3.19}$$

for $E_x \sim 10^3 V/m, d_{12}, d_{23} \sim 1A^\circ, \Delta\omega = 10^{13}$ (infra-red) we get $\Omega_{Resonance-Raman} \sim 10^{-4}$ Hz.

Resonance Raman is a $\mathcal{E}$ and $\mathcal{E}$ process, Ω_1 is $\mathcal{E}$ and Ω_2 is $\mathcal{E}$.

Rabi frequencies of Rayleigh and Raman scattering processes are small, because they are calculated from a single atom. Scattering takes place from large number of atoms N and then the $\mathcal{M} \to N\mathcal{M}$, which can be modest.

3.4 Stimulated Raman, Two Photon microscopy and Coherent Antistokes Raman Spectroscopy (CARS)

3.4.1 Stimulated Raman

Stimulated Raman [46], is like Raman, except we use two lasers k and k_1, where k is absorbed and k_1 emitted. Due to k_1 laser being there the emission process is much stronger with Electric field E_1 instead of E_0. Ofcource strong emission is in direction of k_1.

Fig 3.8A depicts Raman scattering, incoming photon at wavevector k scatters to wavevector k_1. Atom is excited from initial atomic level ϕ to ϕ_1. Fig B, depicts a three level Raman process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted by stimulated emission and we return to new atomic level ϕ_1.

In 3 level system analogy, State $|1\rangle$ is electron-nuclear, photon state ϕ, k and state $|2\rangle$ is electron-nuclear state Φ and State $|3\rangle$ is electron-nuclear, photon state ϕ_1, k_1. $|k| \neq |k_1|$, but $E_1 = E_3$. Ω_1 is a $\mathcal{A}$ process with value

$$\Omega_1 = qE_x \frac{\upsilon}{\omega}, \tag{3.20}$$

where E_x is electric field of incoming EM wave. Ω_2 is a $\mathcal{E}$ process (atomic levels change) with value

$$\Omega_2 = qE_1 d, \tag{3.21}$$

where E_1 is the electric field of second laser which stimulates emitted photon at k_1.

From Eq. 5.27, 5.14, 5.15, we have

$$\mathcal{M} = \frac{q^2 E_x E_1 d}{\hbar} \frac{\upsilon}{\omega^2}. \tag{3.22}$$

or Stimulated Raman Rabi frequency,

$$\Omega_{Stimulated-Raman} = \frac{q^2 E_x E_1 d}{\hbar^2} \frac{\upsilon}{\omega^2}. \tag{3.23}$$

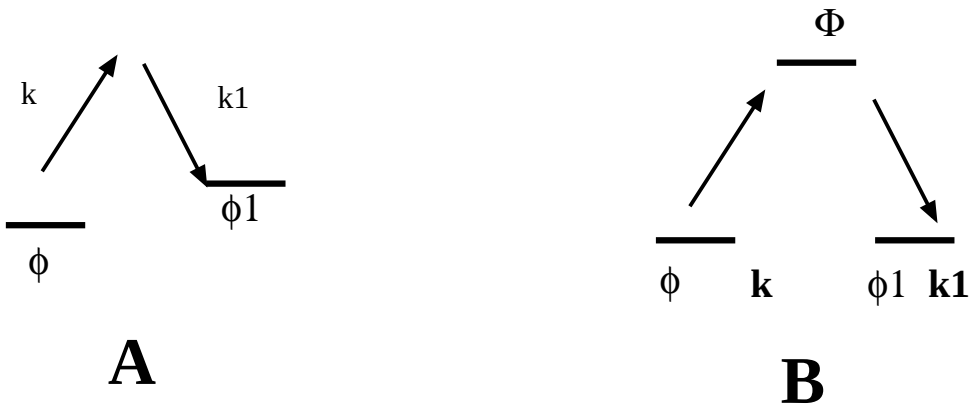


Fig. 3.8 Fig. A, depicts stimulated Raman scattering, incoming photon at wavevector k scatters to wavevector k_1 by stimulation. Fig B, depicts a three level 3 level Rayleigh process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to ϕ_0

for $E_x, E_1 \sim 10^6 V/m$, $\omega = 10^{15}$, we get $\Omega_{Stimulated-Raman} \sim 10$ MHz.

Raman is a $\mathcal{A}$ and $\mathcal{E}$ process, Ω_1 is $\mathcal{A}$ and Ω_2 is $\mathcal{E}$. Ω_2 can be $\mathcal{A}$ but the amplitude is much smaller then.

A nice application of stimulated Raman these days is quantum information processing in Ion trap quantum computer, where ϕ and ϕ_1 are the $|0\rangle$ and $|1\rangle$ qubit states (hyperfine states) of a quantum computer [47].

3.4.2 Coherent Raman AntiStokes (CARS) Spectroscopy

CARS [45] is like Raman, except two Raman back to back. In Fig. 3.9 starting from ground state, pump laser k_p is absorbed and stokes k_s emitted and we transit to excited vibrational level, then, k_p is absorbed and anti-stokes k_{AS} emitted, returning to ground state. k_p, k_s form a Raman pair with amplitude $\mathcal{M}_1$ and k_p, k_{AS} form another Raman pair with amplitude $\mathcal{M}_2$. Mathematically the process is represented as three state variables

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{-i}{\hbar} \begin{pmatrix} 0 & \mathcal{M}_1 & 0 \\ \mathcal{M}_1 & 0 & \mathcal{M}_2 \\ 0 & \mathcal{M}_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{3.24}$$

With $\mathcal{M}_1 \sim \mathcal{M}_2 = \mathcal{M}$, we have transition from $(x_1, x_2, x_3) = (1,0,0)$ to $(0,0,1)$ in $T = \frac{\pi}{\sqrt{M}}$, hence a transition amplitude of $\frac{\mathcal{M}}{\sqrt{2}}$. As before

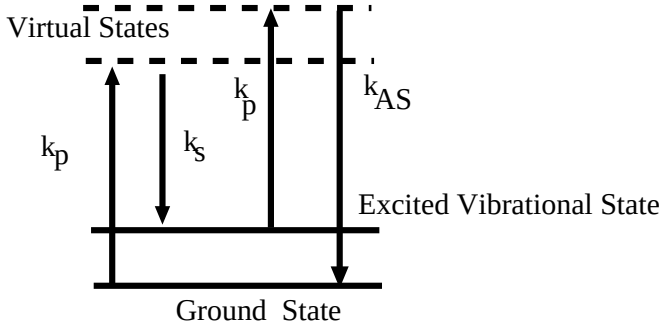


Fig. 3.9 Figure shows the Coherent Raman Antistokes process. Starting from ground state, pump laser k_p is absorbed and stokes k_s emitted and we transit to excited vibrational level, then, k_p is absorbed and anti-stokes k_{AS} emitted, returning to ground state.

$$\mathcal{M} = \frac{q^2 E_x E_0 d}{\hbar} \frac{\upsilon}{\omega_p^2}. \tag{3.25}$$

But there is something nice, for a given k_{AS} there is a whole set of k_s (n in total). Then the transition amplitude to a state with given k_{AS} is $\sqrt{n}M$. n as before can be big as 10^{12} which gives a effective amplitude and Rabi frequency

$$\mathcal{M}_{CARS} = \frac{q^2 \sqrt{\frac{n}{2}} E_x E_0 d}{\hbar} \frac{\upsilon}{\omega_p^2}. \tag{3.26}$$

$$\Omega_{CARS} = \frac{q^2 \sqrt{\frac{n}{2}} E_x E_0 d}{\hbar^2} \frac{\upsilon}{\omega_p^2}. \tag{3.27}$$

For $E_x \sim 10^3 V/m$, $\omega_p = 10^{15}$, we get $\Omega_{CARS} \sim 10$ Hz.

As before k_p is $\mathcal{A}$ process and k_s, k_{AS} are $\mathcal{E}$ process. Scattering with one atom is considered, when probe meets N atoms in a plane perpendicular to probe propogation, we may think $\Omega_{CARS} \to N\Omega_{CARS}$, but different k_s directions donot add over atoms and we only get amplitude $N\mathcal{M}$ not $N\mathcal{M}_{CARS}$. The wavevectors satisfy that transferred momentum $q = k_s + k_{AS} - 2k_p$, satisfy $\sum = k \cdot x_i$ add coherently where x_i are atom locations. Since k_s is degree of freedom, we get scattering from different atom planes at a given desired k_{AS} direction. Clearly an advantage over vanilla Raman microscopy.

Photon is 1 m long has $E_0 = mV/m$, in $E = 10^6 V/m$, we have n photons with $E = \sqrt{n}E_0$, giving $n \sim 10^{18}$. Scattering takes place in $1/10s$, photon travels $\sim 10^7$ m in tis time making a ll in all 10^{25} photons passing for a scattered photon but with N atoms rate improves by N, then $M = 10^{25}/N$ photons for a scattered photon. We know from practice $M \sim 10^7$, giving $N \sim 10^{18}$. With scattering happening over cross section mm^2 and depth λ (500 nm), we have volume $10^{18}(A^\circ)^3$, we indeed have 10^{18} atoms.

3.4.3 Two photon microscopy

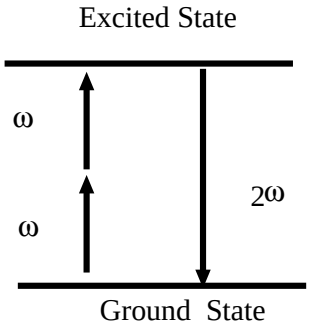


Fig. 3.10 Figure shows a two photon absorbtion process. There is atomic transition at 2ω frequency. A laser a frequency ω is absorbed to a virtual level. Then another photon is absorbed making transition to excited atomic state. This together constitutes a Raman process. This is followed by spontaneous emission to ground state.

Figure, 3.10 shows a two photon absorbtion process [43, 44] . There is atomic transition at 2ω frequency. A laser a frequency ω is absorbed to a virtual level. Then another photon is absorbed making transition to excited atomic state. This together constitutes a Raman process. This is followed by spontaneous emission to ground state. The amplitude of the Raman process as before is

$$\mathcal{M}_1 = \frac{q^2 E_x^2 d}{\hbar} \frac{\upsilon}{\omega^2}. \tag{3.28}$$

amplitude of Spontaneous emission in a given direction is simply

$$\mathcal{M}_2 = qE_0 d \tag{3.29}$$

The net amplitude is smaller of two. For $E_x = 10^3 V/m$, $\omega = 10^{15}$, $E_0 = mV/m$ and $d = 1A^\circ$ and $\frac{\upsilon}{\omega} = 10A^\circ$, we have $\Omega_1 = \frac{\mathcal{M}_1}{\hbar} = 100$ Hz. $\Omega_2 = \frac{\mathcal{M}_2}{\hbar} = 100$ Hz. The two processes are comparable.

As before Raman process is $\mathcal{A}$ and $\mathcal{E}$ while spontaneous emission is $\mathcal{E}$. Two photon microscopy is used to image deeper inside tissue, ω is less scattered than 2ω, we can go deeper upto mm.

3.5 Nonlinear Optical Processes

Figure 3.11 A shows a 4 level process. Energy levels i, with energy E_i. $E_1 = E_4$. Transition frequency Ω_1, Ω_2 and Ω_3 between level $1-2$, $2-3$ and $3-4$. $E_2 - E_1 = \hbar\omega_1$, $E_3 - E_2 = \hbar\omega_2$ and $E_3 - E_4 = \hbar\omega_4$. $\omega_3 = \omega_1 + \omega_2$.

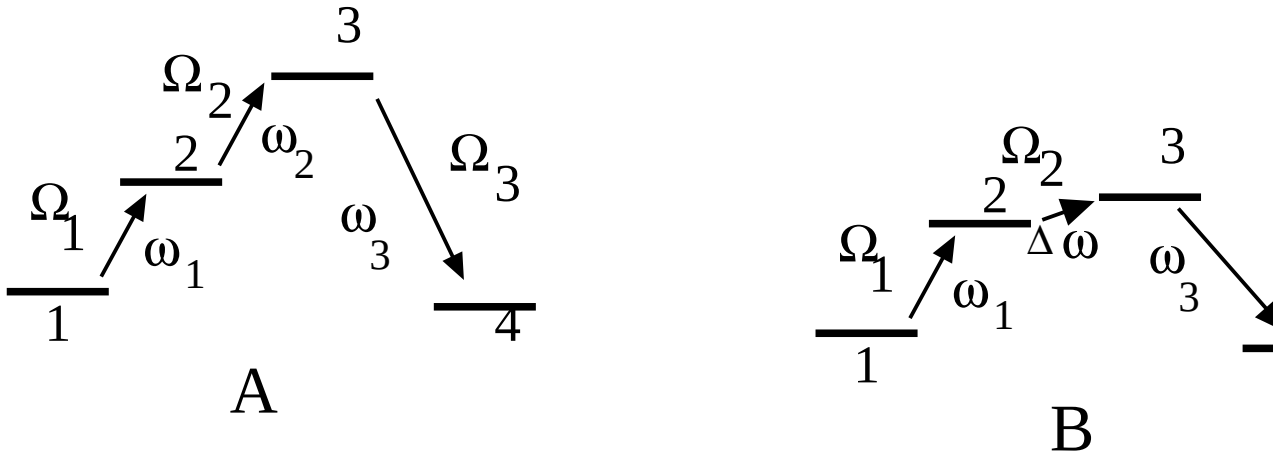


Fig. 3.11 Figure A shows a 4 level process. Energy levels i, with energy E_i. $E_1 = E_4$. Transition frequency Ω_1, Ω_2 and Ω_3 between level $1-2$, $2-3$ and $3-4$. $E_2 - E_1 = \hbar\omega_1$, $E_3 - E_2 = \hbar\omega_2$ and $E_3 - E_4 = \hbar\omega_4$. $\omega_3 = \omega_1 + \omega_2$

Rabi frequency for the whole process is

$$\Omega_{14} = \frac{\Omega_1\Omega_2\Omega_3}{\omega_1\omega_2}, \tag{3.30}$$

In nonlinear optics [48], light of frquency ω gets converted to 2ω, called second harmonic generation (SHG). One photon of 2ω get converted to two one photons of 2ω. More generally photons at ω_1 and ω_2 get converted to $\omega_3 = \omega_1 + \omega_2$. Also called sum frequency generation (SFG).

In Down conversion light of frquency 2ω gets converted to ω. Also called Difference Frequency Generation (DFG), one photon of ω_3 get converted to photons of ω_1 and ω_2 such that $\omega_3 = \omega_1 + \omega_2$.

3.5.1 Second Harmonic Generation and Down Conversion

Fig. 3.12 A shows second harmonic generation (SHG) and Fig. 3.12 B shows down conversion. The 4 level system analogy is as shown in 3.11.

$\Omega_1, \Omega_2, \Omega_3$ are all $\mathcal{A}$ processes

$$\Omega_1 = \Omega_2 = \frac{qE_x\upsilon}{\hbar\omega} \tag{3.31}$$

Ω_3 is spontaneous emission and hence

$$\Omega_3 = \frac{qE_0\upsilon}{\hbar\omega} \tag{3.32}$$

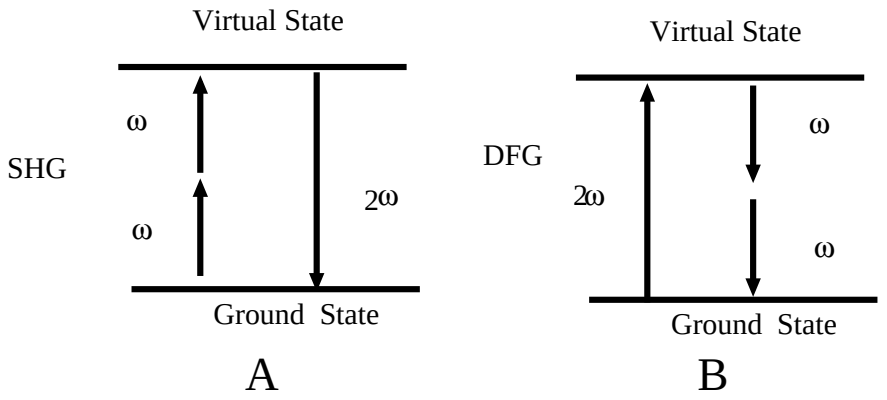


Fig. 3.12 Fig. A shows second harmonic generation (SHG) and Fig. B shows down conversion.

Then

$$\Omega_{14} = (\frac{q\upsilon}{\hbar\omega})^3 \frac{E_x^2 E_0}{\omega^2}, \tag{3.33}$$

For $E_x = 10^6 V/m$, $\omega = 10^{15}$, we get $\Omega_{14} = 10^{-3}$/s.

The transition frequency will be boosted if there is an atomic level close to $E = \hbar\omega$ above ground state, as in Resonance Raman. Figure 3.11 B shows a 4 level process for this. Level 2 and 3 are nearly degenerate with energy difference $\Delta\omega$. Level 3 is now excited atomic level.

Rabi frequency for the whole process is

$$\Omega_{14} = \frac{\Omega_1 \Omega_2 \Omega_3}{\omega_1 \Delta\omega}, \tag{3.34}$$

$$\Omega_1 = \frac{qE_x\upsilon}{\hbar\omega}, \tag{3.35}$$

$$\Omega_2 = \frac{qE_x d}{\hbar}, \tag{3.36}$$

Ω_3 is spontaneous emission and hence

$$\Omega_3 = \frac{qE_0 d}{\hbar}, \tag{3.37}$$

Ω_1 is $\mathcal{A}$ process, Ω_2, Ω_3 are $\mathcal{E}$ processes. For $E_x = 10^6 V/m$, $\omega = 10^{15}$, $\Delta\omega = 1GHz$. we get $\Omega_{14} = 10$/s.

3.6 Photoelectric effect and Compton Scattering

3.6.1 Photoelectric effect

When light of frequency ω is shown on metal surface [49], electrons are ejected. This is photoelectric effect. The electron energy is $E_o = -V_0 + \frac{\hbar^2 k_o^2}{2m} < 0$, where k_0 is electron momentum and $-V_0$ is crystal potential. Momentum of electron changes to k_F and its energy is $E_F = -V_0 + \frac{\hbar^2 k_F^2}{2m} > 0$ (electron is ejected). Change of momentum $\Delta k = k_F - k_0$ accounts for increase of electron energy by $\hbar \upsilon_F \Delta k$ where υ_F is Fermi velocity of electron $\sim 10^5 - 10^6 m/s$. This energy difference is given by photon carrying $\hbar\omega \sim 3 - 5eV$ of energy giving $\Delta k \sim 10^{10}/m$.
This momentum doesn't come from photon which will then carry $\hbar c \Delta k$ energy, which is very large (keV), it comes from phonons in metal, whose energy is $\hbar \upsilon \Delta k$, where $\upsilon \sim 10^3 m/s$ is velocity of sound in metal. Phonon energy is them meV which is negligible.

3.6.2 Compton Scattering

We talked about inelestic scattering process, Raman scattering, where light of wavenumber k gets scattered to k' and the atom transits to different atomic state. Another interesting inelastic scattering of light is due to free electrons [50] in solid, called Compton Scattering.

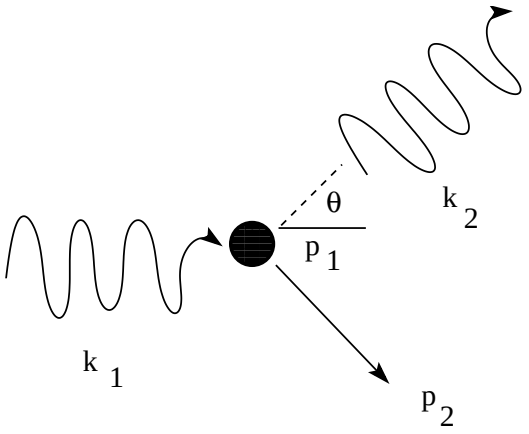


Fig. 3.13 Fig. shows Compton csattering. Light with momentum k_1 hits electron with momentum p_1, with light scattering to k_2 and electron to p_2.

Fig. 6.9 shows Light with momentum k_1 hits electron with momentum p_1, with light scattering to k_2 and electron to p_2 such that momentum and energy are conserved, i.e., $k_1 + p_1 = k_2 + p_2$ and $E_{p_1} + E_{k_1} = E_{p_2} + E_{k_2}$, which gives

$$\lambda_2 - \lambda_1 = \frac{\hbar}{m_0 c}(1 - \cos\theta) \tag{3.38}$$

where $\lambda_c = \frac{\hbar}{m_0 c} \sim .01A^\circ$ is Compton wavelength (m_0 mass of electron). To see a change in wavelength comparable to compton wavelength , we should have short wavelength light (high energy, 10-100 keV, X-rays).

3.6.3 Laser cooling

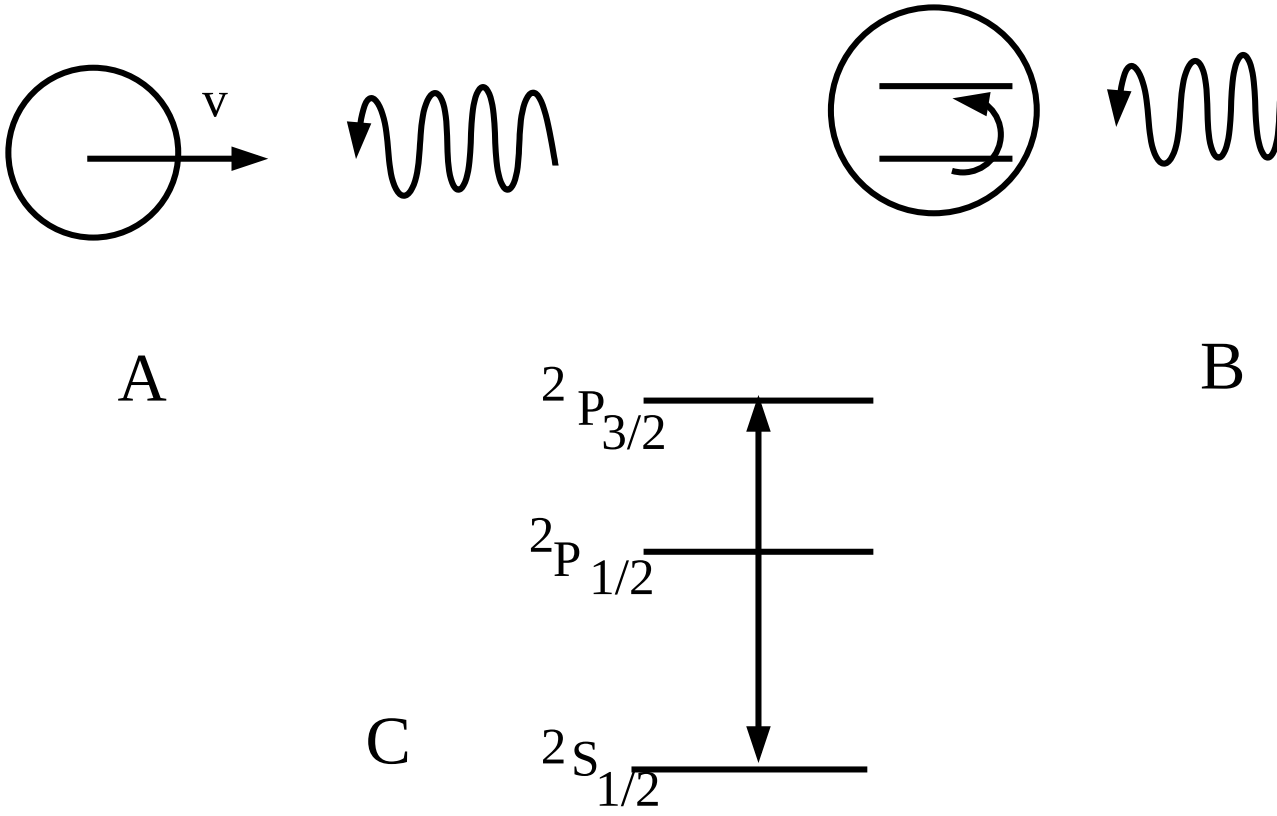


Fig. 3.14 Figure A shows how an atom is hit with light with momentum $\hbar k$ and slows down. Figure B shows atom will absorb light if frequency $\hbar\omega$ matches the difference of energy between the inetrnal energy levels. Figure C shows these energy levels for sodium. These are the electronic states with $n = 3$.

Many experiments in physics require slow atoms [38]. At room temperatures, we have atoms moving at say 300 m/s. We like to slow them down to say 10 m/s. Slow atoms are used to make Bose Einstein condensates (velocity as low as cm/s). How do we slow atoms. We can do it by hitting them with ligh of the right frequency. Atoms have internal energy levels. When the frequency of the laser light matches this, the light is absorbed (optical transition). But light carries momentum $\hbar k$ and therefore on absorbtion atom gets a kick which slows it down. The absorbed photon is spontaneously emitted but in a random direction. When we average over many such absorbtion and emission, the kick is always in the same direction while the recoil due to spontaneous emission is random and averages to zero and in the end atom slows down. If frequency of the laser light is ω an atom moving towards the light source will see the frequency shifted to $\omega(1+\frac{v}{c})$. Then if detune the laser frequency to be slightly less than the internal energy level, due to this doppler shift the atom will see just the right frequency and will absorb it. Atoms moving slowly will have negligible dopper shift and will not absorb light. Hence we will only cool fast atoms and not slow ones. This way we will bring all to same velocity. This is the basic idea of laser cooling. It has been successful in cooling atoms to very low velocities where they form a Bose Einstein condensate. Figure C shows these energy levels for sodium. These are the elctronic states with principal quantum number $n = 3$. The transition shown is called the D_2 line.

Laser cooling is an $\mathcal{E}$ process.

3.7 Semiconductor transitions, ccd cameras, photovoltaics, Light emitting diode (LED)

Until now we considered atomic transitions ϕ to ϕ' where these are atomic levels. In semiconductors [51], we go between electronic bands.

Electron states $\psi_k = \sum_j \exp(ik(ja))\phi_j$ is a electron wave (a, lattice constant) with wavevector k is ϕ band also called Valence band. Similarly $\psi'_k = \sum_j \exp(ik(ja))\phi'_j$ is a electron wave with wavevector k is ϕ' band also called conduction band.

Energy of ψ_k is $E_k = \epsilon_0 - 2t\cos ka$ where ϵ_0 is orbital energy of ϕ. Energy of ψ'_k is $E'_k = \epsilon_1 - 2t\cos ka$ where ϵ_1 is orbital energy of ϕ'.

Photon with momentum q and emergy $\hbar\omega = \hbar cq$ transfers $\psi_k to \psi'_{k+q}$ (q is small else photon has large energy) with photon energy $\hbar\omega = \epsilon_1 - \epsilon_0$.

The transition between valence and conduction band involves atomic level change hence $\mathcal{E}$ and $\mathcal{M}$ for the process is for $d = \langle\phi'|x|\phi\rangle$

$$\mathcal{M} = qE_x d \tag{3.39}$$

When there is spontaneous emission from conduction band to valence band

$$\mathcal{M} = qE_o d \tag{3.40}$$

Examples of semi-conductor transitions include Light emitting diodes (LED), Lasers , Photovoltaics, ccd cameras.

3.8 Radiation heating from sun and infrared sources.

Solids are hot because of lattice vibrations, phonons. Light from sun can generate phonons. Energy of the phonons is 10^{13}Hz. Infrared photons carry this energy. Since atomic levels change it is $\mathcal{E}$ process.

The phonon has energy $\hbar\upsilon k \sim 10meV$, giving $k \sim 10^{10}/m$, giving a momentum $\hbar k$ to phonon. This much momentum light doesn't have. The momentum is balanced by center of mass (CM) of the whole solid recoiling with momentum $-k$ and energy $\frac{\hbar^2 k^2}{2M}$ (M mass of whole solid), which is negligible because of large M. How do CM and phonon momentum talk ? They do it by conduction electrons. All in all this becomes a 4 level process as encountered before in 3.11.

Inverse of this process is heating a solid to generate infrared light as in a Globar source for Infrared spectroscopy or infrared imaging as in night vision cameras.

3.9 Scattering in Quantum Information and Imaging

3.9.1 Slow Light

In Resonance Raman setting, if we have have stimulated Raman, done with a probe beam, then photons from pump will scatter into probe enhancing the effective permittivity of probe, and slowing it down [77]. The atoms will of-course transfer from ground to excited-ground state. The difference in the frequency of the pump and probe pulse is the energy difference of two ground atomic levels. If atoms are moving around there is Doppler shift in frequencies and resonance condition is not exactly met, so we have to work with cold atoms to minimize Doppler or use a pump laser which is broad-banded using a laser cavity with low quality factor.

3.9.2 Møelmer-Sørenson two qubit Gate

Fig. depicts two ions coupled with a vibrational mode. The internal state of the ions encodes a logical bit. By putting a laser at the frequency $\omega_a + \omega_b$, such that we excite both internal mode (atom A) and vibration mode by absorbing a photon and then emitting it we return the atom B onto its ground state performing a swap operation. This is the famous Mølemer-Sørenson gate in Ion trap quantum computing [78].

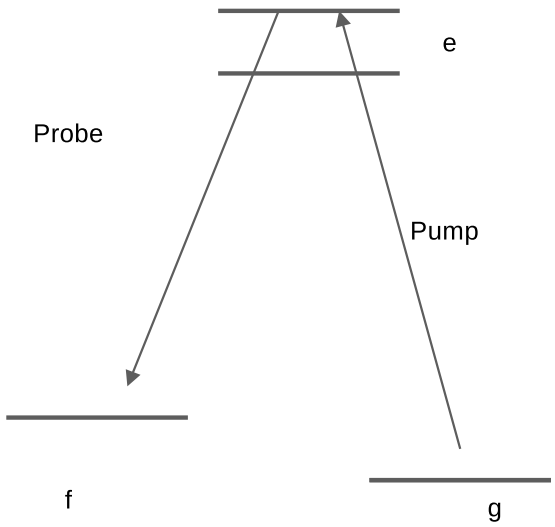


Fig. 3.15 Fig. depicts Stimulated Raman in setting of Resonance Raman, where pump light scatters in probe and slows it down.

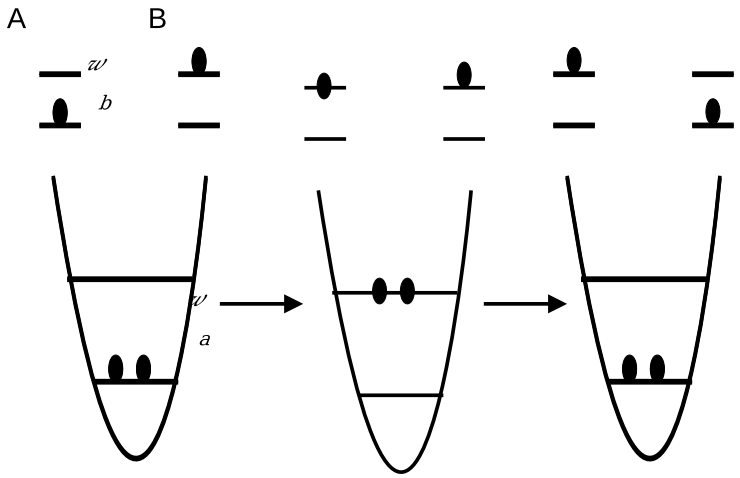


Fig. 3.16 Fig. depicts two ions coupled with a vibrational mode. The internal state of the ions encodes a logical bit. By putting a laser at the frequency $\omega_a + \omega_b$, such that we excite both internal mode (atom A) and vibration mode, by absorbing a photon and then emitting it we return the atom B onto its ground state performing a swap operation.

3.9.3 Optical Pumping a Laser and Solid State Maser

In 3.17, Fig. A depicts optical pumping from s orbital ground state g to s orbital excited state e via a $\mathcal{A}$ process and then decay back via a meta-stable p orbital state f. Decays are $\mathcal{E}$ processes. Fig. B depicts optical pumping from p orbital ground state g to p orbital excited state e via a $\mathcal{A}$ process and then decay back via a meta-stable s orbital state f. Decays are $\mathcal{E}$ processes and we return with spin flipped. This way we can spin polarize the system and the two spin states in ground manifold can be used to make a MASER as in a NV center [79]. The energy difference between two spin states can be tuned with a magnetic field. giving a variable frequency MASER.

3.9.4 Super Resolution Imaging

Fig. 3.18, depicts Stimulated Raman in setting of Resonance Raman where pump light scatters in the probe. Scattering is very directional and more when probe is high intensity. By modulating the pump we can form a very high resolution image of the scattered light [80].

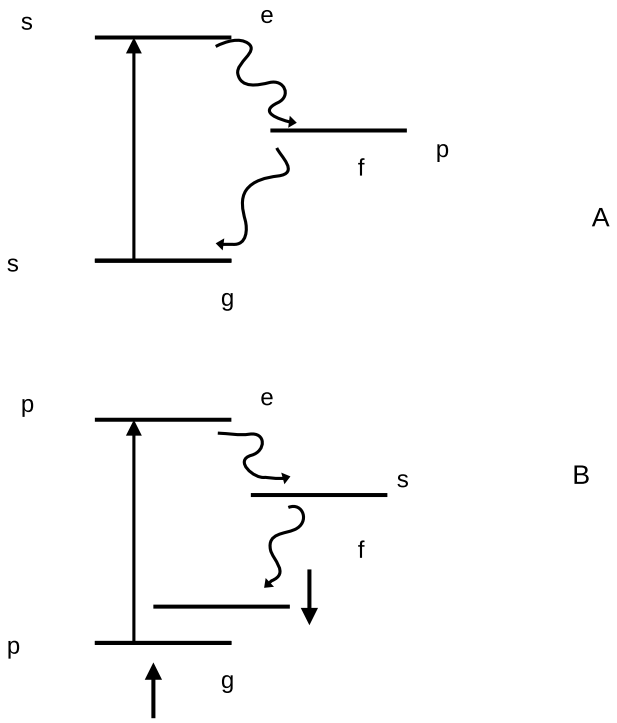


Fig. 3.17 Fig. A depicts optical pumping from s orbital ground state g to s orbital excited state e via a $\mathcal{A}$ process and then decay back via a meta-stable p orbital state f. Decays are $\mathcal{E}$ processes. Fig. B depicts optical pumping from p orbital ground state g to p orbital excited state e via a $\mathcal{A}$ process and then decay back via a meta-stable s orbital state f. Decays are $\mathcal{E}$ processes and we return with spin flipped. This way we can spin polarize the system and the two spin states in ground manifold can be used to make a MASER as in a NV center

3.9.5 Non-demolition measurements and Faraday Rotation

Fig. 3.19 shows a two level atom (g ground, e excited state) with excited state having +1 angular momentum. If we scatter light off this atom in the ground state, atom will transit to intermediate excited state by absorbing a right polarized photon and return back emitting the photon. The linearly polarized photon which is sum of right and left polarization $L+R$ encounters a phase on the right polarized photon and becomes $L+\exp(i\phi_1)R$, which results in net rotation of linearly polarized axis. $\phi_1 \propto \frac{\Omega^2}{\omega-\omega_0}$, where ω_0 is energy difference between g and e, ω light frequency and Ω is Rabi frequency ($\sim qEd$).

If we scatter light off this atom in the excited state it will transit to intermediate ground state by absorbing a left polarized photon and return back emitting the photon. The photon which is sum of right and left polarization $L+R$ encounters a phase on

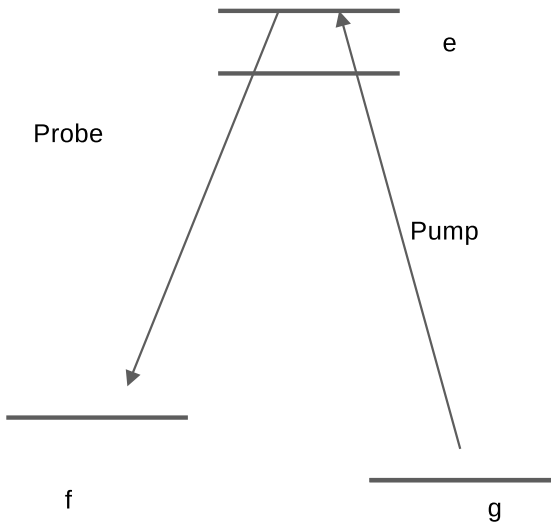


Fig. 3.18 Fig. depicts Stimulated Raman in setting of Resonance Raman where pump light scatters in probe.

the Left polarized photon and becomes $\exp(i\phi_2)L+R$, which results in net rotation of linearly polarized axis in opposite direction $\phi_2 \propto \frac{\Omega^2}{\omega+\omega_0}$.

If atom is in superposition then the two phases almost cancel and we donot get a phase shift. This way we can measure the atom state in a non-demolition way [83].

The two levels can be up and down states of the spin with energy difference arising from the energy $\mu \cdot B$ where B is effective field of other polarized spins. *Faraday rotation* [81] is the shift in phase of the linearly polarized axis. While Faraday rotation is an $\mathcal{A}$ process, no-demolition measurement is typically $\mathcal{E}$ process.

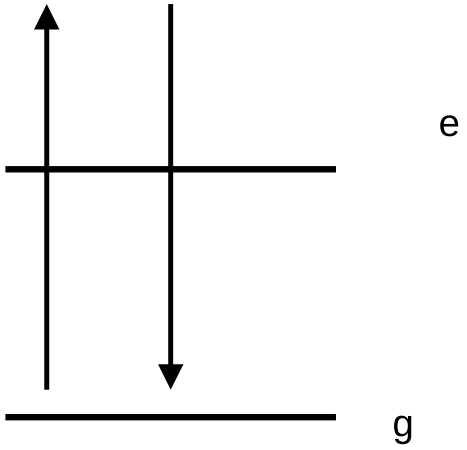


Fig. 3.19 Fig. shows light scattering of a two level atom

3.9.6 Magnetic Resonance a Classical $\mathcal{A}$ process

Electron spin states $\uparrow$ and $\downarrow$ are separated in energy $E = \mu \cdot B_o$ where $\mu = \frac{q\hbar}{m}$. The rf photon induces the transition from state $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\uparrow\rightarrow\begin{pmatrix}\sin\theta+\phi\\ \cos\theta+\phi\end{pmatrix}$, which transition amplitude (Rabi frequency) $q\frac{E\omega\sin\phi}{c} = \hbar\frac{q}{m}B = \hbar\gamma B$ where $\frac{q}{m} = \gamma$ is gyromagnetic ration [**?**].

3.10 Conclusion

In this chapter, we studied optical transitions. The starting point is the electron Hamiltonian, H, in a electromagnetic wave. $H = \sum_{j=x,y,z} \frac{(p_j - qA_j)^2}{2m}$, where q is electron charge, m electron mass, p is electron momentum and A the vector potential. We showed this is not complete. The complete Hamiltonian is $H = \sum_{j=x,y,z} \frac{(-i\hbar\partial_j - qA_j)^2}{2m} + qE \cdot r$, where E is the electric field of electromagnetic wave. In this paper we derived this Hamiltonian. We classified optical transitions into two types, one that are mediated by A ($\mathcal{A}$ process) and other by E ($\mathcal{E}$ process). Equipped with this, we studied many optical processes. These include atomic transitions, elastic and inelastic scattering processes like Rayleigh scattering, Raman scattering, Two photon absorption, stimulated Raman, second harmonic generation, parametric down conversion, photoelectric effect, optical processes in semiconductors. We show the $\mathcal{E}$ process is stronger than the $\mathcal{A}$ process when we transit between atomic levels. The $\mathcal{A}$ process happens when we have virtual levels. In processes like Raman scattering, two photon absorption , we have both the $\mathcal{A}$ and the $\mathcal{E}$ processes. The $\mathcal{A}$ process flips the spin while the $\mathcal{E}$ process doesn't. The $\mathcal{A}$ processes are therefore always circularly polarized. Transition amplitude of the $\mathcal{E}$ process is straightforward and is everyday calculation of Rabi frequency. The transition amplitude of the $\mathcal{A}$ process was done through Dirac equation, the relativistic cousin of Schrödinger equation. To sum it all, virtual transitions $\mathcal{A}$, real atomic level transition $\mathcal{E}$.

Problems

1. Calculate the Rayleigh Rabi frequency if $E_x = 10^2 V/m$ and $\omega = 10^{15}/s$.

2. Calculate the Rayleigh Rabi frequency if $E_x = 10^2 V/m$ and $\omega = 10^{15}/s$ and there is excited level at $\omega_0 = .9 \times 10^{15}$ with $d = 1A^\circ$.

3. Calculate the Raman scattering amplitude if $E_x = 10^2 V/m$, $d = 1A^\circ$ and $\omega = 10^{15}/s$.

4. Calculate the Resonance Raman scattering amplitude if $E_x = 10^2 V/m$, $d_{12} = d_{23} = 1A^\circ$ and $\omega = 10^{15}/s$, $\omega_0 = 1.1 \times 10^{15}/s$.

5. Calculate the stimulated Raman Rabi frequency if $E_x = E_1 = 10^5 V/m$ and $\omega = 10^{15}/s$.

Chapter 4
Aspects of electron scattering, the elastic, and the inelastic.

A electron of mass m, when electrically scatters of nucleus, of mass M, transfers momentum q to the nucleus. The energy lost by electron is more than the energy gained by the nucleus. The resulting energy goes in exciting the atom to a higher energy state as in Frank Hertz experiment and sodium, neon, mercury vapor lamps, or ionization of atom as in bubble and cloud chamber experiments, or just production of X-rays as in Bremsstraulung. In this paper, we study these phenomenon. These experiments are inelastic scattering experiments. We remark, why neutrinos donot scatter and can penetrate earth, why muons travel further than electrons in materials and why a material like lead plate can slow down electrons and positrons efficiently. We look at the elastic scattering of electrons as in electron diffraction and electron microscopes. We look at scattering of electrons in the condensed matter, these phenomenon range from scattering of electrons of periodic potential, to give Bloch waves, scattering of electrons of phonons and impurities to give resistance, scattering of electrons of lattice to give cooper pairs and superconductivity. We study electron scattering from exchange potential as in Fermi liquid theory and resulting T^2 resistance at low temperatures. Electron scattering of exchange potential resulting in chemical reactions. We turn our attention to electron-proton scattering both elastic and inelastic, as in deep inelastic scattering experiments and understand the independence of inelastic cross-section of with respect to transferred momentum. We see, why we can just say that there are three quarks in proton from elastic cross-section. Our main contribution in this article is we are detailed at places, we find literature terse.

4.1 Frank Hertz Experiment

In 1914, James Franck and Gustav Hertz [52, 53] performed a beautiful experiment. They accelerated electrons in a cathode ray tube filled with mercury. They observed that at certain value of the accelerating voltage, the current in the tube dropped. This demonstrated the existence of excited states in the mercury atoms, with quantized energies, and helped to confirm the quantum theory. The values of accelerating

voltage, where the current dropped gave a measure of the energy necessary to force an electron to an excited state.

Let x_1, x_2 be coordinates of atomic electron and nucleus, and $X_1 = \frac{mx_1+Mx_2}{m+M}$ and $X_2 = x_1 - x_2$, center of mass and relative, coordinate. Let k_1, k_2 be momentum of x_1, x_2 coordinates and K_1, K_2 momentum X_1, X_2 coordinates. Then $\sum k_i x_i = \sum K_i X_i$ gives $K_1 = k_1 + k_2$ and $K_2 = \mu\left(\frac{k_1}{m} - \frac{k_2}{M}\right)$, where $\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}$ is the reduced mass.

When q momentum transfers from incident electron to the atomic electron, it can change the atomic wavefunction

$$\exp(iqx_1) = \exp(iq'X_1)\exp(iq''X_2),$$

where $q' = q$ and $q'' = \frac{M}{M+m}q \sim q$.

$$\exp(iq''X_2) = \cos(q''(x_1 - x_2)) + i\sin(q''(x_1 - x_2)).$$

$$d_e = \langle\phi_1|\sin(q''(x_1 - x_2))|\phi_0\rangle$$

is the dipole moment of the transition, from ground to excited state of the atom. When incident atom stops and

$$E_1 - E_0 = \frac{\hbar^2 q^2}{2m},$$

we see the atomic transition from E_0 to E_1.

More generally, $q = p_1 - p_2$ (p_1, p_2 initial and final momentum of incident electron) and

$$E_1 - E_0 = \frac{\hbar^2}{2m}(p_1^2 - p_2^2), \tag{4.1}$$

Transition amplitude for transfer of momentum

$$\mathcal{M} = \frac{e^2}{V\epsilon_0 q^2} \tag{4.2}$$

where $V = l^3$ is volume of incident electron, where $l \sim 10^{-8}$, thermal debroglie wavelength.

Total transition amplitude is

$$\mathcal{M}' = \mathcal{M}d_e \tag{4.3}$$

When q momentum transfers from incident electron to the atomic nucleus, it can change the atomic wavefunction

$$\exp(iqx_2) = \exp(iq'X_1)\exp(iq''X_2),$$

where $q' = q$ and $q'' = \frac{m}{M+m}q$.

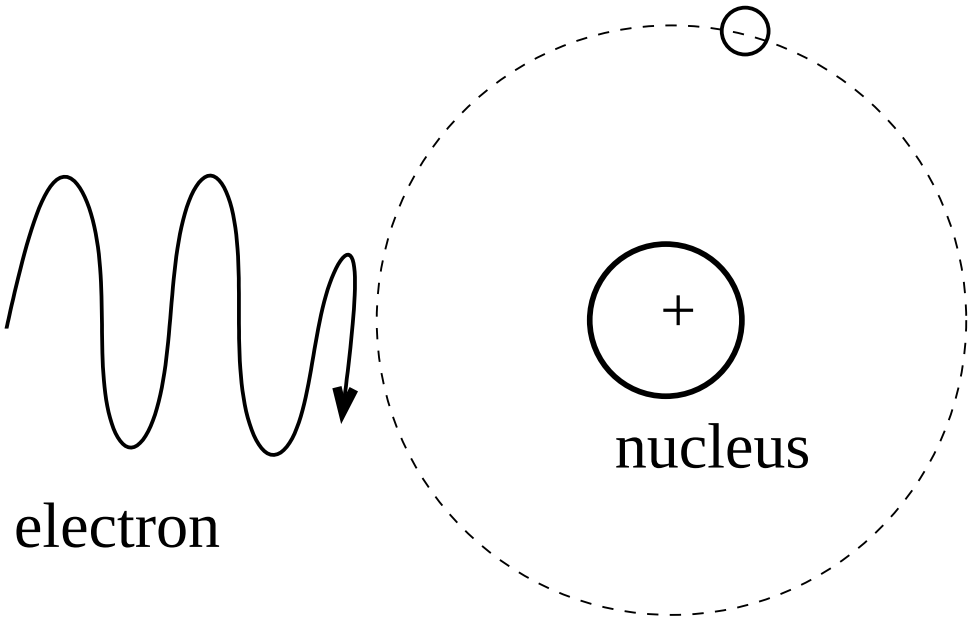


Fig. 4.1 Fig. A depicts Frank Hertz experiment and incident electron makes a atomic transition.

$$d_n = \langle \phi_1 | \sin(q''(x_1 - x_2)) | \phi_0 \rangle$$

is the dipole moment of the transition, from ground to excited state of the atom, which is negligible due to small q''. Ofcourse, transferred momentum went to center of mass (CM) of atomic system.

Frank Hertz phenomenon is ubiquitous in vapor lamps like sodium, neon, mercury vapor lamps, where free electrons are accelerated and bombard atoms exciting them, which subsequently emit light we see.

4.2 Bremsstraulung

Transfer of momentum from electron to atom creates energy deficit, (atom is much heavier than electron). In Frank Hertz this imbalance is paid by exciting the atom. In Bremsstraulung [54], this is paid by free electron radiating. When energy of free electron is in KV range, we produce X-rays with energy in this range. Bremsstraulung stands for breaking radiation, electron decelerates and emits.

When q momentum transfers from incident electron to the atomic electron, it can change the atomic wavefunction

$$\exp(iqx_1) = \exp(iq'X_1)\exp(iq''X_2),$$

where $q' = q$ and $q'' = \frac{M}{M+m} q \sim q$.

$$\exp(iq''X_2) = \cos(q''(x_1 - x_2)) + i\sin(q''(x_1 - x_2)).$$

$$d_o = \langle \phi_0 | \cos(q''(x_1 - x_2)) | \phi_0 \rangle$$

is the elastic moment of self transition, from ground to ground state of the atom (atom doesn't move). When incident atom stops and

$$E_1 - E_0 = \frac{\hbar^2 q^2}{2m},$$

we see the transition.

Transition amplitude for transfer of momentum

$$\Omega_1 = \frac{e^2}{V\epsilon_0 q^2} \tag{4.4}$$

where $V = l^3$ is volume of incident electron, where $l \sim 10^{-8}$, thermal debroglie wavelength.

But there is another amplitude, the radiation by free electron, whose amplitude is

Transition amplitude for transfer of momentum

$$\Omega_2 = qE_o l \tag{4.5}$$

where $V = l^3$ is volume of incident electron, where $l \sim 10^{-8}$, thermal debroglie wavelength. and E_o is electric field of emitted photon,

Then this being a second order process, the net amplitude is

$$\Omega = \frac{\Omega_1 \Omega_2}{E_1 - E_0} \tag{4.6}$$

where $E_1 - E_0$ is as in Eq. (5.6).

A small digression in second order process, shown in Figure 5.9. There are three levels $|1\rangle$, $|2\rangle$ and $|3\rangle$ with energies E_1, E_2, E_3, with $E_1 = E_3$, levels 1 and 3 are degenerate. Ω_1 and Ω_2 are transition amplitudes between level 1 and 2 and level 2 and 3 respectively.

Then gives for $|E_1 - E_2| \gg \Omega_i$, there is transition amplitude of going from $|1\rangle$ to $|3\rangle$, given by,

$$\mathcal{M} = \frac{\Omega_1 \Omega_2}{E_1 - E_2} \tag{4.7}$$

What is E_o , in 5.25. If L is length of photon, then $\frac{\epsilon_0}{E_o}^2 L^3 = \hbar\omega = E_1 - E_0$, but it takes $\hbar\Omega^{-1}$ time for emission in which photon travels a distance $L = c\hbar\Omega^{-1}$. From this we can get E_o.

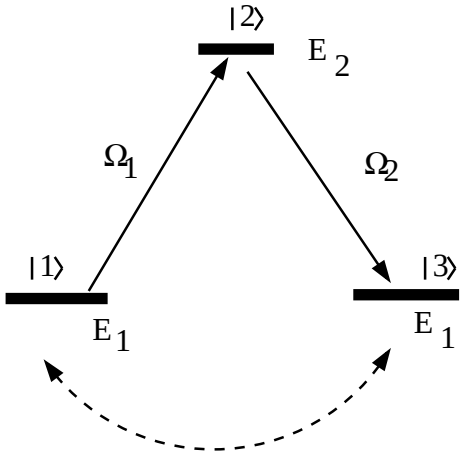


Fig. 4.2 Fig. depicts a 3 level system with transition amplitude Ω_1 and Ω_2 between level 1 and 2 and level 2 and 3 respectively.

Fig. 5.12 shows Bremsstraulung spectra, with characteristic emission frequencies. These characteristic frequencies arise when free electron doesn't emit rather ionizes the atom. When electrons go back, they emit characteristic emission frequencies.

4.3 Muons, neutrinos, cloud and bubble chambers and the lead plate

Electron is light, nucleus heavy, transferred momentum q takes more energy from electron than it gives to nucleus, the deficit, can ionize the atoms as we see in how energetic charged particles in cloud and bubble [55, 56] chambers, in high energy physics experiments leave tracks of ionized atoms when they move through the medium. All this is just Frank Hertz.

Ionization is one way to make up for lost energy during momentum transfer other is radiation as in Bremsstraulung. Electrons and muons when move through solid materials loose energy like this and slow down. Muons travel further than electrons for same initial energy. Why is this, the muon is 200 times heavier. During transfer of momentum to material, electron loses more energy than muon and hence slows down early.

For a material to efficient slower, its nucleus should be very heavy, then energy loss is more, during momentum transfer. This is why the famous experiment of Carl Anderson on discovery of positron [57], uses a lead plate to slow down positrons.

Finally, neutrinos donot slow down at all, they keep going. This is because, they donot have charge they interact through weak interactions. The transition amplitude for momentum transfer q is

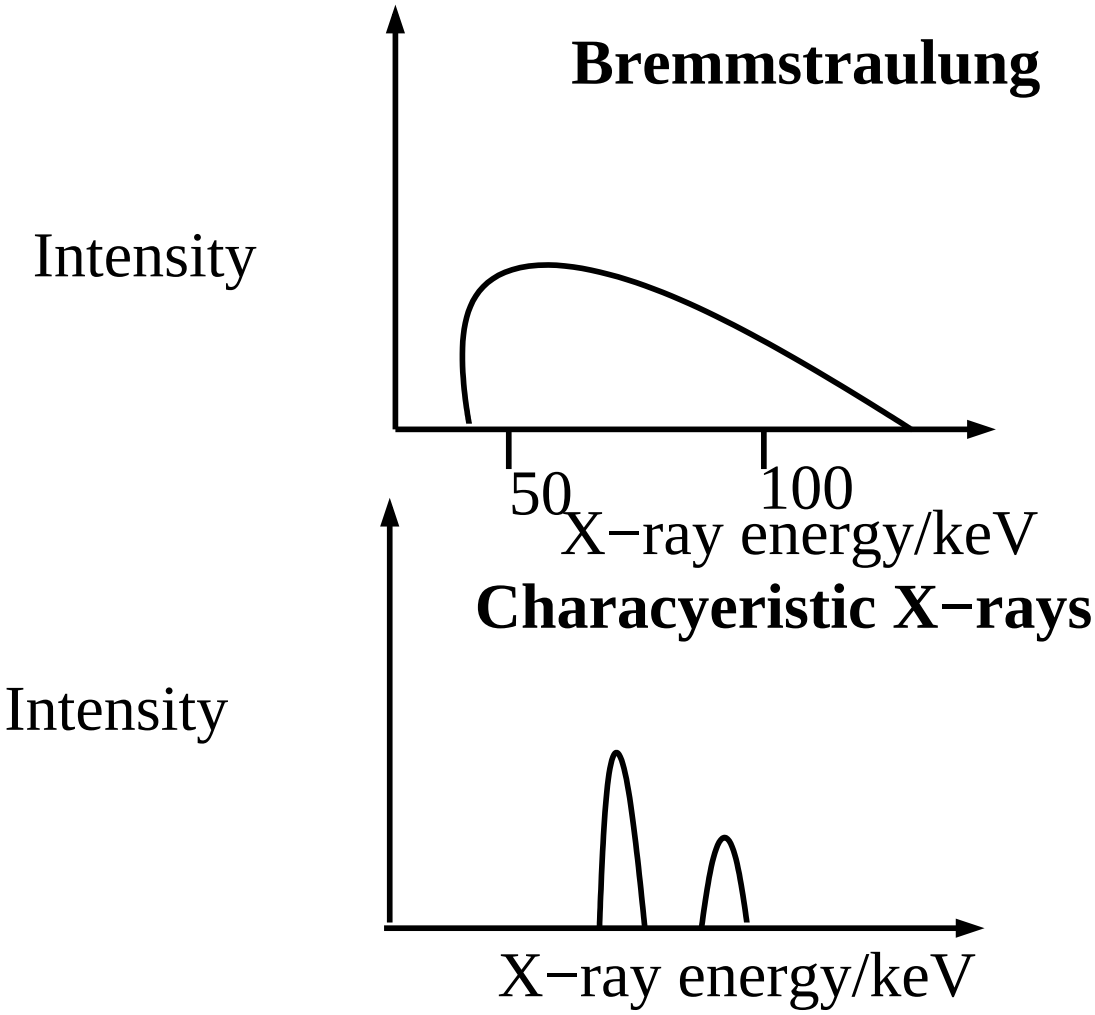


Fig. 4.3 Fig. depicts Bremsstraulung spectra, with characteristic emission frequencies.

$$\Omega' = \frac{\alpha_w (\hbar c)^3}{V M_W^2} \tag{4.8}$$

where V is the volume of incident neutrino, M_ω the mass of W boson (very heavy 90 GeV) and α_w the structure constant for weak interactions (around $1/30$). Very large M_W makes this negligible, hence neutrinos donot interact.

4.4 Elastic Scattering of Electrons

We talked about inelastic scattering electrons. But electrons can elastically scatter from nuclei, change direction, maintaining their kinetic energy.

4.4.1 Electron and Neutron diffraction

Electron diffraction experiments were carried out by Davisson and Germer in 1927 [58]. Experiments showed wave nature of the electron. The electrons fired at an angle to the crystal rebounded at that angle (as classical ball) more so for certain choice of angles. Fig. 5.7A depicts how electrons and neutrons rebound of crystal at certain choice of θ.

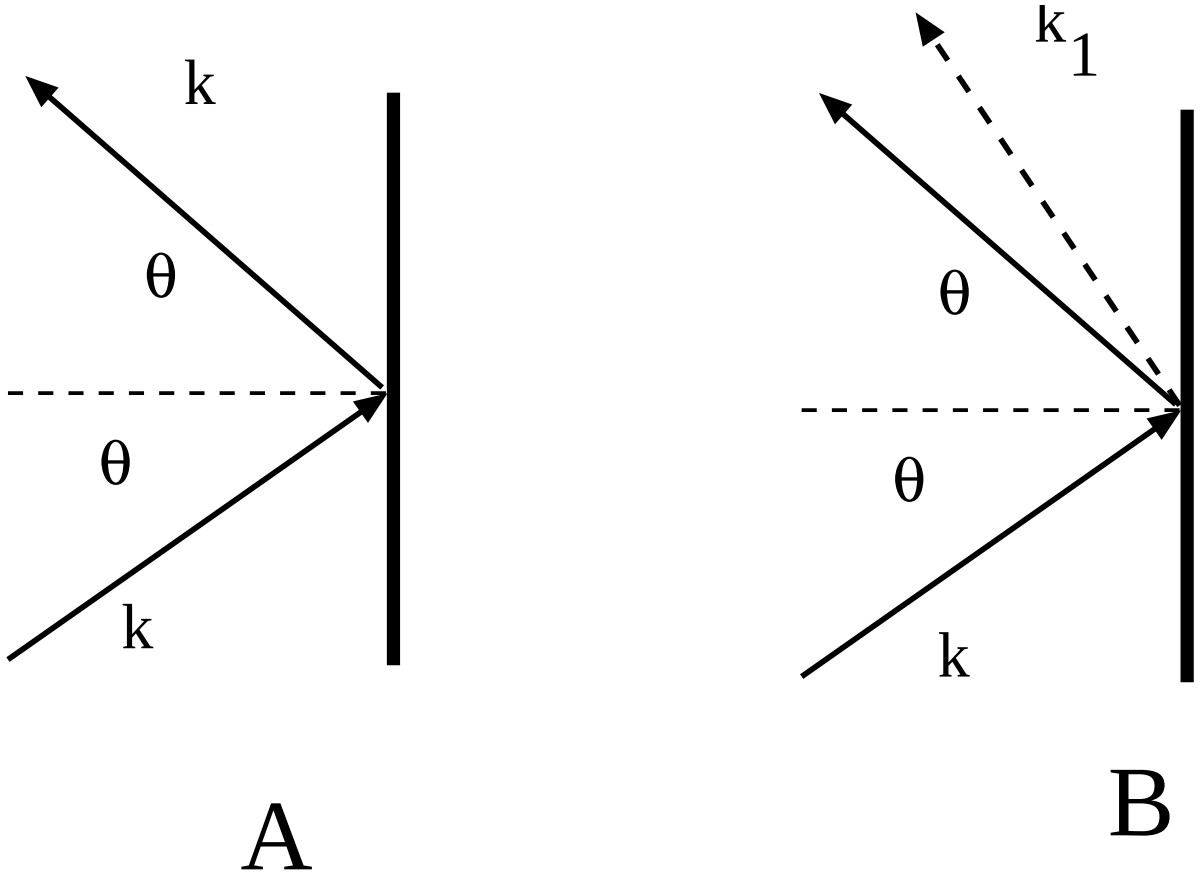


Fig. 4.4 Fig. A depicts how electrons and neutrons rebound of crystal at certain choice of θ. Fig. B depicts inelastic scattering of neutrons to measure phonon spectra.

Coulomb potential of nucleus

$$U(x) = \sum_q \frac{1}{V\epsilon_0 q^2} \exp(iq \cdot x) \tag{4.9}$$

where $1/V = (\Delta q)^3$ (Δq is the step size of q discretization).

The electron wave $exp(ik \cdot x)$ scatters of it to $\exp(ik' \cdot x)$, where $k' = k + q$. Summing amplitude of various sites we get

$$\exp(ik \cdot x) \rightarrow \exp(ik \cdot x) \sum_{x_0} \exp(iq \cdot (x - x_0)) = \exp(ik' \cdot x) \sum_{x_0} \exp(-iq \cdot x_0)$$

For coherent addition of the amplitude $\mathcal{M} = \sum_{x_0} \exp(-iq \cdot x_0)$, we have

$$2k \sin\theta \; d = 2\pi \longrightarrow 2d \sin\theta = \lambda. \tag{4.10}$$

where d lattice spacing and $\lambda = \frac{2\pi}{k}$, wavelength of light.

For solid crystal, $d \sim A^\circ$, giving $\lambda \sim A^\circ$ or $k \sim 10^{10}/m$, which is $10eV$ of energy. Electron accelerated with this much energy bombard the crystal. We get returns at angle θ satisfying Lau condition Eq. 5.20.

Diffraction can also be done with neutrons [60], which are 1000 times heavier and hence for the same wavelength 1000 times less energetic, i.e., 10's of meV.

4.5 Electron Microscopes

Fig. 5.8 depicts how light or electron when bounced of a solid spreads away, if wavelength $\lambda >> l$ and how it comes back straight if $\lambda << l$, where l is object length. This is diffraction limit.

To see smaller and smaller objects we need to use smaller wavelengths. Electrons are naturally small, with wavelength

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}}, \tag{4.11}$$

where v and E are velocity and energy of electrons. At electron energies 200 keV, we have $\lambda \sim$ pico-meter.

In scanning electron microscope, electrons elastically scatter and collecting the scattered electrons we can form an image [59].

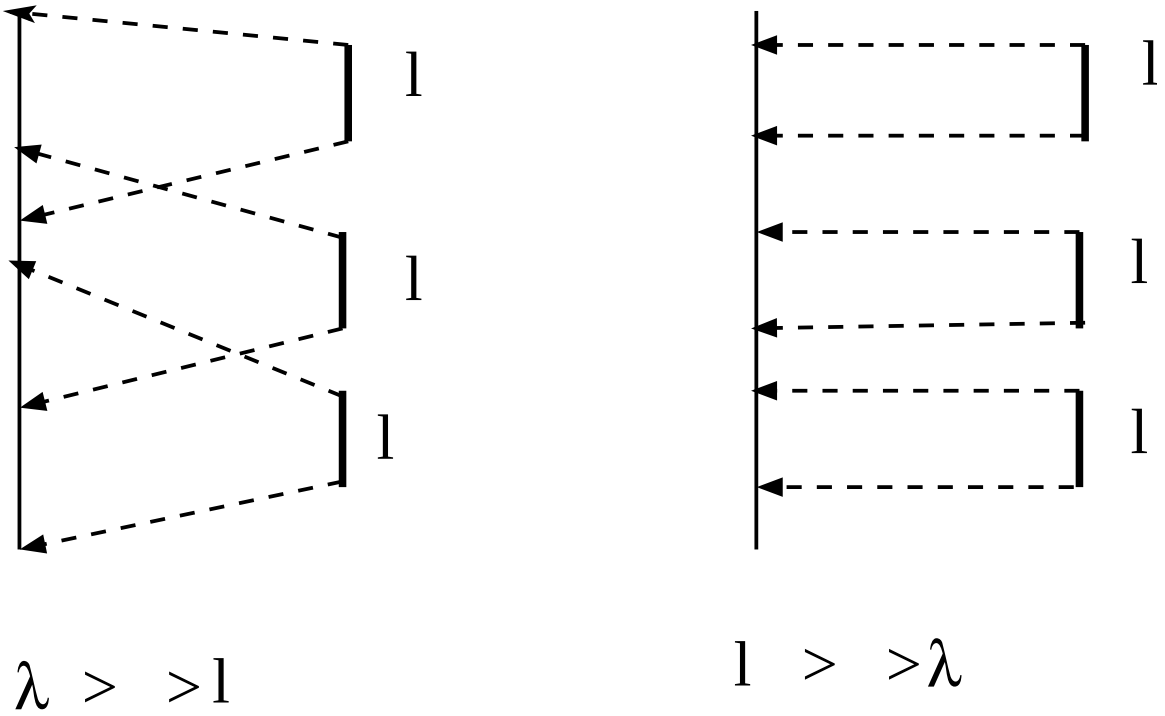


Fig. 4.5 Fig. depicts how light or electron when bounced of a solid spreads away, if wavelength $\lambda >> l$ and how it comes back straight if $\lambda << l$, where l is object length.

4.5.1 Inelastic scattering of neutrons: Phonon spectroscopy

In elastic scattering, neutrons change direction and transfer momentum to nucleus (lattice, which is very big). The energy gained by lattice is negligible (M is huge). But we can excite vibration modes of lattice when we transfer momentum. When mode with momentum k' is excited, it has energy $E(k')$, then in Fig. 5.7B, we have

$$k' = k_1 - k \tag{4.12}$$

$$E(k') = \frac{h^2 k_1^2}{2m_0} - \frac{h^2 k^2}{2m_0} \tag{4.13}$$

This way we can obtain $k', E(k')$ plot , the phonon-dispersion relation. This is inelastic scattering of neutrons, [61].

4.6 Electron Proton Scattering

Beautiful electron proton scattering experiments were carried out by Robert Hofstadter in 1950's [62]. These were Electron scattering experiments can be elastic or inelastic (where we excite internal modes of nucleus).

4.6.1 Elastic scattering

lets first discuss elastic scattering [63] where nucleus internal modes are not excited, so that its mass stays same. This is at electron energies $\ll Mc^2$, (in MeV). Fig. 9.1 depicts how electron scatters of nucleus at certain choice of θ. Let E_0 and E_1 be incident energy of electron and m mass of electron, M mass of nucleus which is at rest, then conserving energy momentum gives

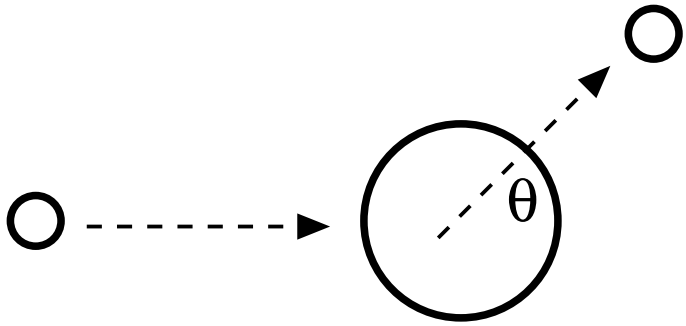


Fig. 4.6 Fig. depicts how electron scatters of nucleus at certain choice of θ.

$$\frac{1}{E_1} - \frac{1}{E_0} = \frac{1-\cos\theta}{Mc^2}, \tag{4.14}$$

when electron energies are relativistic with $E_0 = \frac{hc}{\lambda_0}$ and $E_1 = \frac{hc}{\lambda_1}$, with λ_i de-broglie wavelength's, we have

$$\lambda_1 - \lambda_0 = \frac{h(1-\cos\theta)}{Mc}, \tag{4.15}$$

Proton is three quarks as an approximation equal masses, theta x_1, x_2, x_3 be their coordinates and let $X_1 = \frac{x_1+x_2+x_3}{3}$, $X_2 = x_1 - x_2$ and $X_3 = \frac{x_1+x_2}{2} - x_3$ be Center of Mass and two relative coordinates.

Using $\sum_i k_i x_i = \sum_i K_i X_i$, we have $K_1 = k_1 + k_2 + k_3$, $K_2 = \frac{k_1-k_2}{2}$ and $K_3 = \frac{k_1+k_2-2k_3}{3}$.

When electron transfer momentum q to k_1 we have proton wavefunction ϕ_0 change by

$$\phi_0' = \exp(iqx_1)\phi_0 = \exp(iqX_1)\exp(i\frac{q}{2}X_2)\exp(i\frac{q}{3}X_3)\phi_0$$

The amplitude for this momentum transfer is $\mathcal{M}_0 \propto \frac{ee_1}{q^2\epsilon_0 V}$, where V is electron column and e_1 quark charge. Overlap of new wavefunction with old one is simply

$$\Omega = \langle\phi_0'|\phi_0\rangle = \int \cos(\frac{q}{2}X_2)\phi_0^2 dX_2 \int \cos(\frac{q}{3}X_3)\phi_0^2 dX_3 \sim \frac{1}{q^2}$$

Cross section of scattering $\propto \Omega^2 \propto q^{-4}$, that's it, this is what we find in the experiments, the elastic cross section dies as $\frac{1}{q^4}$ which means there are **three quarks** ! else it will die as $\frac{1}{q^{2n}}$ for n quarks, that's it.

4.6.2 Deep Inelastic Scattering

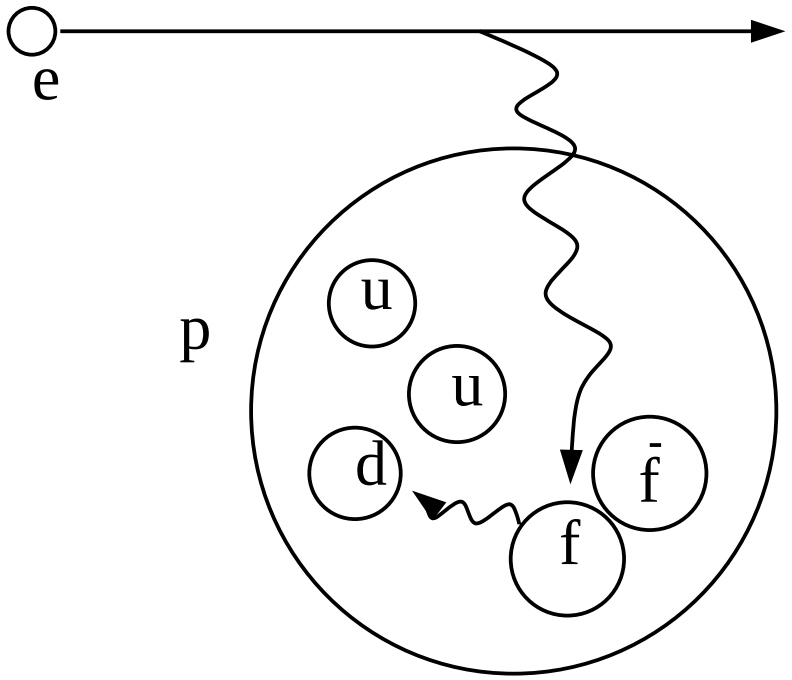


Fig. 4.7 Fig. depicts deep inelastic scattering of electron and proton.

We talked about elastic scattering of electrons and protons. But at high energies we can have in-elastic scattering [64], where by we can excite the internal modes off the proton, such that its internal energy or mass rises from M to W. Since q needed to create new mass is big comparable r_0^{-1} (radius of proton), cross section

will be every small, and will die more with increasing q. But we donot need to talk to proton directly. We can first exchange momentum and create a quark-antiquark $f\bar{f}$ pair (meson) as shown in 9.2, this is creating energy , but we cnnot just do it in thin air, because we cannot balance energy , we do this by further exchanging momentum with the proton and burning this energy in the heavy mass of proton. Scattering amplitude of $f\bar{f}$ has no q dependence, we are just scattering a free particle, so inelastic cross-section is independent of q for a gives x, where

$$x = \frac{q^2}{q^2 + W^2 - M^2}$$

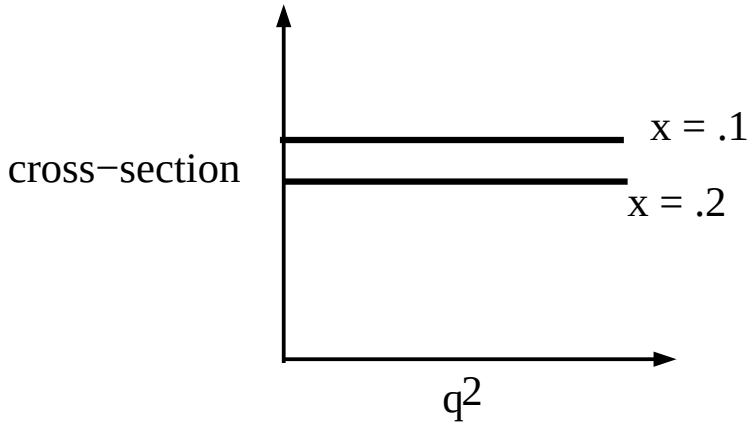


Fig. 4.8 Fig. depicts inelastic cross section as a function of q^2 for given x.

4.7 Electron scattering in solid state

We look at scattering of electrons in condensed matter, these range from scattering of electrons of periodic potential, to give Bloch waves [66, 65], scattering of electrons of phonons and impurities to give resistance, scattering of lattice to give cooper pairs and superconductivity. We study electron scattering from exchange potential as in Fermi liquid theory and resulting T^2 resistance at low temperatures. Electron scattering of exchange potential resulting in chemical reactions.

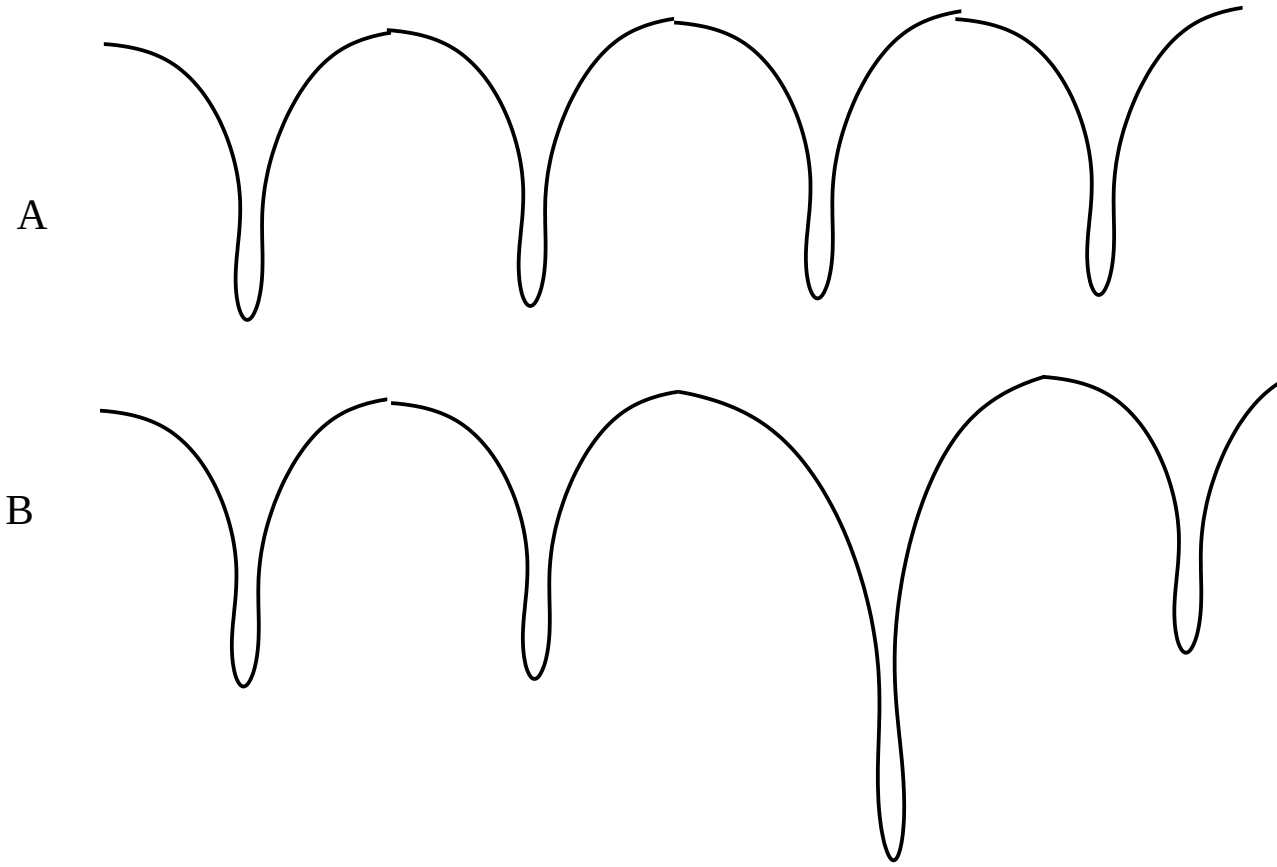


Fig. 4.9 Fig. A depicts periodic Coulomb potential. Fig B depicts periodic Coulomb potential with impurity

4.7.1 Scattering of periodic potential:Bloch waves

In a solid lattice, we have periodic arrangement of nuclei, producing a periodic potential (for simplicity of type)

$$V(x) = V_0 \cos^2(\frac{\pi x}{a}) = \frac{V_0}{4}\left(2 + \exp(-i\frac{2\pi x}{a}) + \exp(i\frac{2\pi x}{a})\right)$$

as shown in 4.9. Free electron wave then $\exp(ikx)$ then scatters to $\exp(i(k \pm \frac{2\pi}{a}x)$). Then the eigenfunction of the period potential has the form

$$\psi_n(k) = \sum_{j=-N}^{N} b_{n,j} \exp(k + \frac{2\pi j}{a})x = \exp(ikx)\phi_n(k),$$

where $\phi_n(k)$ is periodic function of period, the lattice constant a, where $b_n = (b_{n,-N}, b_{n,-(N_1)}, \ldots, b_{n,0}, \ldots, b_{n,N-1}, b_{n,N})$ and

$$H = \begin{bmatrix} \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{\hbar^2(k+\frac{2\pi}{a})^2}{2m} + V_0 & \frac{V_0}{2} & \cdots & 0 \\ 0 & \frac{V_0}{2} & \frac{\hbar^2 k^2}{2m} + V_0 & \frac{V_0}{2} & 0 \\ 0 & 0 & \frac{V_0}{2} & \frac{\hbar^2(k-\frac{2\pi}{a})^2}{2m} + V_0 & \frac{V_0}{2} \\ 0 & \cdots & \cdots & \cdots & \ddots \end{bmatrix}, \tag{4.16}$$

we have

$$Hb_n = \epsilon_n(k) b_n \tag{4.17}$$

, where $\epsilon_n(k)$ is energy of k wave, in the n^{th} energy band, where $k \in (-\frac{\pi}{a}, \frac{\pi}{a})$. Fig. 4.10 depicts energy bands in a periodic potential. In nutshell, eigenfunctions in a periodic potential are of the form $\exp(ikx)\phi_n(k)$.

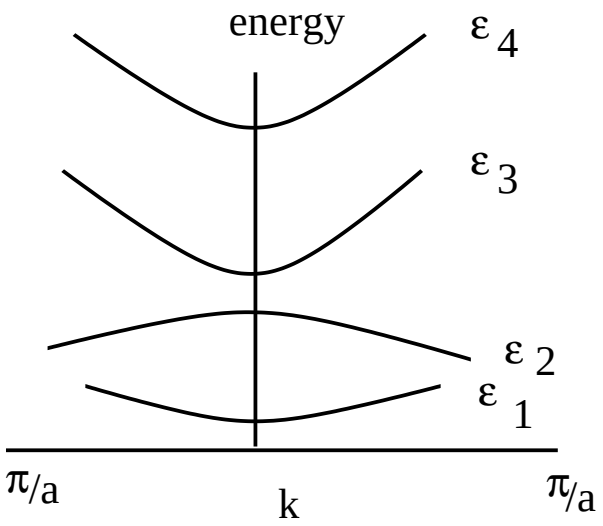


Fig. 4.10 Fig. depicts energy bands in a periodic potential.

4.7.2 Superconductivity

There is very interesting phenomenon that takes place in solid state physics [65, 66, 67, 68] when certain metals are cooled below critical temperature of order of

few kelvin. The resistance of these metals completely disappears, and they become superconducting. This phenomenon, whereby many materials exhibit complete loss of electrical resistance, when cooled below a characteristic critical temperature [69, 70] is called superconductivity. It was discovered in mercury by Dutch physicist Onnes [71] in 1911. For decades, a fundamental understanding of this phenomenon eluded the many scientists who were working in the field. Then, in the 1950s and 1960s, a remarkably complete and satisfactory theoretical picture of the classic superconductors emerged in terms of the Bardeen Cooper Schrieffer (BCS) theory [72].

Electrons want to scatter of phonons but in superconductivity they are bonded by phonon mediated interaction. Scattering would mean breaking this bond costing energy. Lets recapitulate phonon mediated pairing of electrons as in BCS theory. The electron phonon coupling Hamiltonian

$$H = H_e + H_{e-ph} + H_{ph}, \tag{4.18}$$

where H_e is electron Hamiltonian, H_{e-ph} is the electron-phonon coupling Hamiltonian

$$H_{e-ph} = \frac{c}{\sqrt{n^3}} \left(b \exp(ikx) + b^+ \exp(-ikx) \right), \tag{4.19}$$

$$H_{ph} = \hbar\omega b^+ b, \tag{4.20}$$

$c \sim 1$ eV and n^3 is number of lattice points (atoms) , b, b^+ are annihilation and creation operators for the phonon, k is phonon wave-vector, ω is phonon energy. $\omega = \upsilon k$ where $\upsilon = 3000 m/s$ velocity of sound in the solid.

Lets us derive c, assuming a periodic potential of the form

$$V = V_0 \cos^2(\frac{\pi x}{a}) \cos^2(\frac{\pi y}{a}) \cos^2(\frac{\pi z}{a}), \tag{4.21}$$

where a is lattice parameter around $3A^\circ$. V_0 is around $100V$ so that coulomb potential and V have same average value. Consider a phonon wave travelling in the x direction as $A\cos(kx - \omega t)$, the energy of the phonon wave is $E_{ph} = \frac{1}{2} m\omega^2 A^2 n^3$, where m is mass of atom and n^3 is number of lattice points. Equating this to energy of a phonon mode $\frac{1}{2}kT$ we get $A = \sqrt{\frac{kT}{m\omega^2 n^3}}$. For one quanta of phonon excitation, with energy $\hbar\omega_d$, we have $A = \sqrt{\frac{\hbar}{m\omega n^3}}$. Phonon perturbs lattice points, and hence the periodic potential. The perturbation, at $X = (x, y, z)$ is

$$\begin{aligned}
\Delta V(X) &= \frac{\partial V}{\partial x}|_{(X - X_0)} \;\; A\cos(kx_0 - \omega t), \\
&= V_0 \sin(\frac{2\pi(x - x_0)}{a}) \cos^2(\frac{\pi(y - y_0)}{a}) \cos^2(\frac{\pi(z - z_0)}{a}) \;\; A\cos(k(x_0 - x) + \theta), \\
&= f_1(X - X_0)\cos(\theta) + f_2(X - X_0)\sin(\theta),
\end{aligned}$$

where X_0 and x_0 is the coordinate and x-coordinate of atom closest to X. $\theta = kx - \omega t$. f_1 and f_2 are periodic in X with lattice period a and can be expanded into a Fourier series with spatial frequency $\frac{2\pi}{a}$. Taking the zero Fourier coefficient (so that resulting wave is in first Brillouin zone) by averaging $\Delta V(x)$ over unit cell (keeping θ constant) gives,

$$e\Delta V(X) = \frac{1}{\sqrt{n^3}} \underbrace{\frac{eV_0}{8\upsilon}\sqrt{\frac{kT}{m}}}_{c} \cos(kX - \omega t). \tag{4.22}$$

For e, electron charge, $T = 100K$ and $m = 10^{-26}Kg$ (10 proton masses), we have $c \sim 1$ e V. One quanta of phonon excitation, with energy $\hbar\omega_d$, (ω_d is the Debye frequency of 10^{13} rad/s) corresponds to $100K$ temperature and has $c \sim 1$ e V.

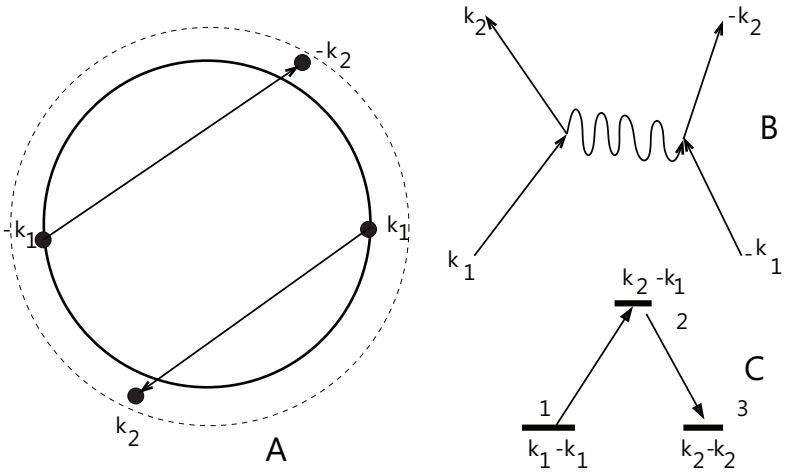


Fig. 4.11 Fig. A depicts the Fermi sphere, and how electron pair $k_1, -k_1$ at Fermi sphere, scatters to $k_2, -k_2$, at the Fermi sphere. Fig. B shows how this is mediated by exchange of a phonon in a Feynman diagram. Fig. C shows a three level system, that captures the various transitions involved in this process.

Let us take two electrons, both at the Fermi surface, one with momentum k_1 and other $-k_1$. Lets see how they interact with phonons. Electron k_1 pulls/plucks on the lattice due to Coulomb attraction and in the process emits a phonon, and thereby recoils to new momentum k_2. The resulting lattice vibration is sensed by electron $-k_1$, which absorbs this phonon and is thrown back to momentum $-k_2$. The total momentum is conserved in the process. This is depicted in Fig. 5.9A. The corresponding Feynman diagram for this process is shown in Fig. 5.9B. The above process, where two electrons interact with exchange of phonon, can be represented as a three level atomic system. Level 1 is the initial state of the electrons $k_1, -k_1$ and level 3 is the final state of the electrons $k_2, -k_2$ and the level 2 is the intermediate state

$k_2, -k_1$. There is transition with strength $\Omega = \frac{c}{\sqrt{n^3}}$ between level 1 and 2, involving emission of a phonon and a transition with strength Ω between level 2 and 3, involving absorption of a phonon. Let E_1, E_2, E_3 be energy of the three levels. Since pairs are at Fermi surface, $E_1 \sim E_3$. Second order effective transition from level $1-3$ is

$$d = \frac{\Omega^2}{E_1 - E_2} = -\frac{c^2}{n^3 \hbar \omega_d}. \tag{4.23}$$

The binding energy of the superposition $\psi = \sum \underbrace{|k_i, -k_i\rangle}_{\phi_i}$, from electron pairs in annulus of width $\hbar\omega_d$ at Fermi surface is then

$$\Delta' = -\frac{c^2}{n^3 \hbar \omega_d} \frac{\hbar \omega_d n^3}{\epsilon_f} = -\frac{c^2}{\epsilon_f}, \tag{4.24}$$

where ϵ_f is the Fermi energy. From above, $\Delta \sim 100$ meV. Energy E_i of $|k_i, -k_i\rangle$ are not same, and vary in annulus of width $\hbar\omega_d$. Hence the binding energy, taking this into account as shown by Cooper [72] is

$$\Delta = \hbar \omega_d \exp(-\underbrace{\frac{\epsilon_f \hbar \omega_d}{c^2}}_{\frac{1}{c'}}). \tag{4.25}$$

From above, $\Delta \sim 10\ meV$. In BCS theory, we have multiple electron wavefunction

$$\Psi = \prod_i (\sin\theta_i + \cos\theta_i \phi_i), \tag{4.26}$$

ϕ_i is in annulus of width ω_d at Fermi surface, with probability of state ϕ_i being occupied is $\cos^2\theta_i$. Optimal θ_i can be calculated and the ground state energy reduces energy by $-\frac{\omega_d}{4}$ per electron, in an annulus of width ω_d at Fermi surface. If a Cooper pair is broken , the cost of elementary excitation is $\sim \Delta$. Scattering of electrons by phonons will break a Cooper pair, leading to energy increase of $\sim \Delta$, which cannot be paid by energy of a low temperature phonon, hence no scattering, leading to absence of resistance.

In high T_c superconductors [73] we have d orbital electron waves (D-waves) , with wave dispersion $\epsilon_0 - 2t\cos(ka)$, where ϵ_0 is the orbital energy, k wavevector, a lattice parameter and t hopping parameter (transfer integral). t is small, as d orbitals are localized, leading to small fermi energies ϵ_f. Lower ϵ_f means larger c' and higher T_c. On the other hand, a metal like Lithium, has high t and hence high ϵ_f and lower c' and lower T_c.

Another interesting aspect of BCS theory is Josephson Junction Tunnelling [74]. When a voltage V is applied between two superconductors, separated by a weak link (insulator), a oscillating current $\propto \cos(\frac{2eVt}{\hbar})$ develops. This oscillation can be explained by a phase that develops between the superconductors and tunnelling of a Cooper pair. But, how will the electron pair with momentum k and -k simultaneously

tunnel. One is moving towards the tunnel junction while other away from it. They cannot tunnel simultaneously.

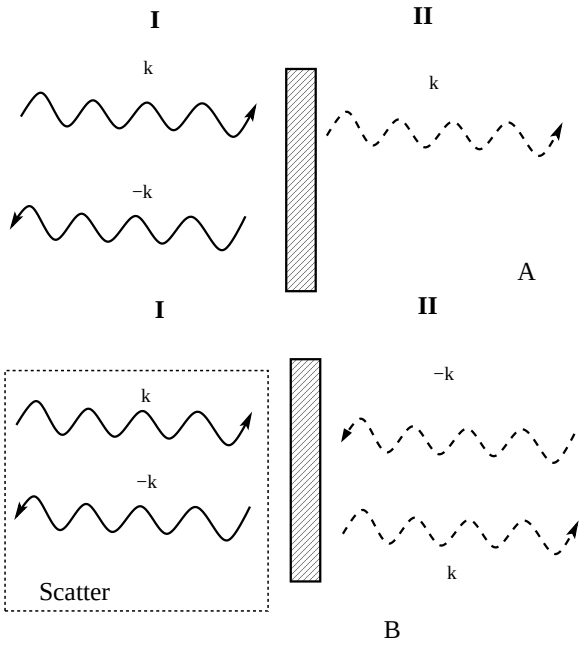


Fig. 4.12 Fig. A depicts how on a tunnel junction between two superconductors, the two electrons in a Cooper pair travel in opposite direction , they cannot simultaneously cross the barrier. Fig. B depicts, k is moving towards the barrier, $-k$ away, but they scatter of the lattice. k changes to $-k$ but by that time it has crossed the barrier. For the other electron $-k$ changes to k and now is travelling towards barrier and crosses it.

Shown in Fig. 4.12A is how on a tunnel junction between two superconductors, the two electrons in a Cooper pair travel in opposite direction , they cannot simultaneously cross the barrier. How do we explain simultaneous crossing. The answer is actually very subtle, k is moving towards the barrier, $-k$ away, but they scatter of the lattice. k changes to $-k$ but by that time it has crossed the barrier. For the other electron $-k$ changes to k and now is travelling towards barrier and crosses it. This is shown in Fig. 4.12B.

4.7.3 Resistance and resonant absorption of phonons

Fig. 4.13 shows how it is when we accelerate electrons with an electric field say in $-x$ direction. The whole Fermi sphere displaces to the right by a small amount. There is net momentum in the x direction and this constitutes the current. How does the

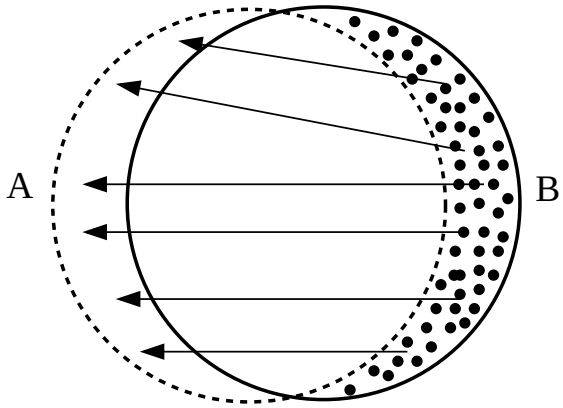


Fig. 4.13 Fig. shows how application of electric field accelerates electrons, shifts the Fermi sphere and how electrons back scatter by absorbing phonons.

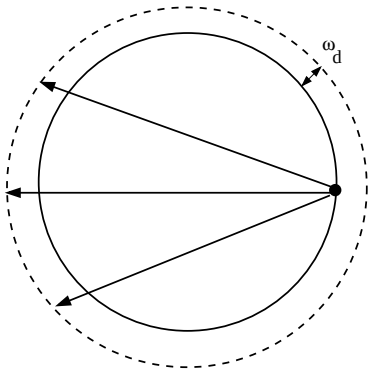


Fig. 4.14 Fig. shows how an electron scatters when it absorbs a phonon of energy $\hbar\omega_d$ and its kinetic energy slighly increases.

current stop. The electrons on the right shown as black dots in Fig. 4.13 are scattered to the left as shown. This scattering is due to absorption of phonons and annuls the forward x-momentum of electrons. How much is this scattering rate. If we absorb a phonon, the electron energy rises by $\hbar\omega_d$ and the electron scatters to states as shown in dotted sphere as shown in Fig. 4.14.

There are n^3 electron states in the Fermi sphere, and n^2 on the annulus , then the scattering rate by *Fermi Golden Rule*, is $\Gamma = \frac{\Omega^2 n^2}{\Delta E}$ where $\Omega = \frac{c}{\sqrt{n^3}}$, $\Delta E = \frac{\epsilon_f}{n}$ is the energy width of electron state (packet).

$$\Gamma = \frac{c^2}{\epsilon_f} \tag{4.27}$$

Taking $c = 1eV$ and $\epsilon_f = 10eV$, we get $\Gamma \sim 10^{14}/s$. This is in agreement with typical relaxation times of $10^{-14} - 10^{-15}$ sec.

4.7.4 Temperature dependence of resistivity and Bloch's T^5 law

In understanding temperature dependence of resistivity, we consider two limits. High temperature limit, when all phonon modes are occupied. Then number of phonons n_k in a mode satisfies $n_k\hbar\omega_k = k_BT$ with $\Omega = \frac{c}{\sqrt{n^3}}$ as in Eq. 4.22, $\Omega \propto \sqrt{n_k}$. Then observe $\Omega^2 \propto k_BT$ and we have $\Gamma \propto T$. Linear variation with T.

There is another regime, the low temperature regime in which only phonons with small wavevectors which satisfy $\hbar\omega = k_BT$ are active. This is as shown by vector OA in fig. 4.15.

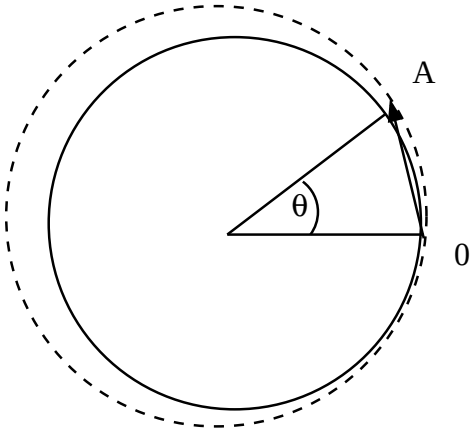


Fig. 4.15 Fig. shows at low temperatures, only the phonons with small wavevectors as OA are active.

The length of wavevector OA that is active $\propto T$ and hence the the surface area on the Fermi sphere that will be active due to phonon scattering is $\propto T^2$ and hence $D \propto T^2$. Since $\Omega^2 \propto T$, we get $\Gamma = \Omega^2 D \propto T^3$. However scattering by an angle θ as shown in fig. 4.15, impedes the current by a factor $1-\cos\theta$ which for small angle theta is $\propto \theta^2$ but from fig. 4.15, $\theta^2 \sim T^2$ or the resisitvity varies as $\propto T^5$. This is called the Bloch's T^5 law.

4.7.5 Scattering of impurity

Solid may have impurities, as in dopants in a semi-conductors. Then then potential is not strictly periodic as shown in Fig. 4.9. This can be thought of as coulomb potential of the impurity on top of a periodic potential. This impurity potential will back-scatter electrons as shown in 4.16, forward going k_f back-scattered to backward going k_i leading to resistance.

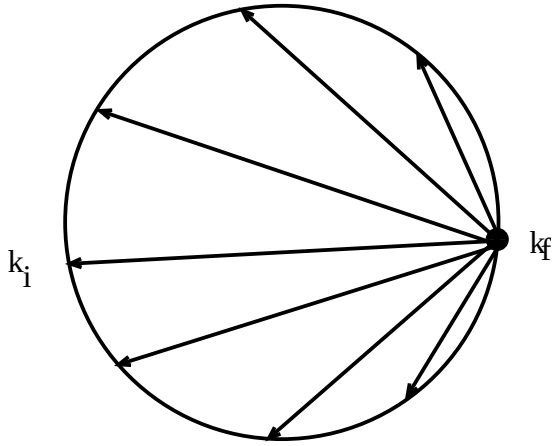


Fig. 4.16 Fig. A depicts how electron momentum k_f backscatters due to Coulomb potential.

4.8 Scattering of exchange potential

We talked of electron scattering of Coulomb potential, two electrons wavefunctions ϕ_1 and ϕ_2 have coulomba energy

$$U = e^2 \int \frac{|\phi_1(r_1)|^2 |\phi_2(r_2)|^2}{4\pi\epsilon_0 |r_1 - r_2|} d^3 r_1 d^3 r_2, \tag{4.28}$$

and exchange energy

$$U_{ex} = e^2 \int \frac{\phi_2^*(r_1)\phi_1^*(r_2)\phi_2(r_2)\phi_1(r_1)}{4\pi\epsilon_0 |r_1 - r_2|} d^3 r_1 d^3 r_2, \tag{4.29}$$

For electron waves with momentum k_1 and k_2 this exchange energy is

$$U_{ex} = e^2 \frac{1}{V^2} \int \frac{\exp(-i(k_1 - k_2)\cdot(r_1 - r_2))}{4\pi\epsilon_0 |r_1 - r_2|} d^3 r_1 d^3 r_2, \tag{4.30}$$

In CM and relative coordinate $R = \frac{r_1 + r_2}{2}$ and $r = r_1 - r_2$, we have for $K = k_1 - k_2$,

$$U_{ex} = e^2 \frac{1}{V} \int \frac{\exp(-iK \cdot r)}{4\pi\epsilon_0 r} d^3 r, \tag{4.31}$$

with $V = l^3$ and $|K| = \frac{n}{l}$ we have

$$U_{ex} = \frac{1}{n^2} \frac{e^2}{4\pi\epsilon_0 l} \tag{4.32}$$

Larger the K, smaller the exchange energy.

4.9 Fermi liquid theory

The electron waves in the solid occupy k states within the Fermi-sphere of radius k_f energy ϵ_f. At very low temperatures, all states below energy ϵ_f are filled and above ϵ_f empty as shown in fig. 4.17A. But due to exchange energy between electrons we have a picture more like fig. 4.17B. We seprate the electrons in k space so that exchange energy is minimized. This puts more electrons outside ϵ_f.

Now when we apply electric field in x direction, Fermi sphere moves to the right and creates crowded k space on the right as shown in fig. 4.18. The exchange energy can be minimized by scattering electrons to the left as shown in the fig. 4.18. This is electron electron scattering. This is principle source of resistance at low temperatures. The scattering amplitude $\propto \frac{1}{V}$ the volume of electron wave, but $\frac{1}{V} = \Delta k_L \Delta k_T^2$, ($\Delta k_L$ is in direction and Δk_T perpendicular to k of packet. But we have $\frac{\hbar^2 \Delta k_T^2}{2m} \sim kT$ and $\frac{\hbar k}{m} \hbar \Delta k_L \sim kT$, giving $\frac{1}{V} \propto T^2$, hence resistance increases $\propto T^2$ at low temperatures, where Fermi liquid theory is main source of resistance [75].

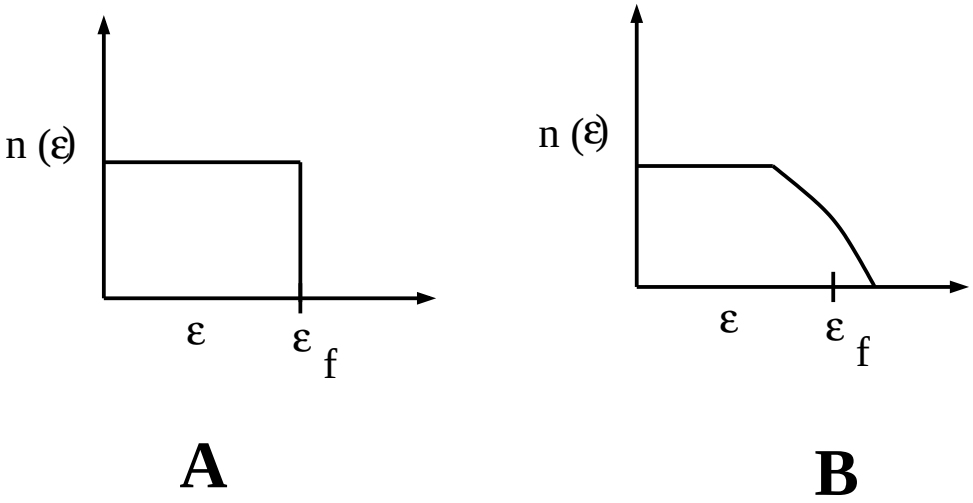


Fig. 4.17 Fig. A depicts density of states as function of energy for non-fermi liquid (without exchange interactions). Fig. A depicts density of states as function of energy for fermi liquid (with exchange interactions).

4.9.1 Exchange interactions and chemical reaction dynamics

Day to day collisions are exchange interactions of electrons. Collisions in chemical reactions are no exceptions [76]. Fig. 4.19 depicts molecules AB and CD moving along y direction with momentum $-p, p$ scatter due to exchange potential and form

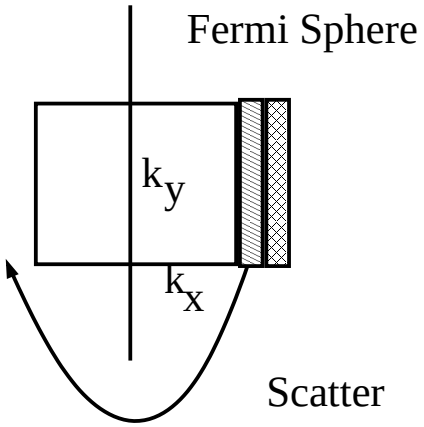


Fig. 4.18 Fig. A depicts displaced fermi sphere, due to exchange potential scattering, excess electrons on right backscatter.

new molecules AC and BD which moves sideways with momentum $-k, k$. The kinetic energy difference accounts for energy difference in bonds.

Momentum is exchanged by exchange interactions. The initial electronic configuration is

$$\phi_0 = \frac{1}{2}(\phi_A(r_1)\phi_B(r_2) + \phi_A(r_2)\phi_B(r_1))\ \ (\phi_C(r_3)\phi_D(r_4) + \phi_C(r_4)\phi_D(r_3)). \quad (4.33)$$

Final electronic configuration is

$$\phi_0' = \frac{1}{2}(\phi_A(r_1)\phi_C(r_3) + \phi_A(r_3)\phi_C(r_1))\ \ (\phi_B(r_2)\phi_D(r_4) + \phi_B(r_4)\phi_D(r_2)). \quad (4.34)$$

Overlap is just $\frac{1}{4}$.

4.10 Conclusion

In this paper, we balance the energy momentum budget in electron-nuclear collisions. Nucleus is much heavier than electron. The transferred momentum takes more energy out of electron than it gives to nucleus, this lost energy can be used for atomic excitation as in Frank Hertz effect, vapor lamps, for ionization of atoms as in cloud and Bubble chambers or as in X-ray production by Bremmstrulung. This extra energy can go into exciting internal modes of a proton as in deep inelastic scattering. We looked at elastic scattering of electrons as in electron diffraction and electron microscopes. We looked at scattering of electrons in condensed matter, these range

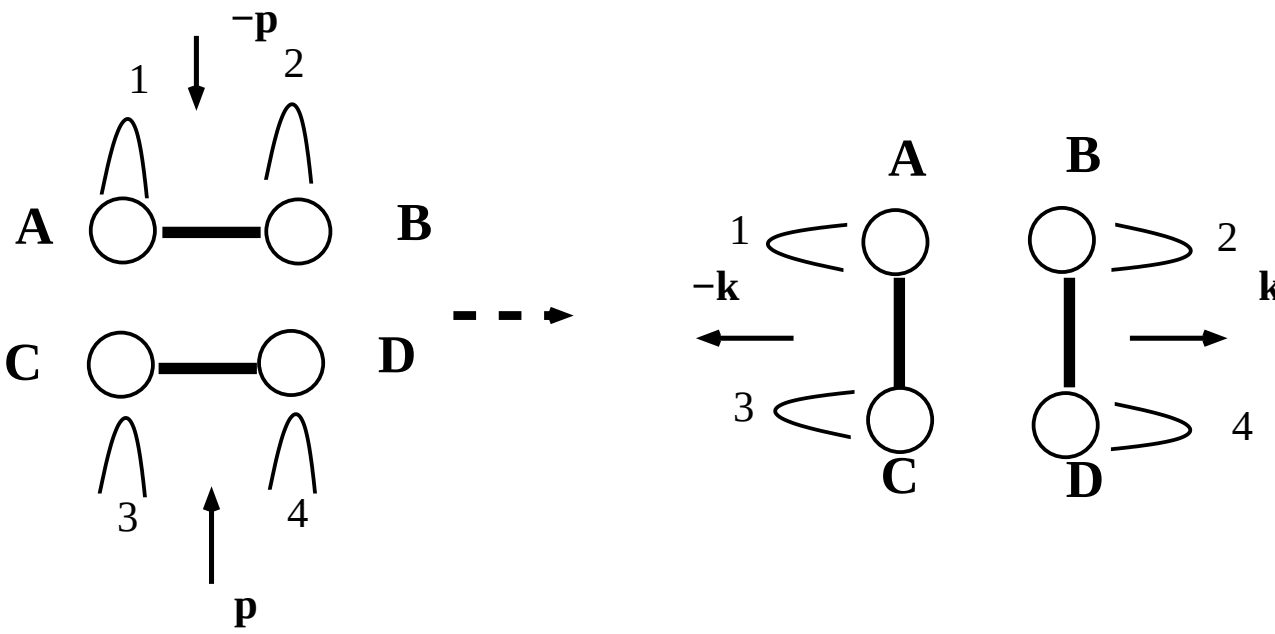


Fig. 4.19 Fig. depicts molecules AB and CD moving along y direction with momentum $-p, p$ scatter due to exchange potential and form new molecules AC and BD which moves sideways with momentum $-k, k$. The kinetic energy difference accounts for energy difference in bonds.

from scattering of electrons of periodic potential, to give Bloch waves, scattering of phonons and impurities to give resistance, scattering of lattice to give cooper pairs and superconductivity. We study electron scattering from exchange potential as in Fermi liquid theory and resulting T^2 resistance at low temperatures. Electron scattering of exchange potential resulting in chemical reactions. We stress again that our main contribution in this paper was to provide details at places where literature is succinct.

Problems

1. In Frank Hertz experiment electrons incident on Mercury atoms with kinetic energy 10 eV will come out with what energy, if the mercury is gets excited with energy difference between excited and ground state is $4.9eV$.
2. What is the outgoing energy of incident electrons with energy $50keV$ that scatters of an atom and radiates $1A^\circ$ wavelength light as Bremmstraulung.
3. For a lattice with spacing $a = 3$ A° find the minimum energy such that we get electron diffraction at angle $\theta = 30^\circ$.

4. Find the minimum energy of neutrons to get a neutron diffraction at an angle $\theta = 30°$.

5. If crystal is though of as a square well with potential $U = -5$ eV. What is the angle and wavelength of refracted electrons when they are incident at an angle 30° with the normal with energy 10 eV.

Chapter 5
Lattices as struck by light, electrons and molecules

What happens when we shine light or fire electrons on a solid material ? Few things can happen. If material is colored, it will just absorb light of right frequency and spontaneously emit it in all the directions. It will of-course back scatter light, which will be missing the color, it absorbed and we will see the complimentary color. If material is metallic, with free electrons, it will elastically scatter light back, and we will get reflection of light. Plastic, wood, walls, plants, humans will all scatter light and scattering amplitude will depend on how close we are to optical resonance; rf, microwaves will easily go through and light will be stopped. There is of-course some light scattered back. This kind of scattering is not of the free electrons, it is more like Rayleigh scattering, where light elastically scatters of an atom. Free electrons can also scatter in-elastically energetic waves like X-rays and that is Compton scattering. If the light has frequency of the bandgap of solid, then it will promote electron from valence to conduction band as in CCD sensors and cameras or photodetectors. Of-course, for materials like glass, the bandgap is very large, so elastic scattering is small and it is transparent. Infrared light of-course talks to phonons and is absorbed heating the solid. Phonons can also be excited with optical light by a Raman process, which results in inelastic scattering of light. Light can also eject electrons from a metal by phenomenon of photoelectric effect. So the light basically sees three things inside a solid material, free electrons, bonded electrons and bonded atoms and scatters of them. At high light intensity, light photon (frequency 2ω) can be absorbed and subsequently emitted as two photons (frequency ω), a phenomenon called down conversion, or two photons (frequency ω) can be absorbed and a photon of frequency 2ω be emitted, a phenomenon termed, higher harmonic generation. These are nonlinear optical processes.

Lets look at how it is when we fire electrons at solid material. Electrons can elastically scatter of the crystal giving rise to diffraction, also seen in neutron and x-rays. Energetic electrons can of-course ionize-excite atoms by Frank-Hertz phenomenon and subsequently emit light also termed scintillation or electrons can exchange momentum with atoms losing energy which they give out as radiation (say X-rays) called Bremmstraulung. Scintillation and ionization forms the basis of many high energy physics detectors and for equipment like cloud and bubble chambers or photographic

emulsions. Energetic Mev rays, α rays (Helium atoms), β rays (electrons) and γ rays can penetrate matter, ionize it lose energy and then stop. Heavier, the ray more momentum it will lose for given ionization energy and hence α rays are less penetrating than β which is less than γ. Neutrinos of course hardly interact and pass through the earth. Surprisingly muons travel farther than electrons, as they lose less energy per unit of transferred momentum. The energy of transferred momentum is just died by heavy metal like lead making them efficient stopping materials. In this paper we provide a quantitative analysis of these phenomenon. Our main contribution in this article is we are detailed at places, we find literature terse.

5.1 Introduction

We have to start with an atom and diatomic molecule. Atom can be struck with photon or electron, which can excite it to higher energy state. The excitation can be direct or be indirect which we call a Raman process. So there are four cases.

Diatomic molecule vibrates, when struck with light or electron it can be vibrationally excited. The excitation can be direct as by infrared light or a Raman process where a higher electronic state is excited and then we go back to higher vibration state but ground electronic state. Again there are four cases. What happens in atom or molecule happens in a crystal. We study the crystal cases in conjunction with atom and molecule to delineate the analogy.

Light can elastically or in-elastically scatter of atom and so off crystal. We delve in these cases one by one.

5.2 Light Atom Interactions, the $\mathcal{E}\cdot x$ gauge

5.2.1 Lagrangian and Hamiltonian

Take a classical electron, with coordinates $(x,y,z) = (x_1,x_2,x_3)$. Its Lagrangian in the electromagnetic field is

$$L = \frac{m}{2}\sum_i \dot{x_i}^2 + q\sum_i A_i\dot{x_i} - qA_0, \tag{5.1}$$

where q and m are electron charge and mass. $\mathcal{A}$ and V are vector and scalar potentials.

The Euler Lagrange equations are the familiar Lorentz force law $m\dot{v} = q(E + v \times B)$, where v is the velocity vector, $E_i = -\frac{\partial A_i}{\partial t} - \frac{\partial V}{\partial x_i}$, $B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}$, the electric and magnetic fields.

The momentum $p_i = \frac{\partial L}{\partial x_i}$ and the Hamiltonian of the system $H = p_i\frac{\partial}{\partial \dot{x}_i} - L$ is

$$H=\sum_{j=x,y,z}\frac{(p_j-qA_j)^2}{2m}+qA_0. \tag{5.2}$$

5.2.2 Dirac and Schrödinger Equation

The Electron Schrödinger Equation [36] is

$$i\frac{\partial\psi}{\partial t}=\left(\sum_{j=x,y,z}\frac{(-i\hbar\frac{\partial}{\partial x_j}-qA_j)^2}{2m}+qA_0\right)\psi, \tag{5.3}$$

where ψ is electron wave-function. This equation is not very tractable, because it is nonlinear in $\mathcal{A}$, lets write a linear equation, which is the Dirac equation [63], which takes the form

$$i\frac{\partial\phi}{\partial t}=\left(\sum_{j=x,y,z}c(-i\,\hbar\frac{\partial}{\partial x_j}-qA_j)\alpha_j+\beta mc^2+qA_0\right)\phi. \tag{5.4}$$

where $\alpha_j=\sigma_z\otimes\sigma_j$ and $\beta=\sigma_x\otimes\mathbf{1}$ are Dirac matrices, where σ_j are the Pauli matrices, $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. ϕ is electron spinor, for a electron wave with momentum k, takes the form $\phi=\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}\otimes\uparrow$, where $\uparrow$ is spin up, $\cos\theta=\frac{\hbar k}{mc}=\frac{\upsilon}{c}$, where $\upsilon=\frac{\hbar k}{m}$, is electron wave group velocity. Electron Orbitals are of size $\sim A^\circ$, their $k\sim 10^{10}m$, then $\upsilon\sim 10^6 m/s$ and $\cos\theta\sim\frac{10^6}{3\times10^8}\sim 10^{-3}$. Electron is non-relativistic, $\cos\theta=\frac{\upsilon}{c}\sim 0$, $\theta\sim\frac{\pi}{2}$, $\phi=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\otimes\uparrow$.

To fix ideas, take incoming EM wave, along z direction, with electric field $E_x\sin(kz-\omega t)$, the Lorentz gauge is $(A_0,A_x,A_y,A_z)=\frac{E_x}{\omega}\cos(kz-\omega t)(0,1,0,0)$. Electron wave with momentum q absorbs the photon with momentum k, and transits to momentum $q+k$. The $\mathcal{A}$ process transition is driven by Dirac matrix α_x, with transition amplitude

$$\mathcal{M}=\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}\otimes\uparrow\underbrace{(\sigma_z\otimes\sigma_x)}_{\alpha_x}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}\otimes\downarrow=qcA_x\frac{\upsilon}{c}=qE_x\frac{\upsilon}{\omega} \tag{5.5}$$

If we have electron orbital ϕ_0 then $k'=\frac{M}{M+m}k$ of photon momentum goes to electron-nuclear relative coordinate, while $k''=k$ momentum to CM (center of mass), where M is nucleus mass. The $\mathcal{A}$ process drives the transition

$$\phi_0\uparrow\;\longrightarrow\;\exp(ik'z)\phi_0\downarrow,$$

with amplitude $\mathcal{M}=qE_x\frac{\upsilon}{\omega}$.

When orbital ϕ_1 is different from ϕ_0 we go to

$$\phi_0\uparrow \longrightarrow \exp(ik'z)\phi_0\downarrow,$$

with amplitude $\mathcal{M} = qcA_x\frac{v}{c}$ whose overlap with ϕ_1 is

$$\mathcal{M}_1 = qcA_x\frac{v}{c}ik'\underbrace{\langle\phi_1|z|\phi_0\rangle}_{d_z} = iqE_xd_z\,\frac{v}{c},$$

where $ck' \sim \omega$.

But this is not suited for study of optical transitions, because we do-not recover the Rabi frequency qE_xd. What we find is orders of magnitude smaller (down by $\frac{v}{c}$). Instead we work with gauge

$$(A_0, A_x, A_y, A_z) = \frac{-E_x}{2}(x\ \sin(kz-\omega t), -\frac{\cos(kz-\omega t)}{\omega}, 0, \frac{x}{c}\sin(kz-\omega t)).$$

Now we have an $\mathcal{E}$ process driven by x term. For the $\mathcal{E}$ process, the amplitude of $\phi_0 \to \phi_0$ is just 0, as $\langle\phi_0|x|\phi_0\rangle = 0$ and the amplitude of $\phi_0 \to \phi_1$ is simply

$$\mathcal{M}_1' = qE_x\underbrace{\langle\phi_1|x|\phi_0\rangle}_{d_x} = qE_xd_x,$$

Dipole elements d_z, d_x are approx, Bohr radius $\sim A^\circ$. Due to the factor $\frac{v}{c} \sim 10^{-3}$, $\mathcal{M}_1 \ll \mathcal{M}_1'$. Therefore transition between different atomic orbitals are largely driven by the $\mathcal{E}$ process.

$\mathcal{A}$ process flips electron spin, $\mathcal{E}$ doesn't. When ϕ_1 and ϕ_0 have different angular momentum $\mathcal{A}$ process cannot drive the transition, but $\mathcal{E}$ process can, with a circularly polarized light and

$$\mathcal{M}_1' = qE\underbrace{\langle\phi_1|r|\phi_0\rangle}_{d} = qEd,$$

Since $\mathcal{A}$ process always flips spin, it is necessarily done with circularly polarized light.

5.3 The Crystal Case and Photo-detector

In atom, we have electron as orbitals, ϕ_0 and ϕ_1 and we can transit between the two with light wave with wave-vector k . In Crystal, these orbitals are banded. We have $\phi_0 \to \sum_l x_l\phi_0^l = \Phi_0(p)$, where $x_l = \exp(jpla)$, where p is wave-vector and a spacing between atoms and l the atom index. ϕ_0^l transits to ϕ_1^l with amplitude $\exp(jkx_0)\mathcal{M}'$, then all in all $\Phi_0(p)$ transits to $\Phi_1(p+k)$ when $\Delta E = E_1 - E_0 = \hbar ck$. Φ_0 is in valence

band and Φ_1 in conduction band and we get a nice photo-detector where a photon promotes an electron to conduction band, where it gives a current. $\Delta E \sim eV$ in a semiconductor crystal.

5.4 Frank Hertz Experiment

In 1914, James Franck and Gustav Hertz [52, 53] performed a beautiful experiment. They accelerated electrons in a cathode ray tube filled with mercury. They observed that at certain value of the accelerating voltage, the current in the tube dropped. This demonstrated the existence of excited states in the mercury atoms, with quantized energies, and helped to confirm the quantum theory. The values of accelerating voltage, where the current dropped gave a measure of the energy necessary to force an electron to an excited state.

Let x_1, x_2 be coordinates of atomic electron and nucleus, and $X_1 = \frac{mx_1+Mx_2}{m+M}$ and $X_2 = x_1 - x_2$, center of mass and relative, coordinate. Let k_1, k_2 be momentum of x_1, x_2 coordinates and K_1, K_2 momentum X_1, X_2 coordinates. Then $\sum k_i x_i = \sum K_i X_i$ gives $K_1 = k_1 + k_2$ and $K_2 = \mu\left(\frac{k_1}{m} - \frac{k_2}{M}\right)$, where $\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}$ is the reduced mass.

When q momentum transfers from incident electron to the atomic electron, it can change the atomic wave-function

$$\exp(iqx_1) = \exp(iq'X_1)\exp(iq''X_2),$$

where $q' = q$ and $q'' = \frac{M}{M+m}q \sim q$.

$$\exp(iq''X_2) = \cos(q''(x_1 - x_2)) + i\sin(q''(x_1 - x_2)).$$

$$d_e = \langle\phi_1|\sin(q''(x_1 - x_2))|\phi_0\rangle$$

is the dipole moment of the transition, from ground to excited state of the atom. When incident atom stops and

$$E_1 - E_0 = \frac{\hbar^2 q^2}{2m},$$

we see the atomic transition from E_0 to E_1.

More generally, $q = p_1 - p_2$ (p_1, p_2 initial and final momentum of incident electron) and

$$E_1 - E_0 = \frac{\hbar^2}{2m}(p_1^2 - p_2^2), \tag{5.6}$$

Transition amplitude for transfer of momentum

$$\mathcal{M} = \frac{e^2}{V\epsilon_0 q^2} \tag{5.7}$$

where $V = l^3$ is volume of incident electron, where $l \sim 10^{-8}$, thermal debroglie wavelength.

Total transition amplitude is

$$\mathcal{M}' = \mathcal{M} d_e \tag{5.8}$$

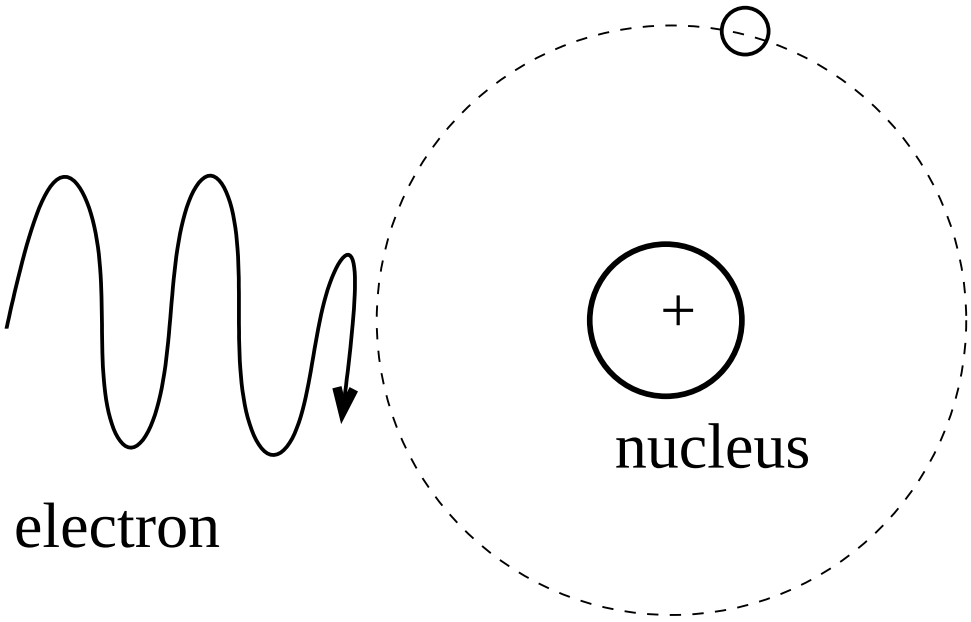


Fig. 5.1 Fig. A depicts Frank Hertz experiment and incident electron makes a atomic transition.

When q momentum transfers from incident electron to the atomic nucleus, it can change the atomic wavefunction

$$\exp(iqx_2) = \exp(iq'X_1)\exp(iq''X_2),$$

where $q' = q$ and $q'' = \frac{m}{M+m}q$.

$$d_n = \langle\phi_1|\sin(q''(x_1 - x_2))|\phi_0\rangle$$

is the dipole moment of the transition, from ground to excited state of the atom, which is negligible due to small q''. Of-course, transferred momentum went to center of mass (CM) of atomic system.

Frank Hertz phenomenon is ubiquitous in vapor lamps like sodium, neon, mercury vapor lamps, where free electrons are accelerated and bombard atoms exciting them,

which subsequently emit light we see. It is the basis of phenomenon of Cathodo-luminescence in which electrons impacting on a luminescent material such as a phosphor, cause the emission of photons which may have wavelengths in the visible spectrum. A familiar example is the generation of light by an electron beam scanning the phosphor-coated inner surface of the screen of a television that uses a cathode ray tube.

5.5 Crystal Case: Scintillators

Scintillator crystals convert electron beams and ionizing radiation X-rays, γ rays into photons that can be detected by photo-detectors. With electron beam, we just do Frank hertz on a crystal and we are done. We have shown in for atom, for a collection of atoms we have already shown the idea. Electron hole pairs are created whose energy is eV range and this subtracts from energy of the electron beam. With light its more subtle, because we have to slow change lights wavelength which is done by a Raman effect,

5.6 Scintillators with ionizing radiation: Raman effect

Electron momentum changes as it scatters an exchange momentum , but with an incoming photon, how does the photon momentum change, it does by a Raman effect. The photon is absorbed and re-emitted with slightly lower wavenumber and the energy loss goes in exciting the atom. This is called in-elastic scattering of electron. We first quickly discuss elastic and in-elastic scattering of photon. Elastic scattering is called Rayleigh scattering and in-elastic Raman.

5.6.1 Rayleigh and Raman Scattering

Atomic transitions are first order, we are on resonance to energy difference between energy levels. Now we discuss some second order processes like Rayleigh [39] and Raman Scattering [40, 41]. Rayleigh is elastic scattering of light, Raman inelastic. In Rayleigh scattering, light of wavevector k is scattered by atom to light of wavevector k_1 such that $|k| = |k_1|$, only direction of light changes. In Raman scattering, light of wavevector k is scattered by atom to light of wavevector k_1 such that $|k| \neq |k_1|$. The difference of the photon energy $\Delta\omega = \hbar c(|k| - |k_1|)$, results in atomic transition with $\Delta E = \hbar\omega$. When $\Delta E > 0$ its Stokes process. When $\Delta E < 0$ its anti-Stokes process.

Calculation of transition amplitude for these processes, involves a second order calculation manifested in a three level system. This is as shown in Figure 5.9. There are three levels $|1\rangle$, $|2\rangle$ and $|3\rangle$ with energies E_1, E_2, E_3, with $E_1 = E_3$, levels 1 and 3

are degenerate. Ω_1 and Ω_2 are transition amplitudes between level 1 and 2 and level 2 and 3 respectively.

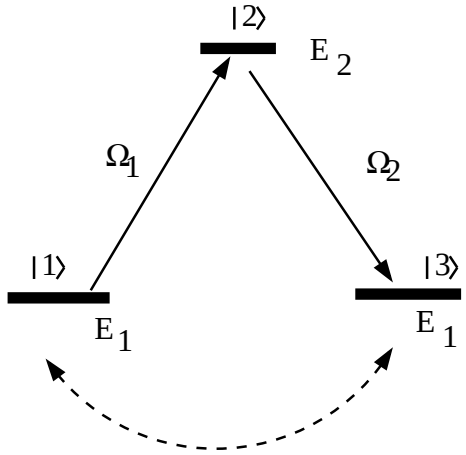


Fig. 5.2 Fig. depicts a 3 level system with transition amplitude Ω_1 and Ω_2 between level 1 and 2 and level 2 and 3 respectively.

Then gives for $|E_1 - E_2| \gg \Omega_i$, there is transition amplitude of going from $|1\rangle$ to $|3\rangle$, given by,

$$\mathcal{M} = \frac{\Omega_1 \Omega_2}{E_1 - E_2} \tag{5.9}$$

Fig 5.3A depicts Rayleigh scattering, incoming photon at wave-vector k scatters to wave-vector k_1. Fig B, depicts a three level 3 level Rayleigh process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to ϕ

In 3 level system analogy, State $|1\rangle$ is electron-nuclear, photon state ϕ, k and state $|2\rangle$ is electron-nuclear state Φ and State $|3\rangle$ is electron-nuclear, photon state ϕ, k'. $|k| = |k'|$, so that $E_1 = E_3$. Ω_1 is a $\mathcal{A}$ process with value

$$\Omega_1 = qE_x \frac{\upsilon}{\omega}, \tag{5.10}$$

where E_x is electric field of incoming EM wave.

$$\Omega_2 = qE_o \frac{\upsilon}{\omega}, \tag{5.11}$$

where E_0 is the electric field of spontaneously emitted photon at k_1. How much is E_0.

Spontaneous emission time Δt, in atomic systems is $\Delta t \sim 10's$ ns, corresponding to a Rabi frequency of $\Omega = 100$ MHz, which is $10^{-26}J$, in which photon travels $\Delta l = c\Delta t \sim 1m$ (leaves the atom). The volume of the photon $\sim 1m^3$. Bandwidth

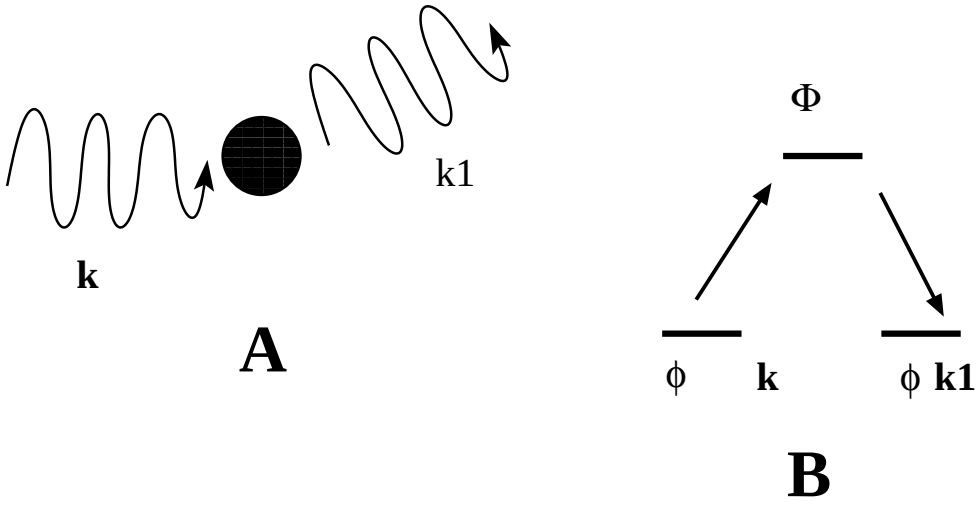


Fig. 5.3 Fig. A, depicts Rayleigh scattering, incoming photon at wave-vector k scatters to wave-vector k_1. Fig B, depicts a three level 3 level Rayleigh process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to ϕ_0

of the photon $\Delta B = \sim \frac{2\pi}{\Delta l}$, how many photons are there in all the directions? Then wave-vector sphere of radius $k_0 = \frac{2\pi}{\lambda_0}$ (say 300 nm photon, $\lambda_0 = 300nm$). Number of direction $n = 4\pi k_0^2/(\Delta B)^2 \sim 10^{12}$.

Rabi frequency $\Omega = q\sqrt{n}E_0 d = 10^{-26}$, giving $E_0 = mV/m$. Ofcourse, energy of the photon $\hbar\omega = \frac{\epsilon_0}{2}E_0^2 V$ with $V \sim 1m^3$, we have for $\omega \sim 10^{15}$, $E_0 \sim mV/m$ (all self consistent).

Coming back to Rayleigh scattering, from Eq. 5.27, 5.10, 5.11, we have

$$\mathcal{M} = \frac{q^2 E_x E_0}{\hbar}\frac{\upsilon^2}{\omega^3}. \tag{5.12}$$

or Rayleigh Rabi frequency,

$$\Omega_{Rayleigh} = \frac{q^2 E_x E_0}{\hbar^2}\frac{\upsilon^2}{\omega^3}. \tag{5.13}$$

for $E_x \sim 10^3 V/m$, $\omega = 10^{15}$, we get $\Omega_{Rayleigh} \sim 10^{-4}$ Hz.

Finally Ω_1 and Ω_2 both $\mathcal{A}$ process.

5.6.2 Raman Scattering

Raman scattering is an inelastic scattering process. Photon as wave-number k is absorbed by atom and k_1 emitted. Energy of absorbed and emitted photon is not same $k \neq k_1$. Deficit goes in exciting atomic transition.

Fig 5.4A depicts Raman scattering, incoming photon at wave-vector k scatters to wave-vector k_1. Atom is excited from initial atomic level ϕ to ϕ_1. Fig B, depicts a three level Raman process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to new atomic level ϕ_1.

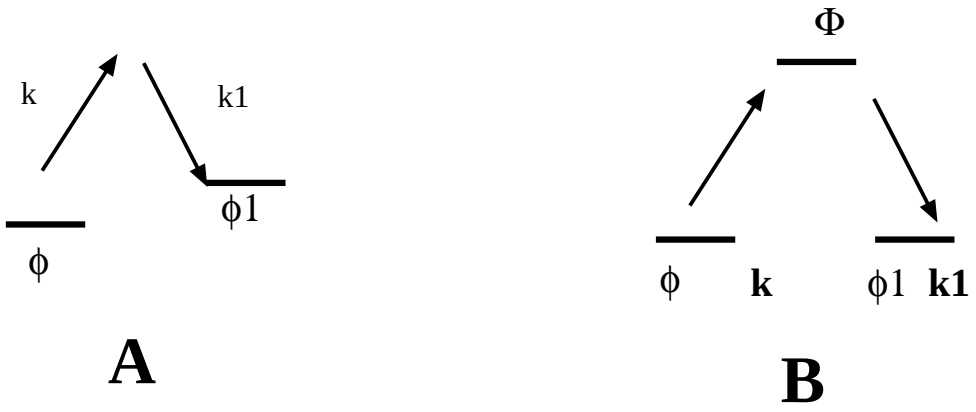


Fig. 5.4 Fig. A, depicts Raman scattering, incoming photon at wave-vector k scatters to wavevector k_1. Fig B, depicts a three level 3 level Rayleigh process, where initial atomic level ϕ absorbs photon k and moves to Φ where CM coordinate gets photon momentum. k_1 is emitted and we return to ϕ_0

In 3 level system analogy, State $|1\rangle$ is electron-nuclear, photon state ϕ, k and state $|2\rangle$ is electron-nuclear state Φ and State $|3\rangle$ is electron-nuclear, photon state ϕ_1, k_1. $|k| \neq |k_1|$, but $E_1 = E_3$. Ω_1 is a $\mathcal{A}$ process with value

$$\Omega_1 = qE_x \frac{\upsilon}{\omega}, \tag{5.14}$$

where E_x is electric field of incoming EM wave. Ω_2 is a $\mathcal{E}$ process (atomic levels change) with value

$$\Omega_2 = qE_o d, \tag{5.15}$$

where E_0 is the electric field of spontaneously emitted photon at k_1.

Coming back to Rayleigh scattering, from Eq. 5.27, 5.14, 5.15, we have

$$\mathcal{M} = \frac{q^2 E_x E_0 d}{\hbar} \frac{\upsilon}{\omega^2}. \tag{5.16}$$

or Raman Rabi frequency,

$$\Omega_{Raman} = \frac{q^2 E_x E_0 d}{\hbar^2} \frac{\upsilon}{\omega^2}. \tag{5.17}$$

for $E_x \sim 10^3 V/m$, $\omega = 10^{15}$, we get $\Omega_{Raman} \sim 10^{-5}$ Hz.

Raman is a $\mathcal{A}$ and $\mathcal{E}$ process, Ω_1 is $\mathcal{A}$ and Ω_2 is $\mathcal{E}$. Ω_2 can be $\mathcal{A}$ but the amplitude is much smaller than.

We have been talking of ϕ and ϕ_1 as atomic levels, which is just fine in principle. In practice, these are vibrational levels of two nuclei making a molecule. Then these are energy levels of their (two nuclei) relative coordinate.

5.7 World of Colors

We are surrounded by beautiful colors. Really beautiful. Our clothes have beautiful colors on them. Plants are green, blood is red. What is the source of this color. Clothes have dyes containing pigments, like paints have pigments. These pigments are primarily inorganic in the sense, they have a transition metal element in them. Transition metal elements are the one found in the center of the periodic table that have electrons in their d-orbitals. These include, for example, Cobalt (Co) , Cadmium (Cd), Chromium (Cr), Manganese (Mn) etc. For example, Cobalt (atomic number 27), has electronic configuration $1s^2 2s^2 2p^6 3s^2 3p^6 3d^7 4s^2$. The d-orbitals are five fold degenerate. These orbitals are $d_{z^2}, d_{x^2-y^2}, d_{xy}, d_{yz}, d_{xz}$. However in a transition metal compound, binding with other atoms called ligands, this degeneracy gets broken. We have orbitals $d_{z^2}, d_{x^2-y^2}$ called e_g manifold at higher energy than the orbitals d_{xy}, d_{yz}, d_{xz} called t_{2g} manifold as shown below

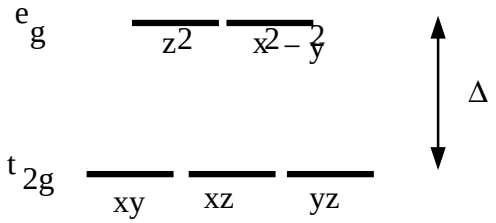


Fig. 5.5 Fig. shows splitting of energy of d-orbitals in e_g manifold and t_{2g} manifold.

The energy difference $\Delta = \hbar\omega_0$ is sub-eV and corresponds to visible wavelength. When we shine light the right color is absorbed, rest scattered back, which we see. We see complimentary colors.

5.7.1 Scattering and Blue Sky

Why is sky Blue? Blue light is more detuned optical resonance in atmospheric molecules, and hence scatters less and reaches us (from sun above us) , red light gets back scattered as less detuned. At evening time we mainly see scattered light (sun on the side) hence red light.

5.8 Metallic reflection, elastic scattering of light

Light bounces back off polished mirrors, metals. We can think of this as elastic scattering (Rayleigh) of light from Free electron. If light travels in z direction with electric field along x then the electron momentum changes along x by q and there is transition element $x\sin(qx)$, electron of-course transits back emitting light backwards and this is the reflection from a mirror.

5.9 Compton effect

As wavelength of light decreases, and reaches Compton wavelength $\frac{h}{m_0 c}$, (m_0 electron mass), the light can ineleastically scatter of a free electron and change wavelength.

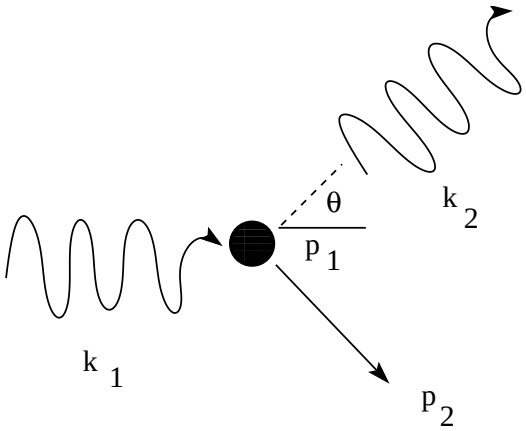


Fig. 5.6 Fig. shows Compton scattering. Light with momentum k_1 hits electron with momentum p_1, with light scattering to k_2 and electron to p_2.

Fig. 6.9 shows Light with momentum k_1 hits electron with momentum p_1, with light scattering to k_2 and electron to p_2 such that momentum and energy are conserved, i.e., $k_1 + p_1 = k_2 + p_2$ and $E_{p_1} + E_{k_1} = E_{p_2} + E_{k_2}$, which gives

$$\lambda_2 - \lambda_1 = \frac{\hbar}{m_0 c}(1 - \cos\theta) \tag{5.18}$$

where $\lambda_c = \frac{\hbar}{m_0 c} \sim .01A^\circ$ is Compton wavelength (m_0 mass of electron). To see a change in wavelength comparable to Compton wavelength , we should have short wavelength light (high energy, 10-100 Kev, X-rays).

5.10 Photelectric effect

When light of frequency ω is shown on metal surface [49], electrons are ejected. This is photoelectric effect. The electron energy is $E_o = -V_0 + \frac{\hbar^2 k_o^2}{2m} < 0$, where k_0 is electron momentum and $-V_0$ is crystal potential. Momentum of electron changes to k_F and its energy is $E_F = -V_0 + \frac{\hbar^2 k_F^2}{2m} > 0$ (electron is ejected). Change of momentum $\Delta k = k_F - k_0$ accounts for increase of electron energy by $\hbar \upsilon_F \Delta k$ where υ_F is Fermi velocity of electron $\sim 10^5 - 10^6 m/s$. This energy difference is given by photon carrying $\hbar\omega \sim 3 - 5eV$ of energy giving $\Delta k \sim 10^{10}/m$.
This momentum doesn't come from photon which will then carry $\hbar c \Delta k$ energy, which is very large (keV), it comes from phonons in metal, whose energy is $\hbar \upsilon \Delta k$, where $\upsilon \sim 10^3 m/s$ is velocity of sound in metal. Phonon energy is them meV which is negligible. If light travels in z direction with electric field along x then the electron momentum changes along x by $2q$ and there is tansition element $x \sin(2qx)$, electron ofcourse transits back by q giving $-q$ to phonon and conserving energy momentum.

5.11 Elastic Scattering of Electrons

We talked about inelastic scattering electrons. But electrons can elastically scatter from nuclei, change direction, maintaining their kinetic energy.

5.11.1 Electron and Neutron diffraction

Electron diffraction experiments were carried out by Davisson and Germer in 1927 [58]. Experiments showed wave nature of the electron. The electrons fired at an angle to the crystal rebounded at that angle (as classical ball) more so for certain choice of angles. Fig. 5.7A depicts how electrons and neutrons rebound of crystal at certain choice of θ.

Coulomb potential of nucleus

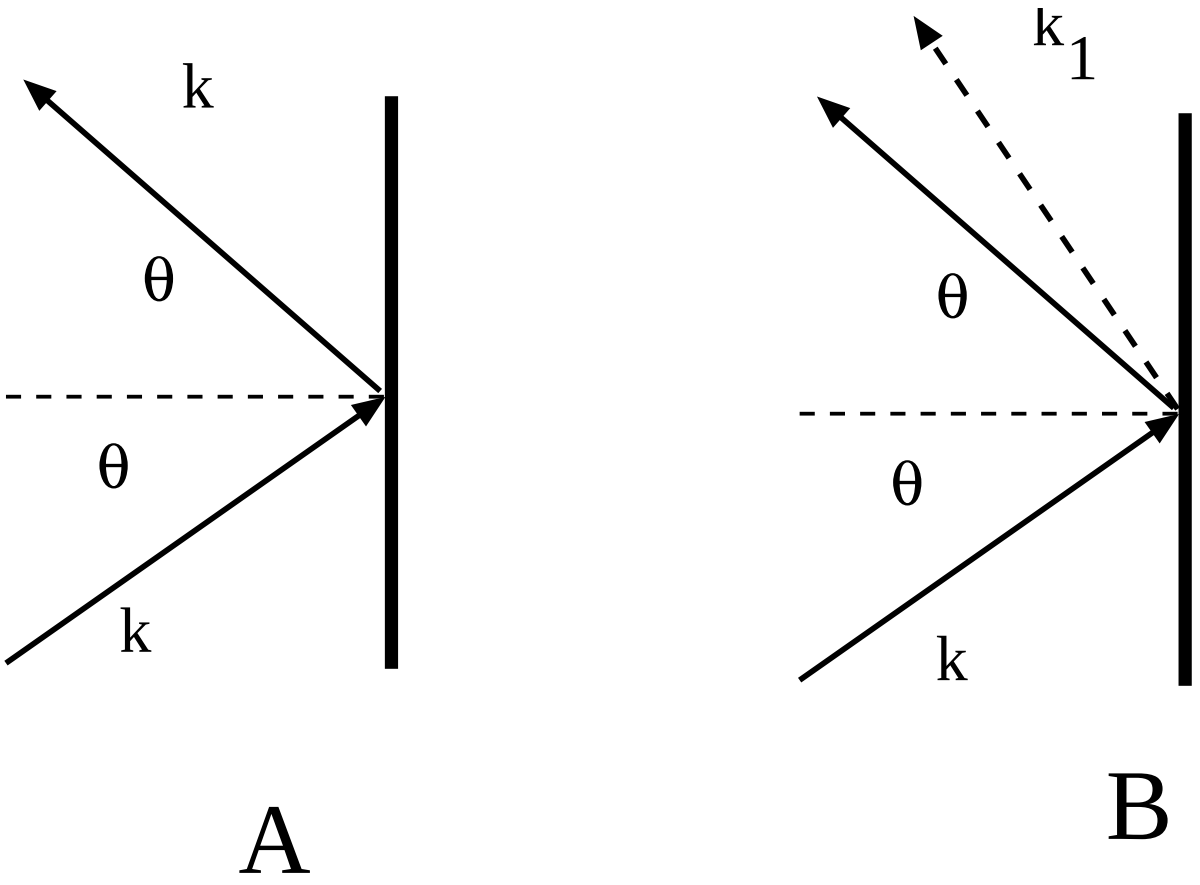


Fig. 5.7 Fig. A depicts how electrons and neutrons rebound of crystal at certain choice of θ. Fig. B depicts inelastic scattering of neutrons to measure phonon spectra.

$$U(x) = \sum_q \frac{1}{V\epsilon_0 q^2} \exp(iq \cdot x) \tag{5.19}$$

where $1/V = (\Delta q)^3$ (Δq is the step size of q discretization).

The electron wave $exp(ik \cdot x)$ scatters of it to $\exp(ik' \cdot x)$, where $k' = k + q$. Summing amplitude of various sites we get

$$\exp(ik \cdot x) \rightarrow \exp(ik \cdot x) \sum_{x_0} \exp(iq \cdot (x - x_0)) = \exp(ik' \cdot x) \sum_{x_0} \exp(-iq \cdot x_0)$$

For coherent addition of the amplitude $\mathcal{M} = \sum_{x_0} \exp(-iq \cdot x_0)$, we have

$$2k \sin\theta \; d = 2\pi \longrightarrow 2d \sin\theta = \lambda. \tag{5.20}$$

where d lattice spacing and $\lambda = \frac{2\pi}{k}$, wavelength of light.

For solid crystal, $d \sim A^\circ$, giving $\lambda \sim A^\circ$ or $k \sim 10^{10}/m$, which is $10eV$ of energy. Electron accelerated with this much energy bombard the crystal. We get returns at angle θ satisfying Lau condition Eq. 5.20.

Diffraction can also be done with neutrons [60], which are 1000 times heavier and hence for the same wavelength 1000 times less energetic, i.e., 10's of meV.

5.12 Electron Microscopes

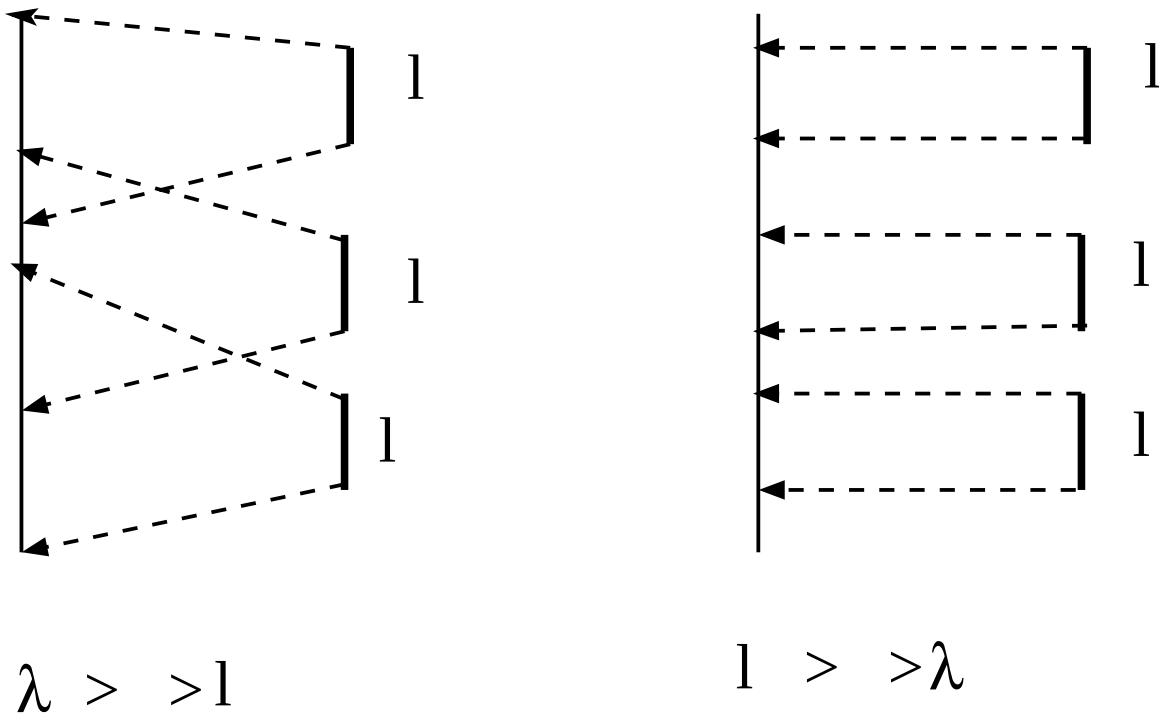


Fig. 5.8 Fig. depicts how light or electron when bounced of a solid spreads away, if wavelength $\lambda >> l$ and how it comes back straight if $\lambda << l$, where l is object length.

Fig. 5.8 depicts how light or electron when bounced of a solid spreads away, if wavelength $\lambda >> l$ and how it comes back straight if $\lambda << l$, where l is object length. This is diffraction limit.

To see smaller and smaller objects we need to use smaller wavelengths. Electrons are naturally small, with wavelength

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}}, \tag{5.21}$$

where v and E are velocity and energy of electrons. At electron energies 200 keV, we have $\lambda \sim$ pico-meter.

In scanning electron microscope, electrons elastically scatter and collecting the scattered electrons we can form an image [59].

5.12.1 Inelastic scattering of neutrons: Phonon spectroscopy

In elastic scattering, neutrons change direction and transfer momentum to nucleus (lattice, which is very big). The energy gained by lattice is negligible (M is huge). But we can excite vibration modes of lattice when we transfer momentum. When mode with momentum k' is excited, it has energy $E(k')$, then in Fig. 5.7B, we have

$$k' = k_1 - k \tag{5.22}$$

$$E(k') = \frac{h^2 k_1^2}{2m_0} - \frac{h^2 k^2}{2m_0} \tag{5.23}$$

This way we can obtain $k', E(k')$ plot , the phonon-dispersion relation. This is inelastic scattering of neutrons, [61].

5.13 Bremsstraulung

Transfer of momentum from electron to atom creates energy deficit, (atom is much heavier than electron). In Frank Hertz this imbalance is paid by exciting the atom. In Bremsstraulung [54], this is paid by free electron radiating. When energy of free electron is in KV range, we produce X-rays with energy in this range. Bremsstraulung stands for breaking radiation, electron decelerates and emits.

When q momentum transfers from incident electron to the atomic electron, it can change the atomic wavefunction

$$\exp(iqx_1) = \exp(iq'X_1)\exp(iq''X_2),$$

where $q' = q$ and $q'' = \frac{M}{M+m}q \sim q$.

$$\exp(iq''X_2) = \cos(q''(x_1 - x_2)) + i\sin(q''(x_1 - x_2)).$$

$$d_o = \langle \phi_0 | \cos(q''(x_1 - x_2)) | \phi_0 \rangle$$

is the elastic moment of self transition, from ground to ground state of the atom (atom doesn't move). When incident atom stops and

$$E_1 - E_0 = \frac{\hbar^2 q^2}{2m},$$

we see the transition.

Transition amplitude for transfer of momentum

$$\Omega_1 = \frac{e^2}{V \epsilon_0 q^2} \tag{5.24}$$

where $V = l^3$ is volume of incident electron, where $l \sim 10^{-8}$, thermal debroglie wavelength.

But there is another amplitude, the radiation by free electron, whose amplitude is

Transition amplitude for transfer of momentum

$$\Omega_2 = q E_o l \tag{5.25}$$

where $V = l^3$ is volume of incident electron, where $l \sim 10^{-8}$, thermal debroglie wavelength. and E_o is electric field of emitted photon,

Then this being a second order process, the net amplitude is

$$\Omega = \frac{\Omega_1 \Omega_2}{E_1 - E_0} \tag{5.26}$$

where $E_1 - E_0$ is as in Eq. (5.6).

A small digression in second order process, shown in Figure 5.9. There are three levels $|1\rangle$, $|2\rangle$ and $|3\rangle$ with energies E_1, E_2, E_3, with $E_1 = E_3$, levels 1 and 3 are degenerate. Ω_1 and Ω_2 are transition amplitudes between level 1 and 2 and level 2 and 3 respectively.

Then gives for $|E_1 - E_2| \gg \Omega_i$, there is transition amplitude of going from $|1\rangle$ to $|3\rangle$, given by,

$$\mathcal{M} = \frac{\Omega_1 \Omega_2}{E_1 - E_2} \tag{5.27}$$

What is E_o , in 5.25. If L is length of photon, then $\frac{\epsilon_0}{E_o}^2 L^3 = \hbar\omega = E_1 - E_0$, but it takes $\hbar\Omega^{-1}$ time for emission in which photon travels a distance $L = c\hbar\Omega^{-1}$. From this we can get E_o.

Fig. 5.12 shows Bremsstraulung spectra, with characteristic emission frequencies. These characteristic frequencies arise when free electron doesn't emit rather ionizes the atom. When electrons go back, they emit characteristic emission frequencies.

We talked about Bremsstraulung from the atom, lattice just being a big atom.

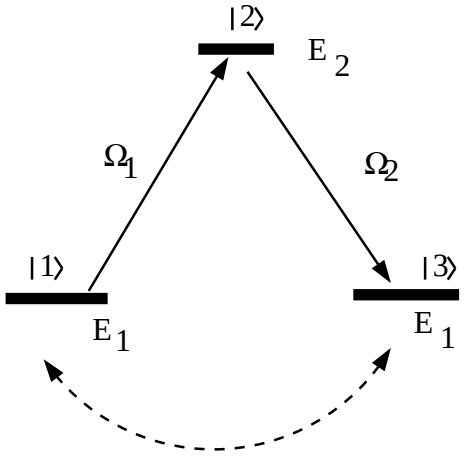


Fig. 5.9 Fig. depicts a 3 level system with transition amplitude Ω_1 and Ω_2 between level 1 and 2 and level 2 and 3 respectively.

5.14 Vibronic excitations of molecules and lattices

5.14.1 Vibronic excitation of molecules with light: direct and indirect

Consider a diatomic molecule with x_1, x_2 as nuclear coordinates and x_3 as electronic co-ordinate. Let k_i be their wave-vectors. Define CM and relative coordinates $X_1 = \frac{\sum x_i}{3}$, $X_2 = x_3 - \frac{x_1+x_2}{2}$ and $X_3 = x_1 - x_2$, with relative wavevectots K_i. Transition $K_2, K_3 \rightarrow K_2.K_3'$ gives vibration energy levels, it comes about as a second order term in the transition $K_2, K_3 \rightarrow K_2' K_3'' \rightarrow K_2, K_3'$. Linear combinations of K_3 gives vibrational orbitals Φ_3 and difference in energy Φ_3 and Φ_3' is in infra-red frequency and infrared light will induce this transition. This is called direct transition.

Transition

$$\Phi_2\Phi_3 \rightarrow \Phi_2'\Phi_3' \rightarrow \Phi_2\Phi_3''$$

are induced by absorbtion and emission of optical photonthe vibration state transits from $\Phi_3 \rightarrow \Phi_3''$ by a Raman Process. This is called indirect transition.

5.14.2 Vibronic excitation of phonons with light: direct or indirect, electron or photon

Vibronic states of crystals are special called phonons, the coordinates of atoms x_i are very localized and can be treated as classical variables. The collective excitation phonon is a wave, $A\cos(kx - \omega t) = A\cos(kla - \omega t)$, with A amplitude and $x = la$ the l^{th} coordinate. How does *phonon talk to photon*.

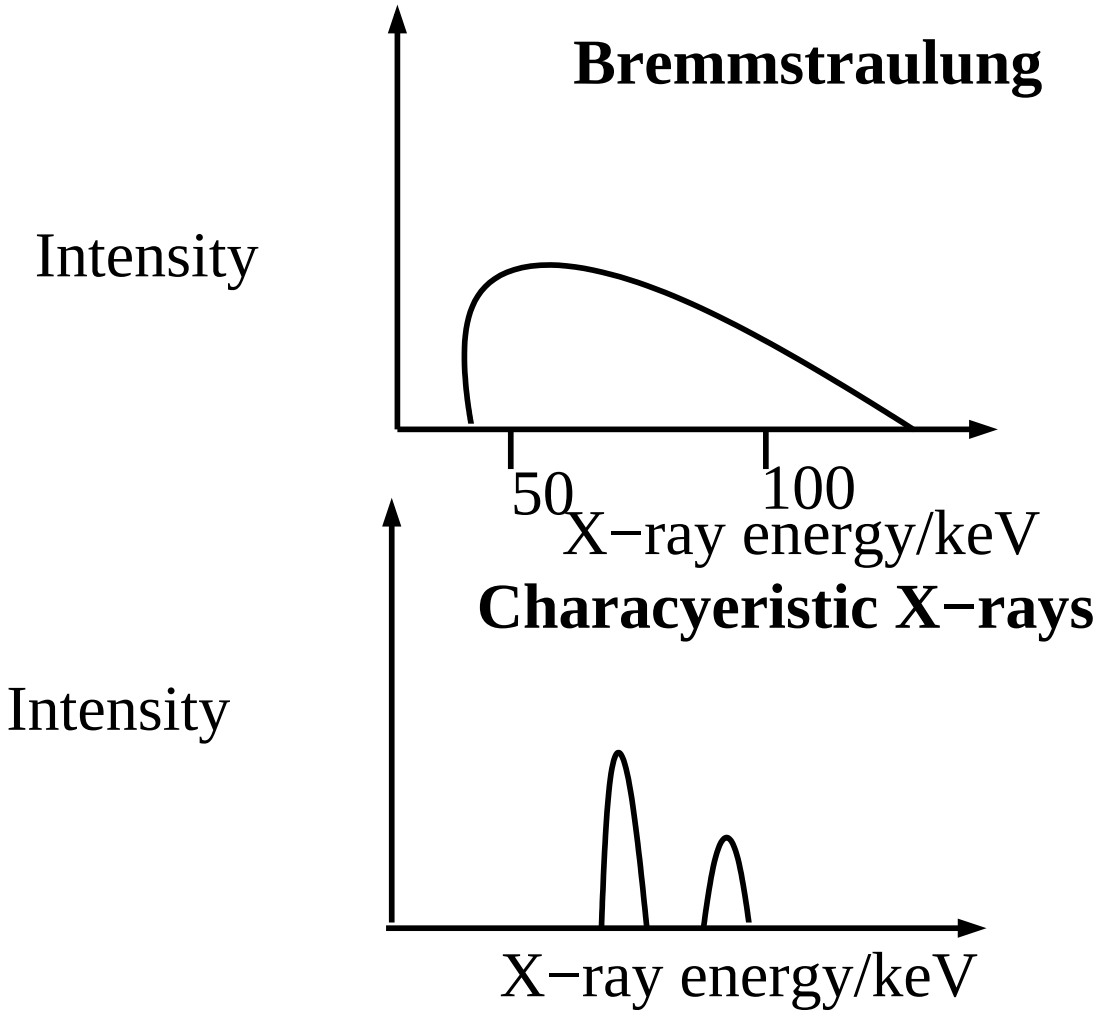


Fig. 5.10 Fig. depicts Bremsstraulung spectra, with characteristic emission frequencies.

Solids are hot because of lattice vibrations, phonons. Light from sun can generate phonons. Energy of the phonons is 10^{13}Hz. Infrared photons carry this energy. The phonon has energy $\hbar\upsilon k \sim 10meV$, giving $k \sim 10^{10}/m$, giving a momentum $\hbar k$ to phonon. This much momentum light doesn't have. The momentum is balanced by center of mass (CM) of the whole solid recoiling with momentum $-k$ and energy $\frac{\hbar^2 k^2}{2M}$ (M mass of whole solid), which is negligible because of large M. How do CM and phonon momentum talk ? There are two ways,

Exchange Photon or Exchange electron. In first case the CM and phonon coordinates exchange momnetum by a photon which is expensive as it has high energy.

A more efficient mechanism is exchange of momentum using a conduction electron which has lower energy, and hence this is a more efficient. Therefore conductors are easy to heat from Sun radiation than insulators.

Ofcourse this is with infrared, but we can heat a solid with optical wavelength, we just follow the above except absorb and emit in CM of lattice paying for the energy deficit, a Raman like process.

Everything done with light can be done with an electron, it can loose energy to lattice vibrations.

5.15 More waves: Surface Plasmons

Electron density can also modulate spatially, creating a wave called plasmon. The restoring force is ofcourse coulonb repulsion between electrons and attrction by lattice. A force of $10eV$ over a angstom distance fives sping constant $k \sim 10^2$ and mass of electron $m \sim 10^{-30}$, giving natural frequency $\omega_0 \sim 10^{15-16}$ and $\upsilon \sim 10^{5-6}$, and we can use a optical phonon of the right frequency to create these waves.

The mechanism is the same, we can absorb the momentum of incoming phootn on the lattice, the energy deficit is paid by exchanging a virtual photon with lattice that creates a surface plasmon whose mometum is negated by the lattice momentum except the energy of the plasmon is same as photon energy. Ofcourse, the plasmon can just be created with an electron, we just have to exchange momentum between electron and lattice.

5.16 More waves: Magnons

Ferrmomagnetic materials have all spins on lattice site salligned, but we can have excitations where we create a Magnetization wave

$$M_z(x) = A\cos(kx - \omega t).$$

The exchange coupling between neighboring spins is $\sim meV \sim 10^{2-3} GHz\,(microwave frequency) \sim \omega_0$

The $\upsilon \sim \omega_0 a \sim 100$, the magnon velocity.

5.17 Air molecules heating the lattice: exchange forces

How does lattic get phonons in first place. The air molecules continuously beat on it. Like an electron, the atomic collision will create a phonon. What is important is

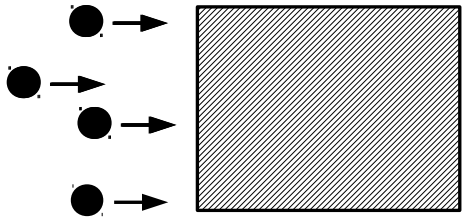


Fig. 5.11 Fig. depicts lattice being bombarded by air molecules.

atom is neutral so it cannot really influence another atom when they collide, except exchange forces between electrons.

Veovity of air molecule is $\sim kT$ which at room temp is $\sim meV$ same as phonon energy, and the molecule eneru can be written as $\hbar\vartheta k$ where $\vartheta \sim \upsilon(forphonon) \sim 10^3$, hence collisions can directly transfer energy and momentum to lattice.

5.18 Coherent electrons and phonons: loaded lattice

Electron waves in hot lattice are localized and have bandwidth. This happens due to molecular collisions. The fast molecule collides with lattice and creates exchange photon, which can excite electron $k_0 \rightarrow k_0'$. We therefore have the transitions $|FAST, k_0\rangle \rightarrow |SLOW, k_1\rangle$ and therefore we create a superposition $\frac{|SLOW,k_0\rangle+|FAST,k_1\rangle}{\sqrt{2}}$, thus we have created a superposition of

$$\phi = \frac{|k_0\rangle + |k_1\rangle}{\sqrt{2}},$$

in general $|k_0\rangle$ broadened to $\frac{1}{\sqrt{N}} \sum_N |k_i\rangle$, which means we have electron braodened in k space and localized in real space. We say lattice is temperature loaded and classical.

5.19 Light from the stars: Exchange Collisions

Light from stars come in all frequencies, radi, microwaves, infrared, visible, ultraviolet, xray. Everytime there is a atomic collision, which exchanges momentum between atoms with an exchange photon the atoms are in excited state and fall back emitting radiation. Temperature of collision determines how excited the atom is and its subsequent emitted wavelength.

Our main contribution in this article is we are detailed at places, we find literature succinct.

Problems

1. In compton scattering, X-ray at wavelength of $1A^\circ$, scatters by 30° to a new wavelength, which is ?

2. In electron proton scattering, electroms at energy of 10 MeV scatters by 30° to a new energy, which is ?

3. In photoelectric eefect if a metal has work function of $3eV$, what is the maximum wavelength photon that will eject electrons at 10^6 m/s.

4. In phonon spectroscopy using neutrons, the temperature of neutron drops by $10K$ after scattering, what is the frequency of the phonon excited.

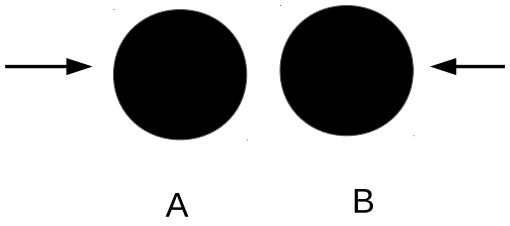


Fig. 5.12 Fig. depicts atomic collision in stars.

5. What is the binding energy of a Cooper pair, for typical lattice paramters in the book.

Chapter 6
Scattering in Quantum Electrodynamics

6.1 Introduction

We first develop an analogy between the three level atomic system so called Λ system and scattering processes in quantum electrodynamics (QED) [8, 9, 10, 13]. In a Λ system as shown in Fig. 6.1 we have two ground state levels $|1\rangle$ and $|3\rangle$ at energy E_1 and excited level $|2\rangle$ at energy E_2. The transition from $|1\rangle$ to $|2\rangle$ has strength Ω_1 and transition from $|2\rangle$ to $|3\rangle$ has strength Ω_2. In the interaction frame of natural Hamiltonian of the system, we get a second order term connecting level $|1\rangle$ to $|3\rangle$ with strength $\frac{\Omega_1\Omega_2}{(E_1-E_2)}$. This term creates an effective coupling between ground state levels and drives transition from $|1\rangle$ to $|3\rangle$. Scattering processes in QED can be modelled like this. Feynman amplitudes are calculation of second order term $\mathcal{M} = \frac{\Omega_1\Omega_2}{(E_1-E_2)}$.

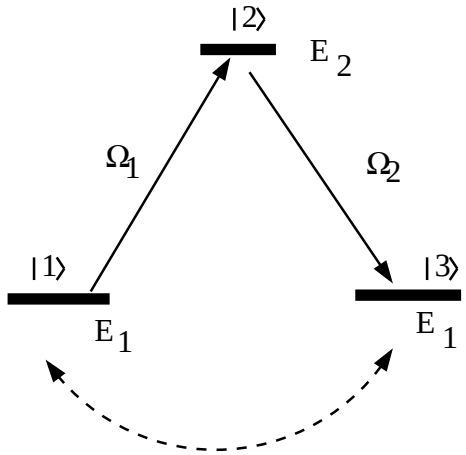


Fig. 6.1 Above Fig. shows a three level Λ system, with two ground state levels $|1\rangle$ and $|3\rangle$ and an excited level $|2\rangle$.

The state of the three level system evolves according to the Schröedinger equation

$$\dot{\psi} = \frac{-i}{\hbar}\begin{bmatrix} E_1 & \Omega_1^* & 0 \\ \Omega_1 & E_2 & \Omega_2^* \\ 0 & \Omega_2 & E_1 \end{bmatrix}\psi. \tag{6.1}$$

We proceed into the interaction frame of the natural Hamiltonian (system energies) by transformation

$$\phi = \exp(\frac{i}{\hbar}\begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_1 \end{bmatrix})\psi. \tag{6.2}$$

This gives for $\Delta E = E_2 - E_1$,

$$\dot{\phi} = \underbrace{\frac{-i}{\hbar}\begin{bmatrix} 0 & \exp(-\frac{i}{\hbar}\Delta E\ t)\Omega_1^* & 0 \\ \exp(\frac{i}{\hbar}\Delta E\ t)\Omega_1 & 0 & \exp(\frac{i}{\hbar}\Delta E\ t)\Omega_2^* \\ 0 & \exp(-\frac{i}{\hbar}\Delta E\ t)\Omega_2 & 0 \end{bmatrix}}_{H(t)}\phi. \tag{6.3}$$

$H(t)$ is periodic with period $\Delta t = \frac{2\pi}{\Delta E}$. After Δt, the system evolution is

$$\phi(\Delta t) = (I + \int_0^{\Delta t} H(\sigma)d\sigma + \int_0^{\Delta t}\int_0^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2 d\sigma_1 + \ldots)\phi(0). \tag{6.4}$$

The first integral averages to zero, while the second integral

$$\int_0^{\Delta t}\int_0^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2 d\sigma_1 = \frac{1}{2}\int_0^{\Delta t}\int_0^{\sigma_1} [H(\sigma_1), H(\sigma_2)]d\sigma_2 d\sigma_1. \tag{6.5}$$

Evaluating it explicitly, we get for our system that second order integral is

$$\frac{-i\Delta t}{\hbar}\begin{bmatrix} 0 & & \frac{\Omega_1^*\Omega_2^*}{E_1 - E_2} \\ 0 & 0 & 0 \\ \underbrace{\frac{\Omega_1\Omega_2}{E_1 - E_2}}_{\mathcal{M}} & 0 & 0 \end{bmatrix}. \tag{6.6}$$

Thus we have created an effective Hamiltonian

$$\begin{bmatrix} 0 & & \frac{\Omega_1^*\Omega_2^*}{E_1 - E_2} \\ 0 & 0 & 0 \\ \frac{\Omega_1\Omega_2}{E_1 - E_2} & 0 & 0 \end{bmatrix}, \tag{6.7}$$

which couples level $|1\rangle$ and $|3\rangle$ and drives transition between them at rate $\mathcal{M} = \frac{\Omega_1\Omega_2}{(E_1 - E_2)}$.

6.2 Coulomb Potential and Møller Scattering

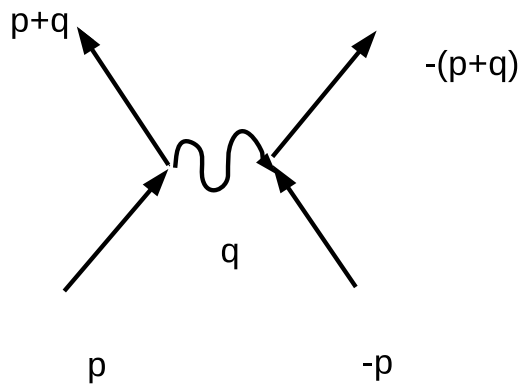


Fig. 6.2 Fig. depicts møller scattering. Two electrons with momentum p and $-p$, scatter by exchange of photon to $p+q$ and $-(p+q)$.

The heart of interactions in high energy physics is the beautiful electron electron scattering of Møller. The coulomb interaction between electrons. Fig. 6.2 shows two electrons with momentum p and $-p$ scatter by exchange of photon say in z direction to $p+q$ and $-(p+q)$. The scattering amplitude is well known, given as

Feynman propagator $\mathcal{M} = \frac{(e\hbar c)^2}{\epsilon_0 V} \frac{\bar{u}(p+q)\gamma^\mu u(p)\ \bar{u}(-(p+q))\gamma_\mu u(-p)}{q^2}$, [9, 10, 63], where V is the volume of the scattering electrons, e elementary charge and ϵ_0 permitivity of vacuum. But this needs to be taken with grain of salt. Since we exchange photon momentum in z direction, we have two photon polarization x, y and hence the true scattering amplitude should be $\mathcal{M}_1 =$

$$\frac{(e\hbar c)^2}{\epsilon_0 V} \frac{\bar{u}(p+q)\gamma^x u(p)\ \bar{u}(-(p+q))\gamma_x u(-p)\ \ + \bar{u}(p+q)\gamma^y u(p)\ \bar{u}(-(p+q))\gamma_y u(-p)}{q^2}.$$

But when electrons are non-relativistic, $\mathcal{M}_1 \sim 0$. This is disturbing, how will we ever get the coulomb potential, where $\mathcal{M} \sim \frac{(e\hbar c)^2}{\epsilon_0 V q^2}$. Where is the problem ? The problem is with the EM gauge used in Dirac equation.

For a plane wave along z direction, with electric field $E_x \sin(kz - \omega t)$, the Lorentz gauge is $(A_0, A_x, A_y, A_z) = \frac{E_x}{\omega} \cos(kz - \omega t)(0, 1, 0, 0)$. But this gauge is not suited for calculating optical transitions, because we don't recover the Rabi frequency $qE_x d$ (d electric dipole moment). What we find is something orders of magnitude smaller. Nor is it suitable for calculating electron electron scattering because we don't recover Coulomb potential. What we find is something orders of magnitude smaller. Instead, we work with $E \cdot x$ gauge

$$(A_0, A_x, A_y, A_z) = \frac{-E_x}{2}(x\ \sin(kz - \omega t), -\frac{\cos(kz - \omega t)}{\omega}, 0, \frac{x}{c}\sin(kz - \omega t))$$

(c light velocity) to find everything correct. What we get is a new Feynman propagator. Lets build up to it.

In Møller scattering, electrons with momentum p_1 and q_1 exchange photon with momentum k and scatter to new momentum states p_2 and q_2. Observe the virtual particle four momentum is k. The Feynman diagram for the process is in 6.3. There are two three level systems associated with this process. Let $P = p_1 + q_1$.

In figure 6.3A, we have the first three level system where the electron with momentum p_1 is annihilated , a electron of momentum p_2 is created and a photon of momentum $q = p_1 - p_2$ is created. Subsequently, the electron with momentum q_1 is annihilated , a electron of momentum q_2 is created and photon of momentum k is annihilated. The amplitude for this process is

$$\Omega_1 = \frac{C}{\sqrt{2E_q}}\, \bar{u}(p_2)\gamma^\nu \epsilon^*_\nu(q) u(p_1), \tag{6.8}$$

$$\Omega_2 = \frac{C}{\sqrt{2E_q}}\, \bar{u}(q_2)\gamma^\nu \epsilon_\nu(q) u(q_1), \tag{6.9}$$

$$E_1 - E_2 = E_{p_1} - E_{p_2} - E_q = q_0 - E_q, \tag{6.10}$$

$$\mathcal{M}_1 = \frac{\Omega_1 \Omega_2}{(E_1 - E_2)}. \tag{6.11}$$

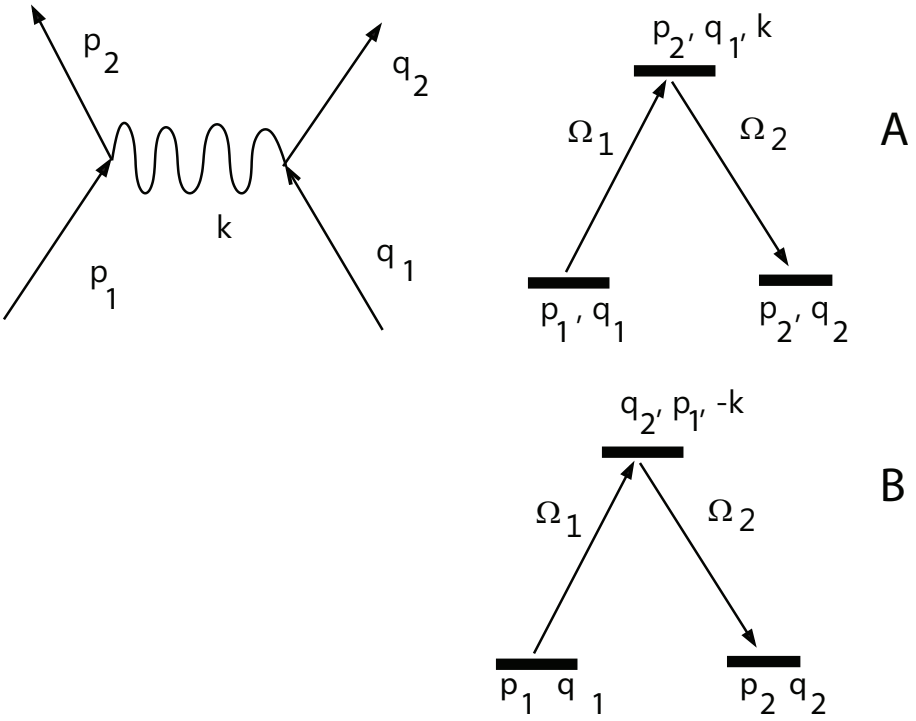


Fig. 6.3 Fig. shows the Feynman diagram for the Møller scattering, and its corresponding three level system. The electron with momentum p_1 emits (absorbs) a photon and scatters to momentum p_2, the photon is absorbed (emitted) by electron with momentum q_1 which scatters to momentum q_2.

Similarly we have another three level system, fig 6.3B in which q_1 emits photon with momentum $-k$ and p_1 absorbs it. This gives

$$\Omega_1 = \frac{C}{\sqrt{2E_q}}\,\bar{u}(q_2)\gamma^\nu \epsilon_\nu(q)u(q_1), \tag{6.12}$$

$$\Omega_2 = \frac{C}{\sqrt{2E_q}}\,\bar{u}(p_2)\gamma^\nu \epsilon^*_\nu(q)u(p_1), \tag{6.13}$$

$$E_1 - E_2 = E_{q_1} - E_{q_2} - E_q = -(q_0 + E_q), \tag{6.14}$$

$$\mathcal{M}_2 = \frac{\Omega_1\Omega_2}{(E_1 - E_2)}, \tag{6.15}$$

where we used conservation of energy $E_{p_2} - E_{p_1} = E_{q_1} - E_{q_2}$. When we add the two amplitudes, we get for $q = p_1 - p_2$ and $q^2 = q_\mu q^\mu$, the total amplitude is

$$\frac{\Omega_1\Omega_2}{q^2} = C^2 \frac{\bar{u}(q_2)\gamma^\nu \epsilon_\nu(q) u(q_1)\ \bar{u}(p_2)\gamma^\nu \epsilon^*_\nu(q) u(p_1)}{q^2} \tag{6.16}$$

We can now sum over photon polarization ϵ say along x and y axis to get

$$\mathcal{M} = \mathcal{M}_x + \mathcal{M}_y = \frac{(\hbar c)^2}{\epsilon_0 V} \frac{\bar{u}(q_2)\gamma^x u(q_1)\ \bar{u}(p_2)\gamma^x u(p_1) + \bar{u}(q_2)\gamma^y u(q_1)\bar{u}(p_2)\gamma^y u(p_1)}{q^2} \tag{6.17}$$

6.3 Scattering with $E \cdot x$ term, the negative sign of amplitude

Now consider Moller scattering with x term

When electron changes momentum by q (say z direction), in Lorentz gauge, photon of momentum $-q$ is emitted. In $\mathcal{E}\cdot x$ gauge, the emitted photon can be more general with momentum $-q+k$, where $k = n\Delta$ ($\Delta = 2\pi$, l length of electron packet) in x or y direction, then the amplitude $\mathcal{M}$ of scattering a momentum exchange q is

$$\mathcal{M}_0 = C u_4^\dagger(p_4) u_3(p_2)\ u_2^\dagger(p_3) u_1(p_1) \tag{6.18}$$

Adding two directions gives

$$C = 2\frac{(e\hbar c)^2}{\epsilon V}\frac{1}{4}\sum_n \frac{1}{n^2}\frac{1}{|q|^2} \sim \frac{(e\hbar c)^2}{\epsilon V |q|^2}, \tag{6.19}$$

We of course have (from A_z) the term

$$\mathcal{M}_z = C u_4^\dagger(p_4)\gamma_z u_3(p_2)\ u_2^\dagger(p_3)\gamma^z u_1(p_1) \tag{6.20}$$

The total amplitude including contribution from A_x, A_y in gauge

$$\mathcal{M} = \frac{1}{4|q|^2}\{u_4^\dagger(p_4)\gamma_\mu u_3(p_2)\ u_2^\dagger(p_3)\gamma^\mu u_1(p_1) - (3\mathcal{M}_0 + 5\mathcal{M}_z)\}. \tag{6.21}$$

The first term is the usual Feynman propagator, scaled by $\frac{1}{4}$, but second term is new and gives big contribution to Coulomb potential.

That's it, we have a new propagator. In its full glory it reads

$$\mathcal{M} = \frac{(e\hbar c)^2}{4\epsilon_0 V q^2}\{u_4^\dagger(p_4)\gamma_\mu u_3(p_2)\ u_2^\dagger(p_3)\gamma^\mu u_1(p_1) - \left(3\mathcal{M}_0 + 5\mathcal{M}_q\right)\}. \tag{6.22}$$

What is remarkable we have been able to get $-\mathcal{M}_0$, which is difficult to explain in Feynman Propagator. However the propagator is not Lorentz invariant, we say our electron-phonon coupling is scaled such that the true Propagator is

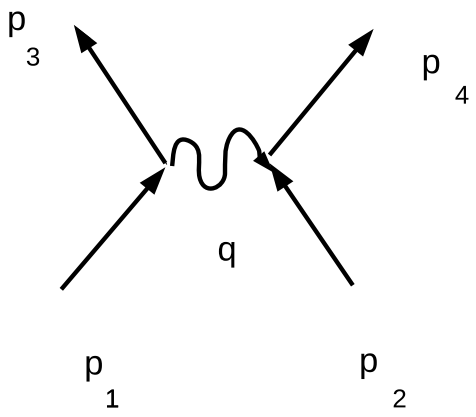


Fig. 6.4 Fig. depicts møller scattering. Two electrons with momentum p_1 and p_2 scatter by exchange of photon to p_3 and p_4.

$$\mathcal{M} = \frac{(e\hbar c)^2}{(\frac{E_1+E_2}{2})^2 \epsilon_0 V q^2} \{\mathbf{u}_4^\dagger(p_4)\gamma_\mu \mathbf{u}_3(p_2)\ \mathbf{u}_2^\dagger(p_3)\gamma^\mu \mathbf{u}_1(p_1)\}. \tag{6.23}$$

Now this Propagator is relativistically invariant, when we boost from CM frame to other frame (OF) , then $\mathcal{M}_{CM} = \frac{\mathcal{M}_{OF}}{\gamma}$. And defining the relativistic discount factor $\eta = \frac{\sqrt{E_1 E_2 E_3 E_4}}{(\frac{E_1+E_2}{2})^2}$, we can write the propagator as $\mathcal{M} = \eta \mathcal{M}_F$ where $\mathcal{M}_F$ is Feynmann Propagator,

$$\mathcal{M}_F = \frac{(e\hbar c)^2}{\epsilon_0 V q^2}\{u_4^\dagger(p_4)\gamma_\mu u_3(p_2)\; u_2^\dagger(p_3)\gamma^\mu u_1(p_1)\}. \tag{6.24}$$

It is this discount factor that plays a important role in our treatment of QED. We will always calculate $\mathcal{M}_F$ and then apply the discount factor η. In CM, $\eta = 1$ amd $\mathcal{M}_{CM} = \mathcal{M}_F$. In CM,

$$\mathcal{M}_q = \frac{e^2}{\epsilon_0 V}\frac{1}{|q|^2} \tag{6.25}$$

This gives a scattering potential

$$V = \sum_q \mathcal{M}_q \exp(-iq(r_1 - r_2)) = \frac{e^2}{(2\pi)^3\epsilon_0}\int d^3q \frac{\exp(-ik(r_1 - r_2))}{|q|^2}. \tag{6.26}$$

For $r = r_1 - r_2$, we have,

$$\begin{aligned}\int d^3q \frac{\exp(-iq(r_1 - r_2))}{|q|^2} &= 2\pi \int d|q| \int_0^\pi \exp(-i|q||r|\cos\theta)\sin\theta d\theta \\ &= \frac{4\pi}{|r|}\int_0^\infty \frac{\sin|q||r|}{|q|}d|q| = \frac{2\pi^2}{|r|}\end{aligned}$$

$$V = \sum_k \mathcal{M}_q \exp(-iq(r_1 - r_2)) = \frac{e^2}{4\pi\epsilon_0|r|} \tag{6.27}$$

The familiar Coulomb potential.

6.4 Bhaba scattering

In quantum electrodynamics, Bhabha scattering is the electron-positron scattering process:

$$e^+e^- \to e^+e^- \tag{6.28}$$

Bhabha scattering is named after the Indian physicist Homi J. Bhabha. The Bhabha scattering rate is used as a luminosity monitor in electron-positron colliders.

How do we understand scattering of an electron and positron.

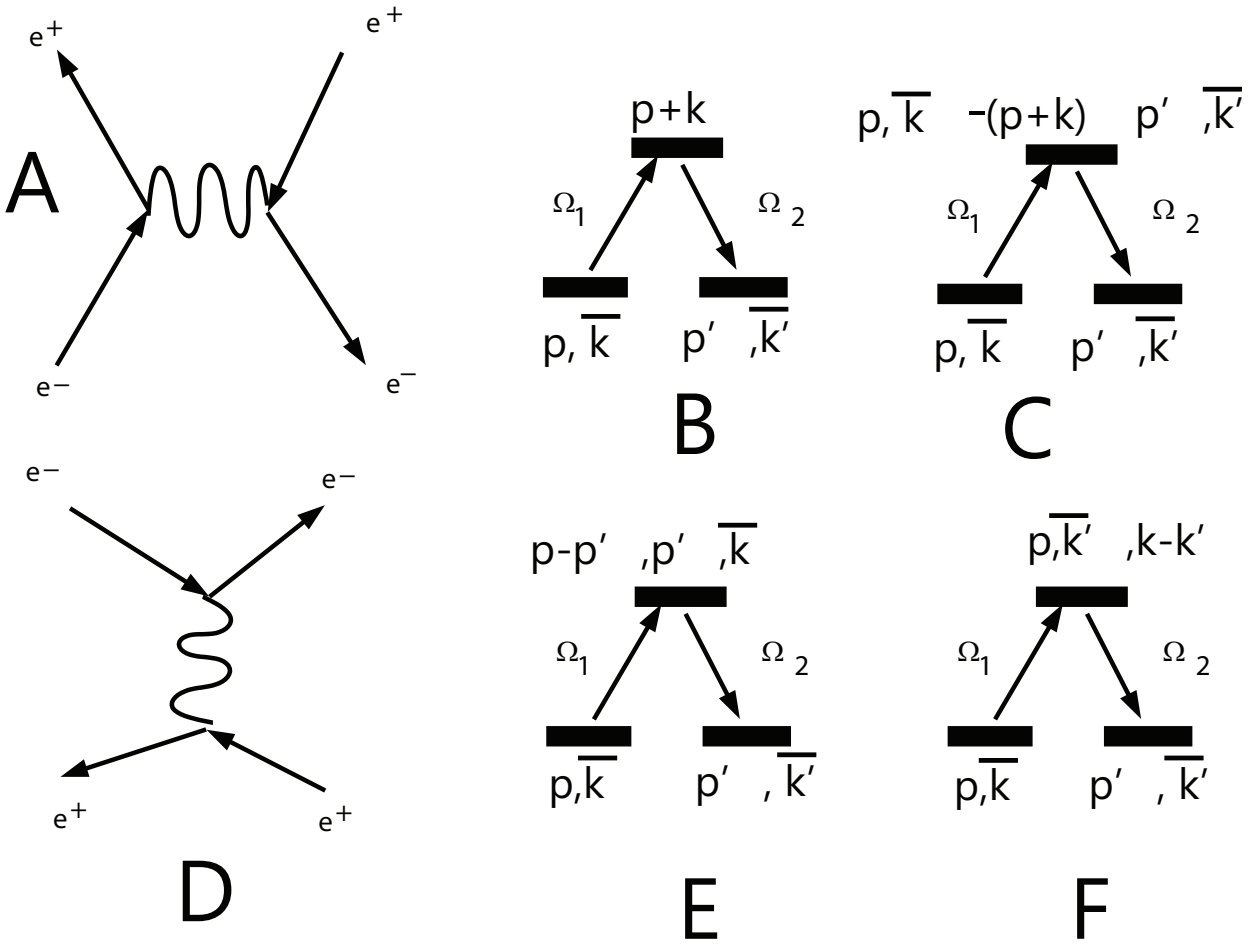


Fig. 6.5 Fig. depicts Bhaba scattering. A is annhilation and D scattering. B and C are three level processes with A and E, F with D

6.4.1 Annihilation

See 6.6. An photon of momentum $p+k$ comes and strikes filled sea of negative energy electrons and ejects a negative energy electron of state k and momentum $-k$ and creates a electron of momentum p and leaves behind missing momentum $-k$ and missing charge $-e$ or positron (hole) with momentum k and charge e . The process is like photoelectric effect where a valence electron is ejected to a free electron. If we read this processs reverse then we have electron-positron pair of momentum p and

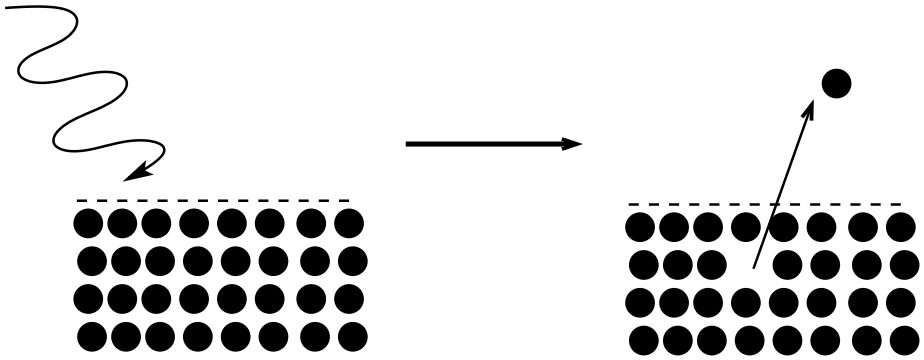


Fig. 6.6 Above Fig. shows how a photon ejects a negative energy electron and creates a electron and positron (hole) pair.

k annihilate to form a photon of momentum $p+k$. Denoting electron and positron helicity by s, s' and t, t' etc., the transition amplitude for this process is

$$\Omega_1 = \frac{C}{\sqrt{2E_{p+k}}}\, \bar{v}_t(k)\epsilon^*_\mu \gamma^\mu u_s(p) \tag{6.29}$$

The photon resulting from annihilation can now do ejection to create electron-positron or electron-hole pair with momentum p' and k' respectively with transition amplitude

$$\Omega_2 = \frac{C}{\sqrt{2E_{p+k}}}\, \bar{u}_{s'}(p')\epsilon_\mu \gamma^\mu v_{t'}(k') \tag{6.30}$$

All this is depicted as a three level process in fig. 6.5B. The associated feynaman diagram is in fig. 6.5A. The energy level difference between the ground and excited states

$$\Delta E_a = E_1 - E_2 = E_p + E_k - E_{p+k}. \tag{6.31}$$

where E_p, E_k and E_{p+k} are electron, positron and photon energies.

There is another three level process in fig. 6.5C associated with Feynman diagram in fig. is 6.5A. In this process, the negative energy electron just emits a photon with momentum $-(p'+k') = -(p+k)$ and a electron and positron with momentum p' and k'. The amplitude of this process is Ω_2 above. The emitted photon then combines with incoming electron and fills the incoming hole (vacancy) with amplitude same as Ω_1. The energy level difference between the ground and excited states

$$\Delta E'_a = E_1 - E_2 = -(E_p + E_k + E_{p+k}). \tag{6.32}$$

where E_p, E_k and E_{p+k} are electron, positron and photon energies. Then the amplitude $\mathcal{M}_1$ of the Feynman diagram in fig. 6.5A is sum of three level process in fig. 6.5B and three level process in fig. 6.5C. Then

$$\mathcal{M}_a = \Omega_1\Omega_2\left(\frac{1}{E_p+E_k-E_{p+k}}-\frac{1}{E_p+E_k+E_{p+k}}\right) \tag{6.33}$$

$$= \Omega_1\Omega_2\ \frac{2E_{p+q}}{(E_p+E_k)^2-E_{p+k}^2} \tag{6.34}$$

Now as in previous section on Moeller scattering, we have,

$$\mathcal{M}_F = C^2\ \frac{\bar{v}_t(k)\gamma^\mu u_s(p)\ \ \bar{u}_{s'}(p')\gamma_\mu v_{t'}(k')}{(p+k)^2}. \tag{6.35}$$

6.4.2 Scattering

There is another picture of positron suitable for scattering which is just electron evolving backward, which makes positive energy states as $v(k)$ instead of $u(k)$.

There is one more Feynman diagram in fig. 6.5D that contributes to this scattering process. There are also two three level processes fig. 6.5E and fig. 6.5F that contribute to this diagram. In fig. 6.5E a electron p scatters to p' giving a photon of momentum $q = p - p'$. This happens with amplitude

$$\Omega_3 = \frac{C}{\sqrt{2E_q}}\ \bar{u}_{s'}(p')\epsilon_\nu^*\gamma^\nu u_s(p) \tag{6.36}$$

and this photon then scatters positron. This happens with amplitude

$$\Omega_4 = \frac{C}{\sqrt{2E_q}}C\ \bar{v}_{t'}(k')\epsilon_\nu\gamma^\nu v_t(k) \tag{6.37}$$

The difference in energies of ground and excited state is

$$\Delta E_b = E_1 - E_2 = E_p - E_{p'} - E_{p-p'}. \tag{6.38}$$

Positron can emit first and electron can absorb a negative momentum photon. This happens with amplitude

$$\Omega_3 = \frac{C}{\sqrt{2E_q}}\ \bar{v}_{t'}(k')\epsilon_\nu^*\gamma^\nu v_t(k) \tag{6.39}$$

This happens with amplitude

$$\Omega_4 = \frac{C}{\sqrt{2E_q}}\ \bar{u}_{s'}(p')\epsilon_\nu\gamma^\nu u_s(p) \tag{6.40}$$

$$\Delta E_b = E_1 - E_2 = E_k - E_{k'} - E_{k-k'}. \tag{6.41}$$

Then the amplitude $\mathcal{M}_2$ of the Feynman diagram in fig. 6.5D is sum of three level process in fig. 6.5E and three level process in fig. 6.5F. Then

$$\mathcal{M}_s = \Omega_3\Omega_4\left(\frac{1}{E_p - E_{p'} - E_{p-p'}} + \frac{1}{E_k - E_{k'} - E_{k-k'}}\right) \tag{6.42}$$

$$= \Omega_3\Omega_4\left(\frac{1}{E_p - E_{p'} - E_{p-p'}} - \frac{1}{E_p - E_{p'} + E_{p-p'}}\right) \tag{6.43}$$

$$= \Omega_3\Omega_4\ \frac{2E_{p-p'}}{(E_p - E_{p'})^2 - E_{p-p'}^2} \tag{6.44}$$

If we denote four momentum $q = p - p'$, and $q^2 = q_\mu q^\mu$. Now as in previous section on Moeller scattering,

$$\mathcal{M}_F = -C^2\ \frac{\bar{u}_{s'}(p')\gamma^\nu u_s(p)\ \ \bar{v}_{t'}(k')\gamma_\nu v_t(k)}{(p-p')^2}. \tag{6.45}$$

The total amplitude then is

$$\mathcal{M}_F = C^2\left(-\frac{\bar{u}_{s'}(p')\gamma^\nu u_s(p)\ \ \bar{v}_{t'}(k')\gamma_\nu v_t(k)}{(p-p')^2} + \frac{\bar{v}_t(k)\gamma^\mu u_s(p)\ \ \bar{u}_{s'}(p')\gamma_\mu v_{t'}(k')}{(p+k)^2}\right) = C^2\mathcal{N}. \tag{6.46}$$

Of-course, $\mathcal{M} = \eta\mathcal{M}_F$, the discount factor.

6.4.3 Cross-section

Fig. (6.7)A shows the schematic of electron positron each of volume $V = l^3$ colliding head on. We can ask what should be the smallest density or the cross-section area $A = l^2$ for the two to scatter at an angle θ with probability 1, when they collide. This is called differential cross-section. We have calculated $\mathcal{M}$ the scattering amplitude, in center of mass frame. Let $\mathcal{M}(p_i)$ denote this as function of outgoing momenta p_i. Then by Fermi Golden rule the probability of scattering P is given by

$$\frac{dP}{dt} = \frac{\sum_i |\mathcal{M}(p_i)|^2}{\hbar\Delta E} \tag{6.47}$$

where ΔE is the energy width of the tessellation of the momentum space volume as shown in Fig. (6.7)C. Note $|\mathcal{M}(p_i)|^2$ carries with it a factor C^4 which has in it $\frac{(\hbar c)^4}{V^2}$. Let $\frac{1}{V} = \frac{d^3k}{(2\pi)^3}$ or $\frac{(\hbar c)^3}{V} = \frac{d^3p}{(2\pi)^3}$. With $E = \sqrt{p^2 + m^2}$ we get $\Delta E = \frac{p\Delta p}{E}$. Then converting sum in 6.47 to integral we get

$$\frac{dP}{dt} = \frac{e^4 c}{\epsilon_0^2 l^3 (2\pi)^3}\frac{|p|^2\Delta p\int |\mathcal{N}(p)|^2 d\Omega}{\Delta E} \tag{6.48}$$

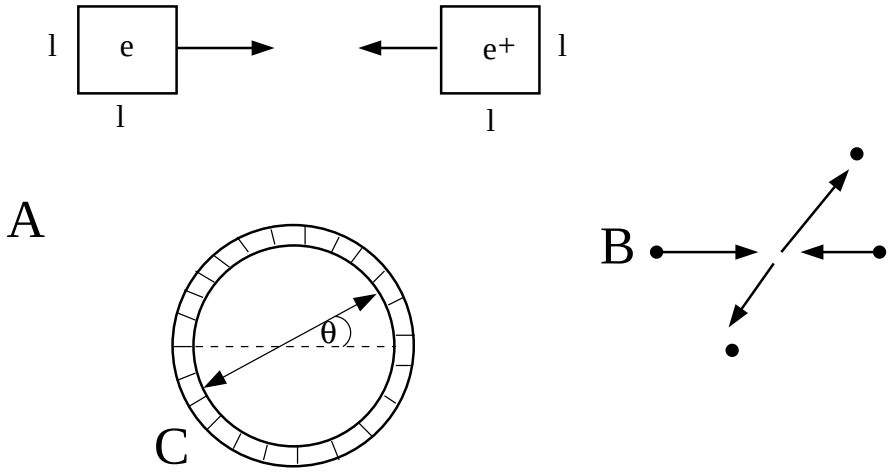


Fig. 6.7 Fig. shows the electron-positron colliding, and scattering to a different angle.

Then it takes $\Delta t = \frac{l}{c}$ for the packets to cross each other and it this time we want

$$\frac{dP}{dt}\frac{l}{c} = 1 \tag{6.49}$$

or we get using ΔE

$$\sigma = l^2 = \frac{e^4}{\epsilon_0^2 (2\pi)^3} E^2 \int |\mathcal{N}(p)|^2 d\Omega \tag{6.50}$$

or

$$\frac{d\sigma}{d\Omega} = \frac{e^4 E^2}{\epsilon_0^2 (2\pi)^3} |\mathcal{N}(p)|^2 \tag{6.51}$$

This is called differential cross-section.

Now we calculate differential cross section by evaluating

$$|\mathcal{N}|^2 = |\mathcal{N}_a + \mathcal{N}_s|^2 \tag{6.52}$$

Infact we evaluate unpolarized cross-section which is to say we average over all possible helicities to get

$$\begin{aligned}\sum_{s,s',t,t'} |\mathcal{N}|^2 &= \sum_{s,s',t,t'} |\mathcal{N}_a + \mathcal{N}_s|^2 \\ &= \sum_{s,s',t,t'} |\mathcal{N}_a|^2 + \sum_{s,s',t,t'} |\mathcal{N}_s|^2 + \sum_{s,s',t,t'} \mathcal{N}_a \mathcal{N}_s^* + \mathcal{N}_a^* \mathcal{N}_s\end{aligned}$$

6.4.4 Relativistic limit

Lets evaluate the unpolarized cross-section

$$\frac{1}{4}\sum_{s,s',t,t'}|\mathcal{N}_s|^2. \tag{6.53}$$

Recall

$$\mathcal{N}_s = -\frac{\bar{u}_{s'}(p')\gamma^\nu u_s(p)\ \ \bar{v}_t(k)\gamma_\nu v_{t'}(k')}{(p-p')^2} \tag{6.54}$$

Let electron and positron approach each other along z and $-z$ direction respectively . Under relativistic limit helicity 1 electron and positron are

$$u(p) = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix};\ \ v(k) = \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix};$$
$$u(p') = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix};\ \ v(k') = \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{bmatrix};$$

Under relativistic limit helicity -1 electron and positron are

$$u(p) = \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix};\ \ v(k) = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix};$$
$$u(p') = \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{bmatrix};\ \ v(k') = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix};$$

Now observe in 6.54, we get zero if we switch either electron and positron helicity. Furthermore under relativistic limit we get

$$(p-p')^2 \sim -2p.p' \sim -2E^2(1-\cos\theta). \tag{6.55}$$

Then substituting all helicities in Eq. 6.53, we get

$$\frac{1}{4}\sum_{s,s',t,t'}|\mathcal{N}_s|^2 = \frac{1}{8E^4}.$$

Now lets evaluate

$$\frac{1}{4}\sum_{s,s',t,t'}|\mathcal{N}_a|^2. \tag{6.56}$$

Recall

$$\mathcal{N}_a = \frac{\bar{v}_t(k)\gamma^\mu u_s(p)\ \ \bar{u}_{s'}(p')\gamma_\mu v_{t'}(k')}{(p+k)^2} \tag{6.57}$$

Now observe under relativistic limit, in 6.57, we get zero if incoming or outgoing pair has same helicity. Then substituting all helicities in Eq. 6.56, we get

$$\frac{1}{4}\sum_{s,s',t,t'}|\mathcal{N}_a|^2=\frac{(1-\cos\theta)^2}{4E^4}.$$

Finally evaluating

$$\frac{1}{4}\sum_{s,s',t,t'}\mathcal{N}_a\mathcal{N}_s^*+\mathcal{N}_a^*\mathcal{N}_s=-\frac{(1-\cos\theta)}{4E^4}. \tag{6.58}$$

where we have only two terms, incoming or outgoing pair has same helicity and helicity cannot switch from incoming to outgoing. Adding everything we get

$$\frac{1}{4}\sum_{s,s',t,t'}|\mathcal{N}|^2=\frac{(1-\cos\theta)^2+\cos^2\theta}{8E^4}. \tag{6.59}$$

For $s=E^2$, we get

$$s\frac{d\sigma}{d\Omega}=\frac{e^4}{\epsilon_0^2}\frac{(1-\cos\theta)^2+\cos^2\theta}{32\,\pi^2} \tag{6.60}$$

We can write the cross-section as $\frac{d\sigma}{d\Omega}=l^2f(\theta)$, where $l=\frac{\alpha\hbar c}{E}$. This is a very useful form. The expression

$$l=\frac{\alpha\hbar c}{E}$$

is very aesthetically appealing. At $E=GeV$, we have cross section $l^2\sim 10^{-8}$ barn, where 1 barn is 10^{-28} m^2.

6.5 Muon scattering

The electron-positron can annihilate to form muon-anti-muon. We can work out the cross section as $\frac{d\sigma}{d\Omega}=l^2f(\theta)$, where

$$l=\frac{\alpha\hbar c}{E}\sqrt{1-\frac{(m_\mu c^2)^2}{E^2}}.$$

There is no scattering term in this collision. At $E=GeV$, we have cross section $l^2\sim 10^{-8}$ barn.

6.6 Compton scattering

Compton scattering is the inelastic scattering of a photon with an electrically charged particle, first discovered in 1923 by Arthur Compton [14]. This scattering process is of particular historical importance as classical electromagnetism is insufficient to describe the process; a successful description requires us to take into account the particle-like properties of light. Furthermore, the Compton scattering of an electron and a photon is a process that can be described to a high level of precision by QED.

In Compton scattering an electron and photon with momentum p and k respectively scatter into momentum p' and k' respectively. We want to calculate the amplitude for this scattering.

First note with $p \sim 0$, at rest, and k say along x, (see Fig. 6.10), we have the energy of the scattered electron $E = \frac{p'^2}{2m} = \hbar^2 \frac{(k-k'\cos\theta)^2 + k'^2 \sin^2\theta}{2m}$, where θ is angle of scattered photon with the x axis. But $E = \hbar c(k-k')$ and we get for $\lambda = \frac{2\pi}{k}$, we have

$$\lambda' - \lambda = \frac{h(1-\cos\theta)}{mc}$$

There are two Feynman diagrams that show mechanism of Compton scattering. They are shown in Fig. 6.8. We can associate each of these with two three level diagrams as shown in Fig. 6.9.

Consider Feynman diagram A in Fig. 6.8, where a electron of momentum p and photon of momentum k are annihilated to give an electron of momentum $q = p + k$ which is then annihilated to create electron and photon with momentum p' and k'. This correspond to three level system Fig. 6.9 A. The scattering amplitude for this system is as follows

$$\Omega_1 = \frac{C}{\sqrt{2E_k}} \bar{u}_s(p+k)\gamma^\nu \epsilon_\nu(k) u(p), \tag{6.61}$$

$$\Omega_2 = \frac{C}{\sqrt{2E_{k'}}} \bar{u}(p')\gamma^\mu \epsilon^*_\mu(k') u_s(p+k), \tag{6.62}$$

$$E_1 - E_2 = q_0 - E_q = E_p + E_k - E_{p+k}, \tag{6.63}$$

$$\mathcal{M}^s_{1a} = \frac{\Omega_1 \Omega_2}{E_1 - E_2}. \tag{6.64}$$

where $E_p = \sqrt{(|p|c)^2 + \mathbf{m}^2}$ and $E_k = |k|c$. Summing over electron polarization we get

$$\mathcal{M}_{1a} = \frac{C^2}{2\sqrt{E_k E_{k'}}} \bar{u}(p')\gamma^u \epsilon^*_\mu(k') \frac{\sum_s u_s(p+k)\bar{u}_s(p+k)}{E_q(q_0 - E_q)} \gamma^\nu \epsilon_\nu(k) u(p). \tag{6.65}$$

There is an associated three level diagram with this as shown in 6.9 B, where we first create electron and photon with momentum p' and k' respectively alongside a

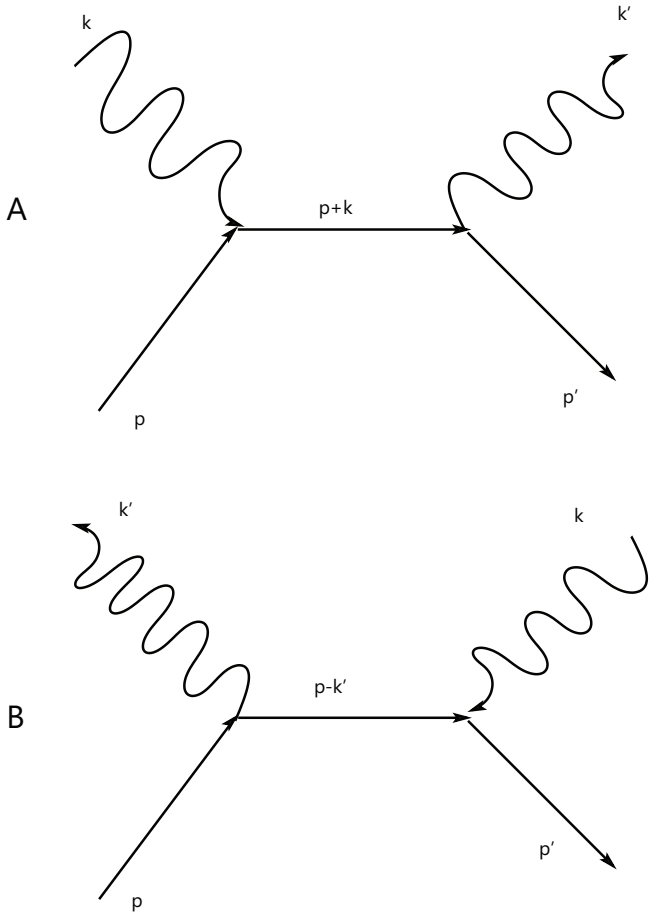


Fig. 6.8 Fig. A shows Fig. B show the two mechanisms for Compton scattering.

positron with momentum $q = -(p' + k') = -(p + k)$ and then annihilate electron and photon with momentum p and k alongside a positron with momentum $-(p + k)$.

The scattering amplitude for this system is as follows

$$\Omega_1 = \frac{C}{\sqrt{2E_{k'}}}\,\bar{u}(p')\gamma^{\mu}\epsilon^{*}_{\mu}(k')u_s(p+k), \tag{6.66}$$

$$\Omega_2 = \frac{C}{\sqrt{2E_k}}\bar{u}_s(p+k)\gamma^{\nu}\epsilon_{\nu}(k)u(p), \tag{6.67}$$

$$E_1 - E_2 = -(q_0 + E_q) = -E_{p+k} - (E_p + E_k), \tag{6.68}$$

$$\mathcal{M}^{s}_{1b} = \frac{\Omega_1\Omega_2}{E_1 - E_2}. \tag{6.69}$$

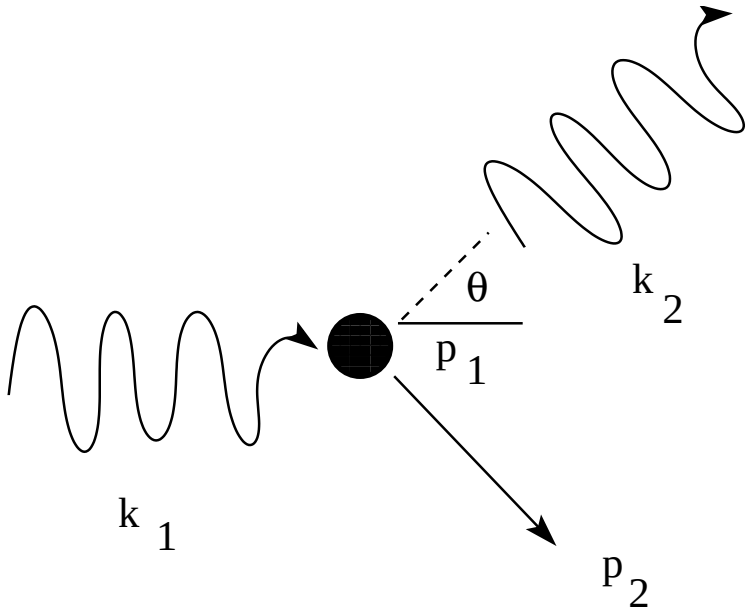


Fig. 6.9 Fig. shows three level systems that go with Feynman diagrams in Fig. (6.8).

Summing over electron polarization we get

$$\mathcal{M}_{1b} = -\frac{C^2}{2\sqrt{E_k E_{k'}}}\,\bar{u}(p')\gamma^\mu \epsilon^*_\mu(k')\frac{\sum_s u_s(p+k)\bar{u}_s(p+k)}{q_0+E_q}\gamma^\nu \epsilon_\nu(k)u(p) \quad (6.70)$$

Adding the two amplitudes $\mathcal{M}_1 = \mathcal{M}_{1a} + \mathcal{M}_{1b}$, we get

$$\begin{aligned}\mathcal{M}_1 &= \frac{C^2}{\sqrt{E_k E_{k'}}}\,\frac{\bar{u}(p')\gamma^\mu \epsilon^*_\mu(k')\sum_s u_s(p+k)\bar{u}_s(p+k)\gamma^\nu \epsilon_\nu(k)u(p)}{q^2-m_0^2},\\ &= \frac{C^2}{2\sqrt{E_k E_{k'}}}\,\frac{\bar{u}(p')\gamma^\mu \epsilon^*_\mu(k')(\not{q}+m_0)\gamma^\nu \epsilon_\nu(k)u(p)}{q^2-m_0^2}. \end{aligned} \quad (6.71)$$

We made use of identity $\sum_s u_s(q)\bar{u}_s(q) = \frac{\not{q}+m_0}{2E_q}$, where $\not{q} = q_j\gamma^j$ (c is implicit). We assume we are in a high energy center of mass frame. Which implies $E_p \sim E_k$ and we can write

Now consider Feynman diagram B in Fig. 6.8, where a electron of momentum p is annihilated and photon of momentum k' is created to give an electron of momentum $q = p - k'$ which is then annihilated along-with the photon of momentum k to create electron with momentum p'. This correspond to three level system Fig. 6.9 C. The scattering amplitude for this system is as follows

$$\Omega_1 = \frac{C}{\sqrt{2E_{k'}}}\,\bar{u}_s(p-k')\gamma^\mu\epsilon^*_\mu(k')u(p), \tag{6.72}$$

$$\Omega_2 = \frac{C}{\sqrt{2E_k}}\bar{u}(p')\gamma^\nu\epsilon_\nu(k)u_s(p-k'), \tag{6.73}$$

$$E_1 - E_2 = q_0 - E_q = E_p - E_{p-k'} - E_{k'}, \tag{6.74}$$

$$\mathcal{M}^s_{2a} = \frac{\Omega_1\Omega_2}{E_1 - E_2}. \tag{6.75}$$

Summing over electron polarization we get

$$\mathcal{M}_{2a} = \frac{C^2}{2\sqrt{E_k E_{k'}}}\,\bar{u}(p')\gamma^\nu\epsilon_\nu(k)\frac{\sum_s u_s(p-k')\bar{u}_s(p-k')}{q_0 - E_q}\gamma^\mu\epsilon^*_\mu(k')u(p) \tag{6.76}$$

There is an associated three level diagram with this as shown in 6.9 D, where we first create electron and annihilate photon with momentum p' and k respectively alongside creating a positron with momentum $-(p-k') = -(p'-k)$ and then annihilate electron and create photon with momentum p and k' alongside annihilate positron with momentum $-(p-k')$.

The scattering amplitude for this system is as follows

$$\Omega_1 = \frac{C}{\sqrt{2E_{k'}}}\,\bar{u}(p')\gamma^\nu\epsilon_\nu(k)u_s(p'-k), \tag{6.77}$$

$$\Omega_2 = \frac{C}{\sqrt{2E_k}}\,\bar{u}_s(p-k')\gamma^\mu\epsilon^*_\mu(k')u(p), \tag{6.78}$$

$$E_1 - E_2 = -(q_0 + E_q) = -E_{p-k'} - E_p + E_{k'}, \tag{6.79}$$

$$\mathcal{M}^s_{2b} = \frac{\Omega_1\Omega_2}{E_1 - E_2}. \tag{6.80}$$

Summing over electron polarization we get

$$\mathcal{M}_{2b} = -\frac{C^2}{2\sqrt{E_k E_{k'}}}\,\bar{u}(p')\gamma^\nu\epsilon_\nu(k)\frac{\sum_s u_s(p-k')\bar{u}_s(p-k')}{q_0 + E_q}\gamma^\mu\epsilon^*_\mu(k')u(p) \tag{6.81}$$

Adding the two amplitudes $\mathcal{M}_2 = \mathcal{M}_{2a} + \mathcal{M}_{2b}$, we get

$$\begin{aligned}\mathcal{M}_2 &= \frac{C^2}{\sqrt{E_k E_{k'}}}\,\bar{u}(p')\gamma^\nu\epsilon_\nu(k)\frac{\sum_s u_s(p-k')\bar{u}_s(p-k')}{q^2 - m_0^2}\gamma^\mu\epsilon^*_\mu(k')u(p),\\ &= \frac{C^2}{2\sqrt{E_k E_{k'}}}\,\frac{\bar{u}(p')\gamma^\nu\epsilon_\nu(k)(\not{q} + m_0)\gamma^\mu\epsilon^*_\mu(k')u(p)}{q^2 - m_0^2}.\end{aligned} \tag{6.82}$$

$$\mathcal{M}_F = \mathcal{M}_1 + \mathcal{M}_2 \tag{6.83}$$

Of-course, $\mathcal{M} = \eta \mathcal{M}_F$, the discount factor.

6.6.1 Cross-section

We have to calculate $|\mathcal{M}|^2$ to find cross-section. Before we do this we can just say that we work in regime (as in original Compton's experiment) where photon energy (10's KeV, wavelength A°) is smaller than rest energy of the electron (Mev, Compton wavelength, .01 A°), we have the cross-section of the form $\frac{d\sigma}{d\Omega} = l^2 f(\theta)$, where

$$l = \frac{\alpha \hbar c}{E}.$$

Calculating $f(\theta)$ is just an exercise where you just have to roll your sleeves. At $E = 10keV$, we have cross-section $\sim$ 100 barn.

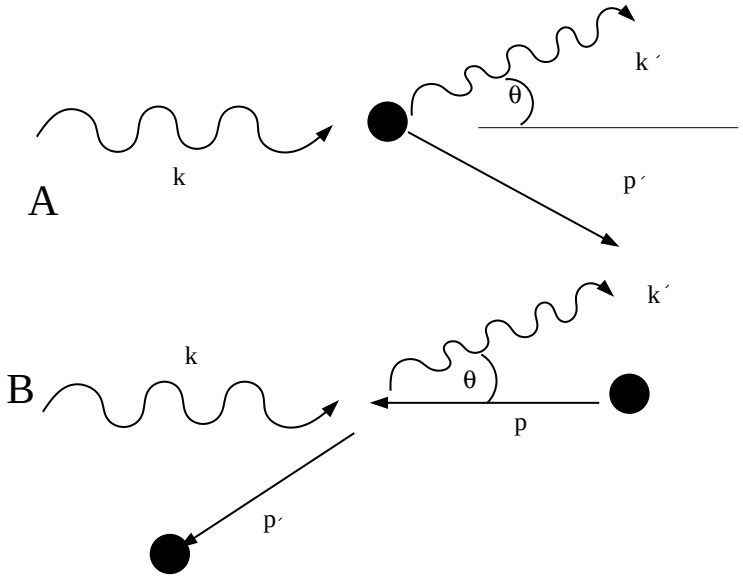


Fig. 6.10 Above Fig. A shows Compton scattering in lab frame. Fig. B shows Compton scattering in center of mass frame.

6.7 Vacuum Polarization

Quantum electrodynamics (QED) is one of the most successful theories of modern physics era [9, 10, 63, 13]. In QED, electrons interact by electromagnetic coupling to

vacuum. Electron emits photon which is absorbed by the second electron leading to momentum exchange between electrons which we call electric force. The emission and absorption changes the energy of the two electrons by what we call the electric potential energy. In calculating this energy, which is a second order calculation, we make use of the energy of photon $E_k = \hbar ck$ where k is its momentum. But this emitted photon can further interact with the vacuum by creating electron positron pairs, which annihilate to give the photon back. This again has its own energy which modifies the energy of the photon E_k to E'_k. we can calculate this modification or correction and we find this will change the electromagnetic potential between two electrons. We may think of this as simply changing ϵ_0 the vacuum permitivity and this is called vacuum polarization, very much like light propagating in a medium polarizes it and changes ϵ_0 and slows down. On another note, an electron can emit and absorb a photon and the process modifies the rest energy of the electron mc^2 to $m'c^2$ a process we call mass correction.

But there is a problem in QED. When we calculate these corrections, we find them divergent. There is a huge body of work in field of QED, that tries to tame these infinities, a process we call renormalization [13]. But, where is the problem ? The problem is when we calculate the modification of photon energy we collide it with a sea electron, but the collision is not in center of mass frame , to get the right amplitude we have to use the correct discount factor η and then we find our answers are finite.

Electron 1 emits photon with momentum k and energy $E_k = \hbar ck$, which is absorbed by electron 2. But this emitted photon can further interact with the vacuum by creating electron positron pairs, which annihilate to give the photon back. This again has its own energy which modifies the energy of the photon E_k to E'_k. we can calculate this modification or correction and we find this will change the electromagnetic potential between two electrons.

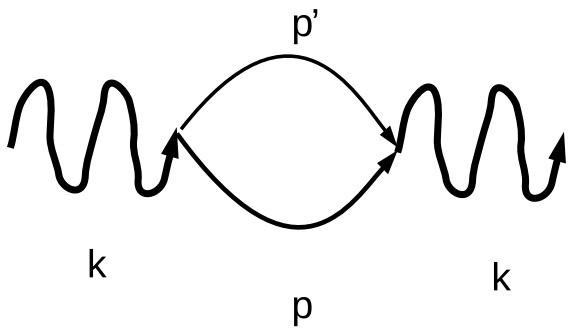


Fig. 6.11 Fig. depicts vacuum polarization. Emitted photon, generates electron-positron pair which recombine to give the photon back.

Creation of electron positron pair is tantamount to Compton scattering, the photon k collides with negative energy, sea, electron with momentum p and energy $-E_p$ and creates a positive energy electron with momentum $p' = p + k$ and positron with momentum $-p$. The name of the game is to sum the amplitude of the process for large values of p.

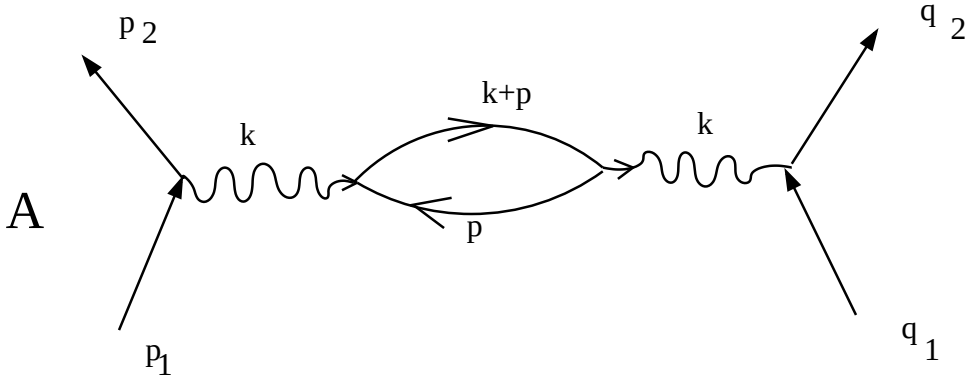


Fig. 6.12 Fig. A shows Feynman diagram for vacuum polarization.

$$\mathcal{M} = \frac{(e\hbar c)^2}{\epsilon_0 V}\frac{E_{p'}+E_p}{E_k}\left(\bar{v}(-p)\,\gamma^\mu u(p')\bar{u}(p')\gamma_\mu\, v(-p)\right)\frac{1}{(E_{p'}+E_p)^2 - E_k^2} \quad (6.84)$$

Simplify by $k \sim 0$ and $p \sim p'$, then two possibilities happen, one when spin of electron and positron are alligned, then in Eq. (6.84), $\mathcal{M} \sim 0$, second possiility when they are antialligned, then for large p, we have one as $\begin{pmatrix}1\\0\end{pmatrix} \otimes \uparrow$ and other $\begin{pmatrix}0\\1\end{pmatrix} \otimes \downarrow$ and again in Eq. (6.84), $\mathcal{M} \sim 0$. We can show spinor part of Eq. (6.84) goes as $\sim \frac{1}{|p|^2}$. Now taking other factors into acount and the discount factor, $\eta \sim \frac{1}{|p|}$ we have

$$\mathcal{M} \sim \frac{1}{E_p^4} = \frac{1}{|p|^4},$$

Summing over $|p|$ we get the Harmonic sum $\int \frac{1}{|p|^2} dp$, which nicely converges.

6.8 Electron self energy

In last section, we talked about vacuum polarization, where a photon splits into an electron-positron pair and recombines. In this section, we discuss another QED

process, the electron self energy. Hereby, an electron of momentum p emits a photon and then reabsorbs it. This is shown in Fig. 6.13A. This process can be represented by two level diagrams as in Fig. 6.13B. In the first one, we have an electron with momentum p emit an photon with momentum k and subsequently reabsorb it. In second one, we have creation of a positron, electron and photon with momentum $-p$, $p-k$ and k respectively and their subsequent annihilation.

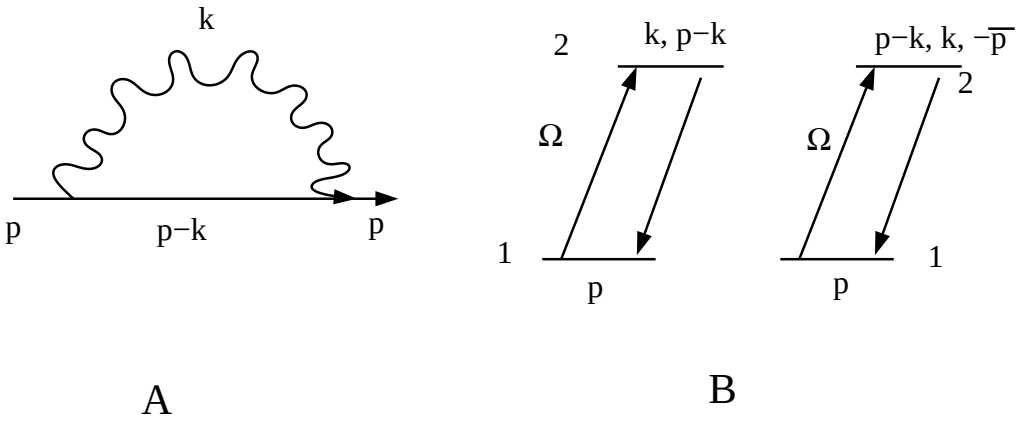


Fig. 6.13 Fig. A shows corrections to electron energy, where an electron emits and absorbs an photon. Fig. B shows two level diagrams for this process.

Electron emits photon, changes momentum $p' = p - k$ and reabsorbs photon to get to its initial state. The name of the game is to sum the amplitude of the process for large values of k.

$$\mathcal{M} = \frac{(\hbar c)^2}{\epsilon_0 V} \frac{E_{p'} + E_k}{E_k} \frac{\bar{u}(p)\gamma^\mu u(p')\; u(p')\gamma_\mu\; u(p)}{(E_k + E'_p)^2 - E_p^2} \tag{6.85}$$

Simplify, choosing $p \sim 0$ and $p' \sim -k$, since p non-relativistic its spin can be chosen aligned with p' and then for large k the spinor part of 6.85 $\sim \frac{1}{|k|}$ and with discount $\eta \sim \frac{1}{|k|}$, and the whole $\mathcal{M} \sim \frac{1}{|k|^4}$, Summing over $|k|$ we get the Harmonic sum $\int \frac{1}{|k|^2} dk$, which nicely converges.

The self energy process may be thought of as a collision between electron and negative energy photon in opposite direction, justifying use of η.

6.9 Vertex Corrections

Consider the Feynman diagram in Fig. 6.14A. It shows moller scattering of incoming electron with momentum p_1 and a heavy particle with momentum r_1. Incoming

electron emits a photon with momentum k that recombines with outgoing electron with momentum p_2. Fig. B shows a equivalent five level system. The incoming particles with momentum p_1, r_1 are at level 1. Emission of a photon with momentum k transits to level 2. Level $2,3,4$ represent the Moller scattering of electron and particle with momentum $p_1 - k$ and r_1 to momentum $p_2 - k$ and r_2 and finally the emitted photon k is reabsorbed and we get to level 5 with outgoing particles with momentum p_2, r_2.

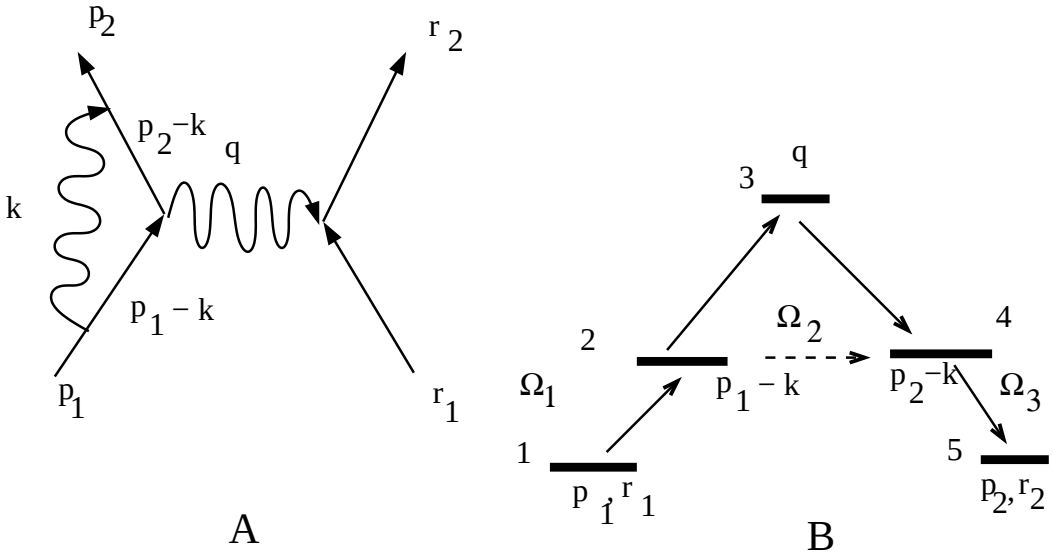


Fig. 6.14 Fig. A shows moller scattering of incoming electron with momentum p_1 and a heavy particle with momentum p_2. Incoming electron emits a photon with momentum k that recombines with outgoing electron with momentum $p_2 - k$. Fig. B shows a equivalent five level system. The incoming particles with momentum p_1, r_1 are at level 1, Emission of a photon with momentum k transits to level 2. Level $2, 3, 4$ represent the Moller scattering and finally the emitted photon k is reabsorbed and we get to level 5.

Lets calculate the scattering amplitude of p_1, r_1 to p_2, r_2 and in the process calculate the new transition amplitude of scattering from p_1 to p_2. This modification of amplitude of scattering from p_1 to p_2 as compared to one studied in section 6.2 is called the *Vertex correction.*

Observe under non-relativistic limit

$$E_{12} = E_1 - E_2 = E_{p_1} - (E_{p_1-k} + E_k) \sim E_{p_2} - (E_{p_2-k} + E_k) = E_{45} \tag{6.86}$$

Then the transition amplitude from level 1 to level 5 is a third order term and simply (see the end of the section)

$$\mathcal{M}_F = \frac{\Omega_1\Omega_2\Omega_3}{E_{12}^2}, \tag{6.87}$$

where Ω_i are as in Fig. 6.14B.

$$\mathcal{M} = \frac{\Omega_1\Omega_2\Omega_3}{(E_{p_1-k}+E_k)^2 - E_{p_1}^2}\frac{E_{p_1-k}+E_k+E_{p_1}}{E_{p_1-k}+E_k-E_{p_1}}, \tag{6.88}$$

where

$$\Omega_1 = \frac{C}{\sqrt{2E_k}}\,\bar{u}(p_1-k)\gamma^\nu\epsilon_\nu^*(k)u(p_1), \tag{6.89}$$

$$\Omega_2 \propto C^2\frac{\bar{u}(r_2)\gamma^\mu\epsilon_\mu(q)u(r_1)\;\bar{u}(p_2-k)\gamma^\mu\epsilon_\mu^*(q)u(p_1-k)}{q^2}, \tag{6.90}$$

$$\Omega_3 = \frac{C}{\sqrt{2E_k}}\,\bar{u}(p_2)\gamma^\nu\epsilon(k)u(p_2-k). \tag{6.91}$$

For large k, we have using discount $\eta \sim \frac{1}{|k|}$, we get $\mathcal{M} = \mathcal{M}_F\eta \sim \frac{1}{|k|^4}$, and hence Summing over $|k|$ we get the Harmonic sum $\int \frac{1}{|k|^2}dk$, which nicely converges.

We end the section by sketching the proof for Eq. 6.87. The state of the four level system (level $1,2,4,5$ in Fig. 6.14B) evolves according to the Schröedinger equation

$$\dot{\psi} = \frac{-i}{\hbar}\begin{bmatrix} E_1 & \Omega_1^* & 0 & 0 \\ \Omega_1 & E_2 & \Omega_2^* & 0 \\ 0 & \Omega_2 & E_2 & \Omega_3^* \\ 0 & 0 & \Omega_3 & E_1 \end{bmatrix}\psi. \tag{6.92}$$

We proceed into the interaction frame of the natural Hamiltonian (system energies) by transformation

$$\phi = \exp(\frac{i\,t}{\hbar}\begin{bmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_1 \end{bmatrix})\psi. \tag{6.93}$$

This gives for $E_{12} = E_2 - E_1$,

$$\dot{\phi} = \underbrace{\frac{-i}{\hbar}\begin{bmatrix} 0 & \exp(-\frac{i}{\hbar}E_{12}\,t)\Omega_1^* & 0 & 0 \\ \exp(\frac{i}{\hbar}E_{12}\,t)\Omega_1 & 0 & \Omega_2^* & 0 \\ 0 & \Omega_2 & 0 & \exp(\frac{i}{\hbar}E_{12}\,t)\Omega_3^* \\ 0 & 0 & \exp(\frac{-i}{\hbar}E_{12}\,t)\Omega_3 & 0 \end{bmatrix}}_{H(t)}\phi. \tag{6.94}$$

$H(t)$ is periodic with period $\Delta t = \frac{2\pi}{E_{12}}$. After Δt, the system evolution is $\phi(\Delta t) =$

$$(I+\int_0^{\Delta t} H(\sigma)d\sigma+\int_0^{\Delta t}\int_0^{\sigma_1} H(\sigma_1)H(\sigma_2)d\sigma_2 d\sigma_1+\int_0^{\Delta t}\int_0^{\sigma_1}\int_0^{\sigma_2} H(\sigma_1)H(\sigma_2)H(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1\ldots)\phi(0). \tag{6.95}$$

The first integral averages to zero, while the second integral doesn't give transition between 1 and 4. The third order does with a contribution

$$\int_0^{\Delta t}\Omega_1\exp(\frac{i}{\hbar}E_{12}\sigma_1)\int_0^{\sigma_1}\Omega_2\int_0^{\sigma_2}\Omega_3\exp(\frac{-i}{\hbar}E_{12}\,\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 = 2\Delta t\frac{\Omega_1\Omega_2\Omega_3}{E_{12}^2}.$$

6.10 Lamb shift

This vertex correction accounts for slight energy shift ~ 1 GHz between $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ energy levels of Hydrogen atom. It was discovered by Willis Lamb in 1951.

6.11 Anomolous magnetic moment of electron

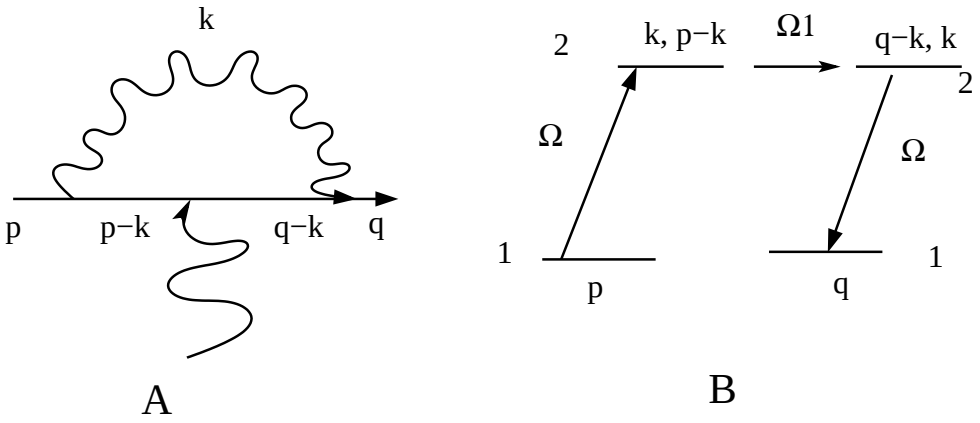


Fig. 6.15 Fig. A shows p electron transiting to $p-k$ by emitting photon, then transiting to $q-k$ by making a Rabi transition and returning bacl to q electron by absorbiting emitted photon. Fig. B shows level diagram for this process where Ω and Ω_1 are vacuum and rabi transitions respectively.

Fig. 6.15 shows the origin to anamolous magnetic moment. Electron with momentum p makes a emission of photon with momentum k and transits to $p-k$ where

it spin flips due to Rabi photon and becomes $q-k$ and then it reabsorbs the emitted photon to return to state q with flipped spin. The spin flip frequency is the Rabi frequency Ω_1 which is modified to to photon emission process and we can say the gyromagnetic ration $\gamma \rightarrow \gamma'$, where $\gamma = g\frac{e}{2m}$, or we can say g changes. To find the change in Ω_1, we have to write the efective transition frequency which is

$$\Omega_{eff} = \frac{\Omega\Omega_1\Omega}{E_k^2},$$

but $\Omega \sim \frac{1}{\sqrt{E_k}}$ and we have the discount $\eta \sim \frac{1}{E_k}$ which gives $\Omega_{eff} = \frac{1}{E_k^4} \propto \frac{1}{|k|^4}$. Summing over $|k|$ we get the Harmonic sum $\int \frac{1}{|k|^2} dk$, which nicely converges.

6.12 Problems

1. Matrices A, B anticommute, if $AB+BA=0$. Show that Dirac matrices $\alpha_1, \alpha_2, \alpha_3, \beta$ all anticommute.

2. A electron velocity is $(v_x, v_y, v_z) = v(\sin\theta, 0, \cos\theta)$. If $v = .9c$, find the two electron spinors with positive energy.

3. In the above problem find the two electron spinors with negative energy.

4. A electron with energy E travelling along z direction, collides with an positron with energy E travelling along $-z$ direction. If $E \gg m_e c^2$, (m_e is rest mass of electron) find the differential cross section $\frac{d\sigma}{d\Omega}$.

5. A photon with wavelength λ moving along z direction collides with an electron at rest and starts traveling with velocity $c(\sin\theta, 0, \cos\theta)$, find the new wavelength λ' of the photon.

Chapter 7
Scattering in Weak Interactions

7.1 Massive Fields

EM photon is not the only photon. EM vacuum is not the only vacuum. We have other photons that mediate so called weak interactions as these photons are heavy and we call them W-Z bosons. In this chapter we develop the theory of W-Z bosons and interactions they mediate.

We equip massive A with a dynamics by defining Lagrangian as density

$$L = \epsilon_0 \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\frac{\mathbf{m}c}{\hbar})^2 A_\mu A^\mu \right). \tag{7.1}$$

We just write

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu. \tag{7.2}$$

where recall

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.3}$$

The energy density of this field is

$$H = -F_{0\mu} F^{0\mu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu. \tag{7.4}$$

Variation of L gives

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{7.5}$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) + m^2 A^\nu = 0 \tag{7.6}$$

Observe

$$\partial_{\mu\nu} F^{\mu\nu} = 0 \tag{7.7}$$

which gives

$$\partial_\mu A^\mu = 0 \tag{7.8}$$

$$\partial_\mu \partial^\mu A^\nu + m^2 A^\nu = 0 \tag{7.9}$$

or

$$\left(\frac{\partial^2}{c^2 \partial t^2} - \nabla^2 + m^2\right) A^\nu = 0 \tag{7.10}$$

Solution is $\epsilon \exp(j(kx - \omega t))$, where

$$k_0 = \frac{\omega}{c} = \sqrt{k^2 + m^2} \tag{7.11}$$

$$\partial_\mu A^\mu = 0 \rightarrow k_\mu \epsilon^\mu = 0 \tag{7.12}$$

Consider field in z direction. There are three independent polarization directions

$$\varepsilon_1 = (0,1,0,0) \tag{7.13}$$
$$\varepsilon_2 = (0,0,1,0) \tag{7.14}$$
$$\varepsilon_3 = \frac{1}{m}(k,0,0,k_0) \tag{7.15}$$

For example, consider a massive photon

$$A\varepsilon_{1,2} \cos(k \cdot z - \omega t), \tag{7.16}$$

propagating in z direction with $\frac{\omega}{c} = \sqrt{k^2 + m^2}$. From 7.4, the energy of this photon is $\frac{\epsilon_0 A^2 \omega^2}{2c^2} V$. Therefore for $\frac{\epsilon_0 A^2 \omega^2}{2c^2} V = \hbar\omega$, we have the photon

$$A = c\sqrt{\frac{2\hbar}{V\epsilon_0 \omega}} \varepsilon_{1,2} \cos(k \cdot z - \omega t) = c\sqrt{\frac{\hbar}{2\epsilon_0 \omega V}} \varepsilon_{1,2} \left(\exp i(k \cdot z - \omega t) + \exp -i(k \cdot z - \omega t) \right). \tag{7.17}$$

Consider the massive photon

$$A\varepsilon_3 \cos(k \cdot z - \omega t), \tag{7.18}$$

propagating in z direction. The energy of this photon is $\frac{\epsilon_0 A^2 m^2}{2} V$. Therefore for $\frac{\epsilon_0 A^2 m^2}{2} V = \hbar\omega$, we have the photon

$$A = \sqrt{\frac{2\hbar\omega}{V\epsilon_0 m^2}} \varepsilon_3 \cos(k \cdot z - \omega t) \sim c\sqrt{\frac{\hbar}{2V\epsilon_0 \omega}} \varepsilon_3 \cos(k \cdot z - \omega t). \tag{7.19}$$

where last approximation true when $k << m$.

7.2 Charged Weak Interaction

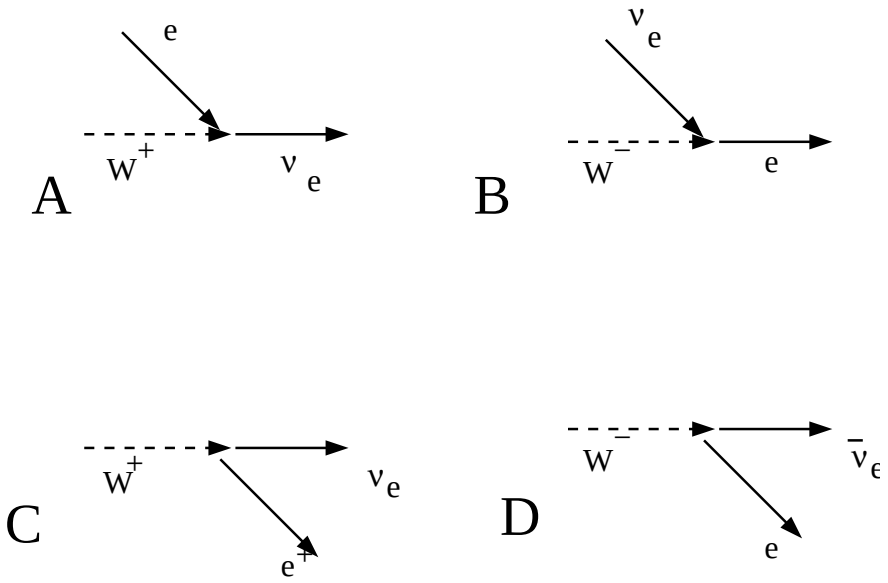


Fig. 7.1 Fig. shows vertices for charged weak interactions

There are two charged massive bosons that mediate charged weak interaction. The Boson W^+ with momentum k takes in a electron of momentum p and emits a neutrino of momentum $p+k$ as shown in Fig. 7.1A. The amplitude for the transition is

$$\Omega = \frac{C}{\sqrt{2m}} \bar{u_2}(p+k)\bar{\gamma}^\nu \epsilon_\nu(k) u_1(p), \tag{7.20}$$

where $C = \frac{\hbar c g_w}{\sqrt{V}}$. Here g_w is weak coupling constant and analogous to $\frac{e}{\sqrt{\epsilon_0}}$ in QED, and

$$\bar{\gamma}^\nu = \gamma^\nu \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} = \gamma^\nu \frac{1-\gamma^5}{2} \tag{7.21}$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. This ensures only left ψ_L of the spinor $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$ takes part in weak interaction. This is called a V-A vertex of weak interaction and arises from parity violation in weak interaction as explained subsequently.

In Fig. 7.1C, the Boson W^+ with momentum k emits a positron of momentum $-p$ and emits a neutrino of momentum $p+k$ as shown in Fig. 7.1A. The amplitude for the transition is

$$\Omega = \frac{C}{\sqrt{2m}} \bar{u_2}(p+k)\bar{\gamma}^\nu \epsilon_\nu(k) v_1(p). \tag{7.22}$$

In Fig. 7.1B we consider Boson W^- instead of W^+. The Boson W^- with momentum k takes in a neutrino of momentum p and emits a neutrino of momentum $p+k$ as shown in Fig. 7.1A. The amplitude for the transition is

$$\Omega = \frac{C}{\sqrt{2m}} \bar{u_2}(p+k)\bar{\gamma}^\nu \epsilon_\nu(k) u_1(p). \tag{7.23}$$

7.3 Inverse Muon Decay

Consider the following process mediated by weak force.

$$e + \nu_\mu \rightarrow \nu_e + \mu \tag{7.24}$$

Electron and muon neutrino with momentum p_1 and p_2 collide to produce electron neutrino and muon at momentum p_3 and p_4. Let k and q denote the on-shell and off-shell momenta of mediator W boson.

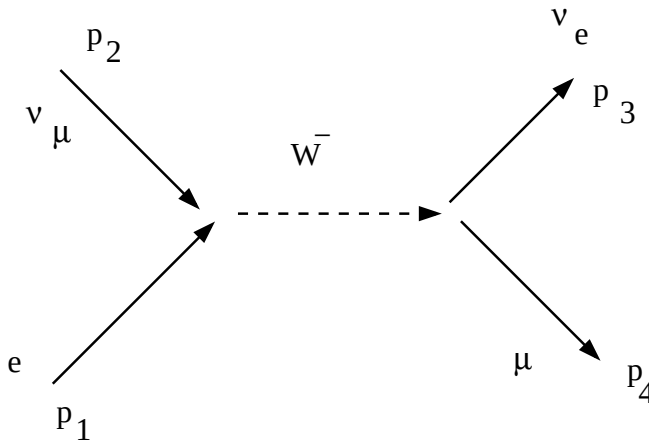


Fig. 7.2 Fig. shows inverse muon decay $e + \nu_\mu \rightarrow \mu + \nu_e$

With

$$\Omega_1 = \frac{C}{\sqrt{2m}}\bar{u}(p_3)\bar{\gamma}^\nu \epsilon^*_\nu(k)u(p_1) \tag{7.25}$$

$$\Omega_2 = \frac{C}{\sqrt{2m}}\bar{u}(p_4)\bar{\gamma}^\mu \epsilon_\mu(k)u(p_2). \tag{7.26}$$

The amplitude for the process

$$\begin{aligned}
\mathcal{M} &= \Omega_1\Omega_2\left(\frac{1}{E_e(p_1)+E_{\nu_e}(p_3)-E_{W^-}(q)} - \frac{1}{E_{W^+}(-q)+E_\mu(p_4)-E_{\mu\nu}(p_2)}\right)\\
&= \Omega_1\Omega_2\left(\frac{1}{E_e(p_1)+E_{\nu_e}(p_3)-E_{W^-}(q)} - \frac{1}{E_{W^+}(-q)+E_e(p_1)-E_{\nu_e}(p_3)}\right)\\
&= \Omega_1\Omega_2\left(\frac{2E_{W^-}(q)}{q^2-m_W^2}\right)\\
&\sim \frac{C^2}{q^2-m_W^2}\ \bar{u}(p_4)\bar{\gamma}^\mu\epsilon_\mu(k)u(p_2)\ \bar{u}(p_3)\bar{\gamma}^\nu\epsilon^*_\nu(k)u(p_1)
\end{aligned}$$

Now we have to sum over the polarization ϵ. Lorentz invariance arguments given in QED dictate that amplitude $\mathcal{M}$ be

$$\mathcal{M} \sim \frac{C^2}{m_W^2}\bar{u}(p_4)\bar{\gamma}^\mu u(p_2)\ \bar{u}(p_3)\bar{\gamma}_\mu u(p_1) \tag{7.27}$$

$$= \frac{C^2}{4m_W^2}\bar{u}(p_4)\gamma^\mu(1-\gamma^5)u(p_2)\ \bar{u}(p_3)\gamma_\mu(1-\gamma^5)u(p_1). \tag{7.28}$$

where we use approximation $q^2 \ll m_W^2$,

Not worrying too much about the spinor part we can just write the cross section in CM frame, where we neglect electron mass, we find, $E_1 = E_2 = E$ and $E_3 \sim E_4 = E$ and $E_4^2 - E_3^2 = m_\mu^2$.

$$l = \frac{\alpha_w \hbar c E}{m_W^2}$$

or more precise taking muon mass into account

$$l = \frac{\hbar c E}{m_W^2}(1 - \frac{m_\mu^2}{E^2}).$$

For collisions at Gev, we get cross section $\sim 10^{-14}$ barn. For those onterested in spinor part,

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{16m_W^2})^2 \frac{1}{E_1E_2E_3E_4} Tr(\not{p}_3\gamma^\mu(1-\gamma^5)(\not{p}_1+m_e)\gamma^\nu(1-\gamma^5)) \times Tr((\not{p}_4+m_\mu)\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)). \tag{7.29}$$

With lot of algebra,

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{m_W^2})^2 \frac{(p_1\cdot p_2)\ (p_3\cdot p_4)}{E_1E_2E_3E_4} \tag{7.30}$$

In The CM frame where we neglect electron mass, we find, $E_1 = E_2 = E$ and $E_3 \sim E_4 = E$ and $E_4^2 - E_3^2 = m_\mu^2$. $E \sim \sqrt{|\mathbf{p}|^2 + m_\mu^2/2}$, where p is the Muon momentum.

$$\sum_s |\mathcal{M}|^2 = (\frac{2C^2}{m_W^2})^2 (1 - \frac{m_\mu^2}{2E^2}) \tag{7.31}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\alpha_w}{m_W^2}\ \hbar c\ E\ (1 - \frac{m_\mu^2}{2E^2})\right)^2 \tag{7.32}$$

7.4 Muon Decay

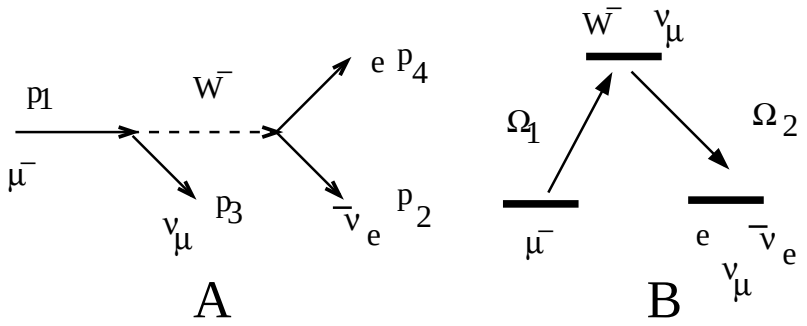


Fig. 7.3 Fig. depicts a muon decay $\mu \to \nu_\mu + e + \bar{\nu}_e$

Fig. 7.4A shows the decay of a muon where the amplitude of the Feynman diagram is In Fig. 7.4B

$$\Omega_1 = \frac{C}{\sqrt{2}m_W} \bar{u}(p_3)\bar{\gamma}^\nu \epsilon_\nu^*(k) u(p_1) \tag{7.33}$$

$$\Omega_2 = \frac{C}{\sqrt{2}m_W} \bar{u}(p_4)\bar{\gamma}^\mu \epsilon_\mu(k) v(p_2). \tag{7.34}$$

$$\begin{aligned}\mathcal{M} &= \Omega_1\Omega_2\left(\frac{1}{E_\mu(p_1)-E_{W^-}(q)-E_{\nu_\mu}(p_3)}-\frac{1}{E_{W^+}(-q)+E_{\bar{\nu}_e}(p_2)+E_e(p_4)}\right)\\ &= \Omega_1\Omega_2\left(\frac{1}{E_\mu(p_1)-E_{W^-}(q)-E_{\nu_\mu}(p_3)}-\frac{1}{E_{W^+}(-k)+E_\mu(p_1)-E_{\nu_\mu}(p_3)}\right)\\ &= \frac{C^2}{4m_W^2}\bar{u}(p_4)\gamma^\mu(1-\gamma^5)v(p_2)\;\bar{u}(p_3)\gamma_\mu(1-\gamma^5)u(p_1).\end{aligned}$$

where last equality follows after polarization sum.

With k as momentum of ν_e and electron momentum as $k+\frac{l}{2}$ as in Fig. (7.4),

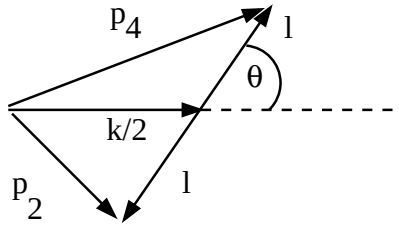


Fig. 7.4 Fig. shows momentum conservation in muon decay

Writing

$$\frac{(\hbar c)^6}{V^2} = \frac{k^2\Delta k}{(2\pi)^3}\frac{l^2\Delta l}{(2\pi)^3}d\Omega_1 d\Omega_2 \tag{7.35}$$

Integrating above over Ω_1 and Ω_2 we get $\frac{1}{8\pi^4}\underbrace{(kl)^2\Delta k\Delta l}_{\Sigma}\,d\theta$. Let $r=\sqrt{k^2+l^2}$ and $\frac{l}{k}=\tan\theta_1$

$$\Sigma(\theta_1) = r^5\cos^2\theta_1\sin^2\theta_1\Delta r\Delta\theta_1 \tag{7.36}$$

$$m_\mu = E = E_2+E_3+E_4 = k+\sqrt{\frac{k^2}{4}+l^2+kl\cos\theta}+\sqrt{\frac{k^2}{4}+l^2-kl\cos\theta} = rf(\theta,\theta_1). \tag{7.37}$$

Decay rate Γ =

$$\pi \frac{\int \sum_s |\mathcal{M}|^2}{\Delta E} = A\ \alpha_w^2\ \frac{m_\mu^5}{m_W^4}. \tag{7.38}$$
$$A = \frac{2}{\pi}\int_0^{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta_1 \sin^2\theta_1}{f^6(\theta,\theta_1)} d\theta_1 d\theta.$$

With $A \sim .01$, $\alpha_w^2 \sim 10^{-3}$ and $m_w \sim 100$ GeV and $m_\mu \sim 100$ MeV, we have decay time $\sim \mu$s.

If we care spinor contribution,

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{16 m_W^2})^2 \frac{1}{E_1 E_2 E_3 E_4} Tr(\not{p}_3 \gamma^\mu (1-\gamma^5)(\not{p}_1 + m_\mu)\gamma^\nu (1-\gamma^5)) \times Tr((\not{p}_4 + m_e)\gamma_\mu (1-\gamma^5)\not{p}_2 \gamma_\nu (1-\gamma^5)). \tag{7.39}$$

With lot of algebra,

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{m_W^2})^2 \frac{(p_1 \cdot p_2)\ (p_3 \cdot p_4)}{E_1 E_2 E_3 E_4} \tag{7.40}$$

$$p_1 \cdot p_2 = E_1 E_2 \tag{7.41}$$
$$p_3 \cdot p_4 = E_3 E_4 (1 + \frac{l\cos\theta + \frac{k}{2}}{\sqrt{\frac{k^2}{4} + l^2 + kl\cos\theta}}) \tag{7.42}$$

$$\frac{(p_1 \cdot p_2)\ (p_3 \cdot p_4)}{E_1 E_2 E_3 E_4} = (1 + \frac{\tan\theta_1 \cos\theta + \frac{1}{2}}{\sqrt{\frac{1}{4} + \tan^2\theta_1 + \tan\theta_1 \cos\theta}}) = g(\theta,\theta_1). \tag{7.43}$$

Decay rate $\Gamma =$

$$\pi \frac{\int \sum_s |\mathcal{M}|^2}{\Delta E} = A\ \alpha_w^2\ \frac{m_\mu^5}{m_W^4}. \tag{7.44}$$
$$A = \frac{2}{\pi}\int_0^{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta_1 \sin^2\theta_1 g(\theta,\theta_1)}{f^6(\theta,\theta_1)} d\theta_1 d\theta$$

7.5 Pion Decay

Consider charged Pion decay as shown in Fig. (7.5).

$$\pi^- \rightarrow \pi_0 + e + \bar{\nu}_e \tag{7.45}$$

It is same a muon decay except now instead of emitting a muon neutrino we emit a neutral pion. However the amplitude of the process is same as in Eq. (7.40)

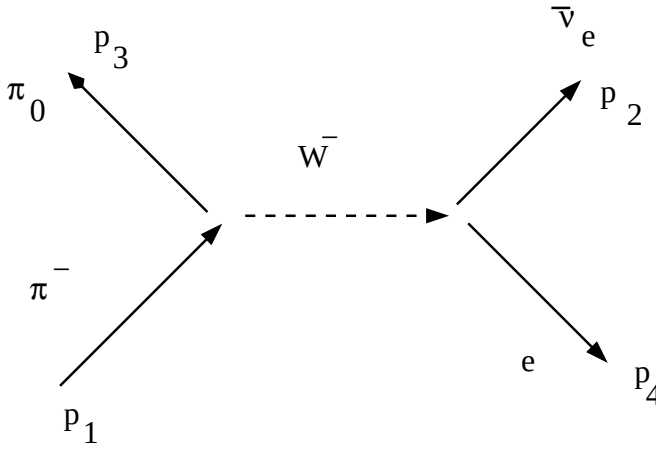


Fig. 7.5 Fig. shows pion decay $\pi^- \to \pi_0 + e + \bar{\nu}_e$

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{m_W^2})^2 (spinor) \tag{7.46}$$

Writing

$$\frac{(\hbar c)^6}{V^2} = \frac{k^2 \Delta k}{(2\pi)^3} \frac{l^2 \Delta l}{(2\pi)^3} d\Omega_1 d\Omega_2 \tag{7.47}$$

$$\Sigma(\theta_1) = r^5 \cos^2\theta_1 \sin^2\theta_1 \Delta r \Delta\theta_1 \tag{7.48}$$

$$m_{\pi^-} = E = E_2 + E_3 + E_4 = m_{\pi_0} + \sqrt{\frac{k^2}{4} + l^2 + kl\cos\theta} + \sqrt{\frac{k^2}{4} + l^2 - kl\cos\theta} = m_{\pi_0} + r f(\theta, \theta_1). \tag{7.49}$$

Decay rate $\Gamma =$

$$\pi \frac{\int \sum_s |\mathcal{M}|^2}{\Delta E} = A\ \alpha_w^2\ \frac{(m_{\pi^-} - m_{\pi_0})^5}{m_W^4}. \tag{7.50}$$

$$A = \frac{2}{\pi} \int_0^{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta_1 \sin^2\theta_1}{f^6(\theta, \theta_1)} d\theta_1 d\theta$$

$A \sim .01$, $\alpha_w^2 \sim 10^{-3}$ and $m_w \sim 100$ GeV and $m_{\pi^-} \sim 139$ MeV, $m_{\pi_0} \sim 135$ MeV, we have decay time ~s.

If we care the spinor part,

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{m_W^2})^2 \frac{(p_1 \cdot p_2)\ (p_3 \cdot p_4)}{E_1 E_2 E_3 E_4} \tag{7.51}$$

$$p_1 \cdot p_2 = E_1 E_2 \tag{7.52}$$

$$p_3 \cdot p_4 = E_3 E_4 (1 + k \frac{l\cos\theta + \frac{k}{2}}{\sqrt{m_e^2 + \frac{k^2}{4} + l^2 + kl\cos\theta}\sqrt{m_{\pi_0}^2 + k^2}}) \tag{7.53}$$

$$\sim E_3 E_4 (1 + \frac{k}{m_{\pi_0}} \frac{l\cos\theta + \frac{k}{2}}{\sqrt{\frac{k^2}{4} + l^2 + kl\cos\theta}}) \tag{7.54}$$

$$\frac{(p_1 \cdot p_2)\ (p_3 \cdot p_4)}{E_1 E_2 E_3 E_4} = (1 + \frac{r\cos\theta_1}{m_{\pi_0}} \frac{\tan\theta_1 \cos\theta + \frac{1}{2}}{\sqrt{\frac{1}{4} + \tan^2\theta_1 + \tan\theta_1 \cos\theta}}) = (1 + \frac{r}{m_{\pi_0}} g(\theta, \theta_1)). \tag{7.55}$$

$$\pi \frac{\int \sum_s |\mathcal{M}|^2}{\Delta E} = A\ \alpha_w^2\ \frac{(m_{\pi^-} - m_{\pi_0})^5}{m_W^4}. \tag{7.56}$$

$$A = \frac{2}{\pi} \int_0^\pi \int_0^{\frac{\pi}{2}} \cos^2\theta_1 \sin^2\theta_1 \frac{1 + \frac{\frac{m_{\pi^-}}{m_{\pi_0}} - 1}{f} g(\theta, \theta_1)}{f^5(\theta, \theta_1)} d\theta_1 d\theta$$

7.6 More Pion Decay

$$\pi^- \to e + \bar{\nu}_e \tag{7.57}$$

$$d + \bar{u} \xrightarrow{W^-} e + \bar{\nu}_e \tag{7.58}$$

π^- is a bound state of d quark and u antiquark. The bound state can be written as sum of states like

$$\phi = \exp(ik \cdot (\frac{r_1 + r_2}{2})) \exp(il \cdot (\frac{r_1 - r_2}{2})) = \exp(ip_1 r_1) \exp(-ip_3 r_2)$$

with different $l's$ as shown in 7.8A corresponding to different θ_1. Then $p_1 = k/2 + l$ and $-p_3 = k/2 - l$. as in 7.8B. For pion at rest $k = 0$. The energy of the pion then is

$$m_\pi = E = \sqrt{m_u^2 + p_1^2} + \sqrt{m_d^2 + p_3^2} = \sqrt{m_u^2 + \frac{k^2}{4} + l^2 + kl\cos\theta_1} + \sqrt{m_d^2 + \frac{k^2}{4} + l^2 - kl\cos\theta_1} \sim 2l \tag{7.59}$$

when $k = 0$.

Let us calculate the decay rate for one configuration $\theta_1 = 0$. The total decay rate then is the average over θ_1 which by symmetry is just as for $\theta_1 = 0$.

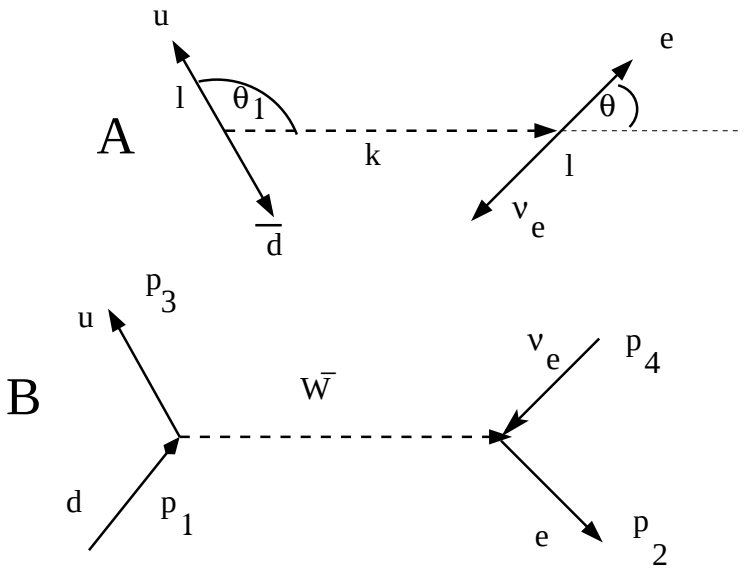


Fig. 7.6 Fig. shows the pion decay $\pi^- \to e + \bar{\nu}_e$

$$\mathcal{M} \sim \frac{C^2}{m_W^2}(spinor) \tag{7.60}$$

$$m_\pi = E = E_3 + E_4 = \sqrt{m_e^2 + \frac{k^2}{4} + l^2 + kl\cos\theta} + \sqrt{\frac{k^2}{4} + l^2 - kl\cos\theta} \sim 2l. \tag{7.61}$$

$$\Delta E = 2\Delta l \tag{7.62}$$

$$\frac{(\hbar c)^3}{V^3} = \frac{l^2 \Delta l}{(2\pi)^3} d\Omega_1 \tag{7.63}$$

Decay rate $\Gamma =$

$$\pi \frac{\int \sum_s |\mathcal{M}|^2}{\Delta E} = \frac{\pi}{2} \frac{\alpha_w^2 m_\pi^2}{m_W^4} \frac{(\hbar c)^3}{V_0} \tag{7.64}$$

(7.65)

With V_0 corresponding to pion radius of 1 fm. We find $\frac{(\hbar c)^3}{V_0} \sim 200(Mev)^3$, and decay time $\sim 10^{-8}$s.

If we care spinors,

$$\mathcal{M} \sim \frac{C^2}{m_W^2} \bar{u}(p_4)\bar{\gamma}^\mu u(p_2)\ \bar{v}(p_3)\bar{\gamma}_\mu u(p_1) \tag{7.66}$$

$$= \frac{C^2}{4m_W^2} \bar{u}(p_4)\gamma^\mu(1-\gamma^5)u(p_2)\ \bar{v}(p_3)\gamma_\mu(1-\gamma^5)u(p_1). \tag{7.67}$$

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{16m_W^2})^2 \frac{1}{E_1E_2E_3E_4} Tr(\not{p}_3\gamma^\mu(1-\gamma^5)(\not{p}_1+m_e)\gamma^\nu(1-\gamma^5)) \times Tr((\not{p}_4+m_\mu)\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)). \tag{7.68}$$

With lot of algebra,

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{m_W^2})^2 \frac{(p_1\cdot p_2)\ (p_3\cdot p_4)}{E_1E_2E_3E_4} \tag{7.69}$$

$$= (\frac{C^2}{m_W^2})^2 (1-\cos^2\theta) \tag{7.70}$$

$$= (\frac{C^2}{m_W^2})^2 \sin^2\theta \tag{7.71}$$

$$\pi \frac{\int \sum_s |\mathcal{M}|^2}{\Delta E} = \frac{\pi}{2} \frac{\alpha_w^2 m_\pi^2}{m_W^4} \frac{(\hbar c)^3}{V_0} \int \sin^2\theta d\theta. \tag{7.72}$$

$$= (\frac{\pi}{2})^2 \frac{(\hbar c)^3}{V_0} \frac{\alpha_w^2 m_\pi^2}{m_W^4} \tag{7.73}$$

7.7 Neutral Weak Interactions

7.7.1 Elastic Neutrino-electron scattering

$$\nu_\mu + e \xrightarrow{Z} \nu_\mu + e \tag{7.74}$$

This is mediated by a Z Boson, as no charge exchange takes place in interaction,

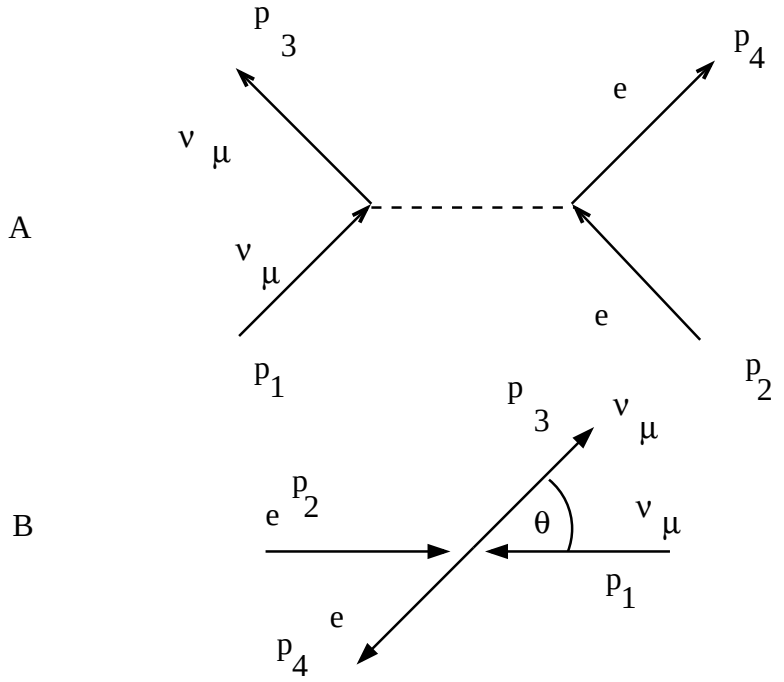


Fig. 7.7 Fig. shows the neutral weak scattering $\nu_e + \mu \rightarrow \nu_e + \mu$

$$\mathcal{M} \sim \frac{C^2}{m_Z^2}(spinor) \tag{7.75}$$

We can just write the cross section as $l = \frac{\alpha_w \hbar c E}{m_Z^2}$, which at GeV energy is 10 femto-barn.

If we care spinor, the Z vertex is bit more complicated as we will see subsequently, we can say, it is combination of left spinor and right spinor,

$$\mathcal{M} \sim \frac{C^2}{4m_Z^2}\bar{u}(p_4)\gamma^\mu(c_V - c_A\gamma^5)u(p_2)\ \bar{u}(p_3)\gamma_\mu(1-\gamma^5)u(p_1). \tag{7.76}$$

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{16m_Z^2})^2 \frac{1}{E_1E_2E_3E_4} Tr((\not{p}_3 + m_e)\gamma^\mu(c_V - c_A\gamma^5)(\not{p}_1 + m_e)\gamma^\nu(c_V - c_A\gamma^5))$$
$$\times Tr(\not{p}_4\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)).$$

With lot of algebra, and E as CM energy

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{2m_Z^2})^2 \frac{((c_A+c_V)^2(p_1\cdot p_2)\ (p_3\cdot p_4) + ((c_A-c_V)^2(p_1\cdot p_4)\ (p_3\cdot p_3) + m_e^2(c_A^2 - c_V^2)(p_1\cdot p_3)}{E_1E_2E_3E_4}$$
$$= (\frac{C^2}{m_Z^2})^2((c_A+c_V)^2 + ((c_A-c_V)^2\cos^4\frac{\theta}{2})$$
$$\frac{d\sigma}{d\theta} = 4\pi(\frac{\alpha_w\hbar cE}{m_Z^2})^2((c_A+c_V)^2 + ((c_A-c_V)^2\cos^4\frac{\theta}{2})$$

7.7.2 Electron Positron scattering

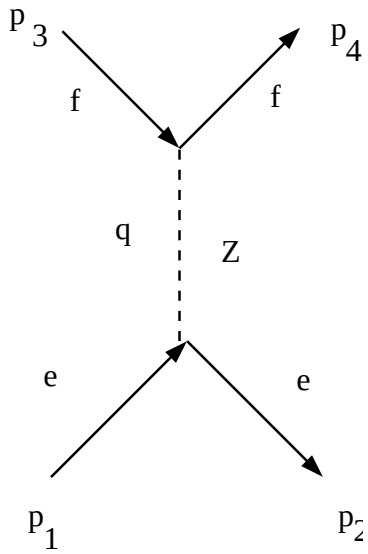


Fig. 7.8 Fig. shows weak electron-positron scattering.

Boson mediated interaction has amplitude,

$$\mathcal{M} = \frac{C^2}{4(q^2 - m_Z^2)}(spinor). \tag{7.77}$$

We can just write the cross section as

$$l = \frac{\alpha_w \hbar c E}{M_z^2},$$

which is as shown before, 10 femto-barn at energy 1 GeV.

If we care all spinors,

$$\mathcal{M} = \frac{C^2}{4(q^2 - m_Z^2)}\bar{u}(p_4)\gamma^\mu(c_V^f - c_A^f\gamma^5)v(p_3)\ \bar{v}(p_2)\gamma_\mu(c_V^e - c_A^e\gamma^5)u(p_1) \tag{7.78}$$

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{16(q^2 - m_Z^2)})^2 \frac{1}{E_1E_2E_3E_4} Tr(\not{p}_4\gamma^\mu(c_V^f - c_A^f\gamma^5)\not{p}_1\gamma^\nu(c_V^f - c_A^f\gamma^5))$$
$$\times Tr(\not{p}_2\gamma_\mu(c_V^e - c_A^e\gamma^5)\not{p}_1\gamma_\nu(c_V^e - c_A^e\gamma^5)).$$

$$\sum_s |\mathcal{M}|^2 = \frac{1}{2}(\frac{C^2}{2(q^2 - m_Z^2)})^2 \frac{1}{E_1E_2E_3E_4}\{\ (\ (c_A^e)^2 + (c_V^e)^2\ )(\ (c_A^f)^2 + (c_V^f)^2)\ [(p_1 \cdot p_2)\ (p_3 \cdot p_4) + (p_1 \cdot p_4)\ (p_2 \cdot p_3)]$$
$$+ 4c_V^e c_A^e c_V^f c_A^f\ [(p_1 \cdot p_2)\ (p_3 \cdot p_4) - (p_1 \cdot p_4)\ (p_3 \cdot p_3)]\ \}$$

In CM frame it reduces to

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{2((2E)^2 - m_Z^2)})^2 \frac{1}{E_1E_2E_3E_4}\{\ (\ (c_A^e)^2 + (c_V^e)^2\ )(\ (c_A^f)^2 + (c_V^f)^2)\ \left[(1 + \cos^2\theta)\right]$$
$$- 8c_V^e c_A^e c_V^f c_A^f \cos\theta\ \}$$

The differential cross-section

$$\sum_s |\mathcal{M}|^2 = (\frac{C^2}{2((2E)^2 - m_Z^2)})^2\{(\ (c_A^e)^2 + (c_V^e)^2\ )(\ (c_A^f)^2 + (c_V^f)^2)\ \left[(1 + \cos^2\theta)\right]$$
$$- 8c_V^e c_A^e c_V^f c_A^f \cos\theta\ \}$$

The differential cross section is

$$\frac{d\sigma}{d\theta} = \pi(\frac{\alpha_w \hbar c E}{(2E)^2 - m_Z^2})^2\{(\ (c_A^e)^2 + (c_V^e)^2\ )(\ (c_A^f)^2 + (c_V^f)^2)\left[(1+\cos^2\theta)\right] - 8c_V^e c_A^e c_V^f c_A^f \cos\theta\ \}$$

7.8 Electroweak Unification, Parity violation and mass

7.8.1 Introduction

Beginning with the seminal work of Yang and Lee [1] and its experimental verification by Wu [2], it is well known that weak interactions do not preserve parity. In the theory of weak interactions, this is manifested by coupling only the left handed components of the fermion doublet. The work of parity violation began with Yang and Lee's observations on K-mesons which led them to question parity conservation in weak interactions. This led them to devise many experiments that would test parity conservation. The first of these was carried out by Wu [2], which confirmed parity violation in weak interactions.

The basic experiment of Wu was β decay of a Cobalt CO^{60} nucleus that had its nuclear spin oriented by Magnetic field along z direction. After β decay the nucleus changed to Ni^{60} by neutron changing to proton. Electron and neutrino were emitted, with both having spin along z direction but electron (relativistic) could have been moving along or opposite to z and it always turned out it wa salong z , which is violation of parity. The explanation now is of-course obvious the vertex has projection on the left spinor so we donot see the alternatively. Fig. 7.9 shows the Cobalt decay experiment of Wu, where electrons are always emitted in one direction.

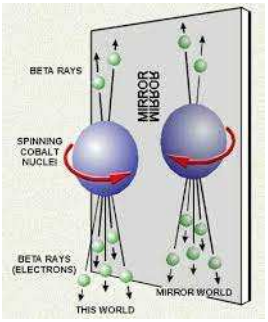


Fig. 7.9 Fig. shows the Cobalt decay experiment of Wu, where electrons are always emitted in one direction.

Further developments in the theory of weak interactions include invention of Higg's mechanism which gives masses to vector bosons and fermions [3, 4, 5] and the theory of electroweak unification [6, 7]. Historical facts suggest that work on

parity violation preceded the work on Higg's mechanism and electroweak unification. In this chapter, we take a different viewpoint. We suggest that parity violation in weak interactions can be predicted on pure theoretical grounds. In this paper, we show that parity violation is a natural consequence of gauge invariance. In a theory where there is no parity violation, we cannot assign masses to fermions in a gauge invariant way using the Higg's mechanism because Higg's field transforms in a quadratic way under gauge transformation. However when we violate parity and only couple the left handed components of the fermions, Higg's field transforms in a linear way under gauge transformation and it becomes possible to give masses to fermions in a gauge invariant manner.

The section is organized as follows. We first review the basics of Higg's mechanism for giving masses to vector bosons and fermions [8, 9, 10]. We then go through the exercise of showing how the theory is gauge invariant, when we have parity violation. Then we work through a theory where there is no parity violation and show we cannot assign masses to fermions in a gauge invariant way using the Higg's mechanism.

7.8.2 Theory

We consider the Higg's doublet

$$\Phi = \begin{bmatrix} \Phi_A \\ \Phi_B \end{bmatrix}. \tag{7.79}$$

The field is coupled to electromagnetic field and W,Z bosons with gauge coupling, with Lagrangian density

$$\mathcal{L}_\Phi = D_\mu \Phi^\dagger D^\mu \Phi - V(\Phi^\dagger \Phi), \tag{7.80}$$

where

$$D_\mu = \partial_\mu + i\frac{g_1}{2} B_\mu + i\frac{g_2}{2} \mathbf{W}_\mu, \tag{7.81}$$

with B_u and $\mathbf{W}_\mu$ the vector potential for EM and Weak interactions respectively and g_1 and g_2 as the corresponding coupling constants and

$$V(\Phi^\dagger \Phi) = \frac{m^2}{2\phi_0^2}[(\Phi^\dagger \Phi) - \phi_0^2]^2, \tag{7.82}$$

where the ground state of the Higg's field is

$$\Phi_{ground} = \begin{bmatrix} 0 \\ \phi_0 \end{bmatrix}, \tag{7.83}$$

and the excited state

$$\Phi = \begin{bmatrix} 0 \\ \phi_0 + \frac{h(x)}{\sqrt{2}} \end{bmatrix}. \tag{7.84}$$

Substituting Φ in Eq. (7.80) gives masses to W, Z bosons and the Higgs boson via gauge coupling.

$$D_\mu \Phi = \begin{pmatrix} 0 \\ \frac{\partial_\mu h}{\sqrt{2}} \end{pmatrix} + i\frac{g_1}{2}\begin{pmatrix} 0 \\ B_\mu(\phi_0 + \frac{h(x)}{\sqrt{2}}) \end{pmatrix} + i\frac{g_2}{2}\begin{pmatrix} \sqrt{2}W_\mu^+(\phi_0 + \frac{h(x)}{\sqrt{2}}) \\ -W_\mu^3(\phi_0 + \frac{h(x)}{\sqrt{2}}) \end{pmatrix}, \tag{7.85}$$

and

$$\mathcal{L}_\Phi = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{g_2^2}{4}(W_\mu^{+\prime} W^{+\mu} + W_\mu^{-\prime} W^{-\mu})(\phi_0 + \frac{h(x)}{\sqrt{2}})^2 + \frac{g_1^2 + g_2^2}{4} Z_\mu Z^\mu (\phi_0 + \frac{h(x)}{\sqrt{2}})^2 - V(h), \tag{7.86}$$

where $W_\mu^+ = \frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}$ and $W_\mu^- = \frac{W_\mu^1 + iW_\mu^2}{\sqrt{2}}$ are the W bosons and

$$Z_\mu = W_\mu^3 \cos\theta_w - B_\mu \sin\theta_w, \tag{7.87}$$

the Z boson and the massless photon

$$A_\mu = W_\mu^3 \sin\theta_w + B_\mu \cos\theta_w, \tag{7.88}$$

where θ_w is the Weinberg angle

$$\cos\theta_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \quad \sin\theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}. \tag{7.89}$$

The field couples to fermions as follows. Let us consider the the neutrino-electron doublet written as a four vector

$$L = \begin{bmatrix} \nu_R \\ \nu_L \\ e_L \\ e_R \end{bmatrix}. \tag{7.90}$$

Using the notation $\sigma_j = (\sigma_x, \sigma_y, \sigma_z)$, $\sigma_u = (\sigma_0, \sigma_x, \sigma_y, \sigma_z)$, and $\tilde{\sigma}_u = (\sigma_0, -\sigma_x, -\sigma_y, -\sigma_z)$, the doublet evolves ($\hbar$ and c are implicit) as $i\frac{dL}{dt} =$

$$\begin{bmatrix} i\partial_j\sigma_j & m_\nu & 0 & 0 \\ m_\nu & -i\partial_j\sigma_j + \frac{1}{2}(g_2 W_\mu^3 - g_1 B_\mu)\sigma_\mu & \frac{g_2}{\sqrt{2}}W_\mu^+\sigma_\mu & 0 \\ 0 & \frac{g_2}{\sqrt{2}}W_\mu^-\sigma_\mu & -i\partial_j\sigma_j - \frac{1}{2}(g_2 W_\mu^3 + g_1 B_\mu)\sigma_\mu & m_e \\ 0 & 0 & m_e & i\partial_j\sigma_j - g_1 B_\mu \tilde{\sigma}_\mu \end{bmatrix} \begin{bmatrix} \nu_R \\ \nu_L \\ e_L \\ e_R \end{bmatrix}. \tag{7.91}$$

where $m_e = c_e(\phi_0 + \frac{h(x)}{\sqrt{2}})$ and $m_\nu = c_\nu(\phi_0 + \frac{h(x)}{\sqrt{2}})$, with c_e, c_ν as coupling of electron and neutrino to Higg's boson.

When we express the above equation in terms of the fields Z_μ, A_μ, it takes the form

$i\frac{dL}{dt} =$

$$\begin{bmatrix} i\partial_j\sigma_j & m_\nu & 0 & 0 \\ m_\nu & -i\partial_j\sigma_j + \frac{e}{\sin 2\theta_w} Z_\mu\sigma_\mu & \frac{g_2}{\sqrt{2}} W^+_\mu\sigma_\mu & 0 \\ 0 & \frac{g_2}{\sqrt{2}} W^-_\mu\sigma_\mu & -i\partial_j\sigma_j - e(A_\mu + \cot 2\theta_w Z_\mu)\sigma_\mu & m_e \\ 0 & 0 & m_e & i\partial_j\sigma_j - e(A_\mu - \tan\theta_w Z_\mu)\tilde{\sigma}_\mu \end{bmatrix} \begin{bmatrix} \nu_R \\ \nu_L \\ e_L \\ e_R \end{bmatrix}. \tag{7.92}$$

where $g_1 \cos\theta_w = g_2 \sin\theta_w = e$, with $-e$, the electron charge.

Now we look at how equations (7.80) and (7.91) transform when we make a Gauge transformation on $\mathbf{W}$ and B. The transformations are for $U \in SU(2)$, we have

$$\mathbf{W}_\mu \to U(x)\mathbf{W}_\mu U^\dagger(x) + \frac{i\partial_\mu U(x) U^\dagger(x)}{g_2/2}, \tag{7.93}$$

$$B_\mu \to B_\mu - \frac{\partial_\mu\theta(x)}{g_1/2}. \tag{7.94}$$

Then the Higg's doublet transforms as

$$\Phi \to \Theta(x)\Phi \tag{7.95}$$

where $\Theta(x) = \exp(i\theta(x))U(x)$.

In terms of field Φ the equation for L takes the form $i\frac{dL}{dt} =$

$$\begin{bmatrix} i\partial_j\sigma_j & c_\nu\Phi^*_A & c_\nu\Phi^*_B & 0 \\ c_\nu\Phi_A & -i\partial_j\sigma_j + \frac{1}{2}(g_2W^3_\mu - g_1B_\mu)\sigma_\mu & \frac{g_2}{\sqrt{2}} W^+_\mu\sigma_\mu & -c_e\Phi^*_B \\ c_\nu\Phi^*_B & \frac{g_2}{\sqrt{2}} W^-_\mu\sigma_\mu & -i\partial_j\sigma_j - \frac{1}{2}(g_2W^3_\mu + g_1B_\mu)\sigma_\mu & c_e\Phi^*_A \\ 0 & -c_e\Phi_B & c_e\Phi_A & i\partial_j\sigma_j - g_1B_\mu\tilde{\sigma}_\mu \end{bmatrix} \begin{bmatrix} \nu_R \\ \nu_L \\ e_L \\ e_R \end{bmatrix}. \tag{7.96}$$

where under the gauge transformation L transforms as

$$\begin{bmatrix} \nu_L \\ e_L \end{bmatrix} \to \exp(-i\theta(x))U(x) \begin{bmatrix} \nu_L \\ e_L \end{bmatrix} \tag{7.97}$$

$$e_R \to \exp(-i2\theta(x))e_R \tag{7.98}$$

In equation (7.91) only e_L and ν_L are coupled. e_R and ν_R are not coupled. That is to say we have parity violation. We now show that this physical law is infact a consequence of the fact that it is not possible to give masses to fermions in a manner that is gauge invariant (as above), if we donot violate parity.

To see this lets reorganize the doublet as

$$M = \begin{bmatrix} \nu \\ e \end{bmatrix}, \quad \nu = \begin{bmatrix} \nu_L \\ \nu_R \end{bmatrix}, \quad e = \begin{bmatrix} e_L \\ e_R \end{bmatrix}. \tag{7.99}$$

$$i\frac{dM}{dt} = \left(\begin{bmatrix} -i\partial_j\alpha_j + \frac{1}{2}(g_2W_\mu^3 - g_1B_\mu)\alpha_\mu & \frac{g_2}{\sqrt{2}}W_\mu^+\alpha_\mu \\ \frac{g_2}{\sqrt{2}}W_\mu^-\alpha_\mu & -i\partial_j\alpha_j - \frac{1}{2}(g_2W_\mu^3 + g_1B_\mu)\alpha_\mu \end{bmatrix} + \underbrace{\begin{bmatrix} m_\nu & 0 \\ 0 & m_e \end{bmatrix}}_{C}\beta \right) \begin{bmatrix} \nu \\ e \end{bmatrix}, \tag{7.100}$$

where, $\beta = \sigma_x \otimes \sigma_0$ and $\alpha_u = \sigma_z \otimes \sigma_\mu$ (α_0 is identity), with σ_μ Pauli matrices. If we plan to write this Eq. (7.100), in terms of Higg's field Φ, then we find that Φ enters the term C above. To make it gauge invariant, this term should be of the form

$$C(\Phi) = \Theta(x) \begin{bmatrix} m_\nu & 0 \\ 0 & m_e \end{bmatrix} \Theta^\dagger(x), \tag{7.101}$$

In the above, $C(\Phi)$ cannot be expressed in terms of Φ alone. The best we can write it is

$$C(\Phi) = \begin{bmatrix} c_\nu\Phi_B^* & c_e\Phi_A \\ -c_\nu\Phi_A^* & c_e\Phi_B \end{bmatrix} \begin{bmatrix} \exp(i\theta(x)) & 0 \\ 0 & \exp(-i\theta(x)) \end{bmatrix} U^\dagger(x), \tag{7.102}$$

which is still not just Φ dependent. Hence when $m_e \neq m_\nu$, we cannot make our equations gauge invariant unless we do a *parity violation*. Therefore parity violation arises as a consequence of gauge invariance.

7.9 Gauge Potential

Energy of Gauge Potential W_μ is

$$W_{\mu\nu} = F_{\mu\nu} + \frac{i}{g}[W_\mu, W_\nu]$$

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$$

$$E = tr\ W_{\mu,\nu}W^{\mu,\nu}$$

If we define gauge transformation as

$$W \rightarrow W' = UWU' - ig\ \partial\mu U\ U^\dagger.$$

Then we have $W'_{\mu,\nu} = UW_{\mu,\nu}U^\dagger$ and energy doesnot take.

Now if we have three photons 1, 2, 3, whose gauge potential forms a $su(2)$ algebra, then the interaction energy density of the three photon can be written as $E = E_1 + E_2$,

$$E = \frac{1}{g} W_3^{\dagger} F_{\mu\nu}^1 W_2. \tag{7.103}$$

If the momentum of three photons add to zero then density integrates to finite interaction energy.

7.10 Renormalizing the W-Z Boson mass

Fig. 7.11 shows a Z photon transit to $W+$ and $W-$ photon, which recombine to give back Z boson. There is correction to the Boson energy due to this Feynmann diagram.

We calculate this in limit $|k|$ becomes large, then in expression for E scale like $\frac{1}{E_k}$, then the total amplitude

$$\mathcal{M} = \eta \frac{E_2^2}{E_k} = \frac{\eta}{E_k^3} \sim \frac{1}{E_K^4},$$

which is summable and renormalizable.

The effective mass of Boson increases as result of this normalization

$$M_Z' > M_z.$$

7.11 Problems

1. Mass of W Boson is $M_W = 80GeV/c^2$. Find its energy if its momentum $p = 1GeV/c$.

2. Mass of Z Boson is $M_Z = 90GeV/c^2$. Find its energy if its wavelength is $1 femtometer$.

3. In inverse muon decay electron and muon neutrino collide at energy of $1Gev$ each along z axis. Find the differential cross section of muon production given mass $m_\mu = 107MeV/c^2$.

4. find the decay rate of a muon to electron and antineutrino , in particular find numerically the coeffecient A discussed in the chapter.

5. Find the decay rate of

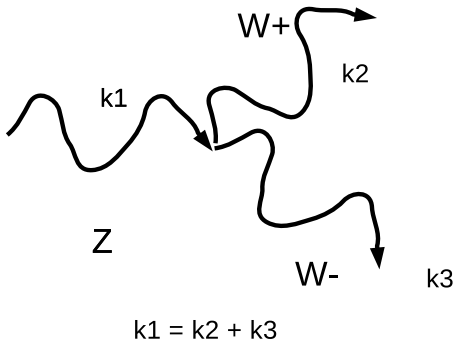


Fig. 7.10 Fig. shows a 3 vertex, a Z photon transit to $W+$ and $W-$ photon, the interaction energy is as in Eq. (7.103)

$$\pi^- \rightarrow \pi^0 + e + \bar{\nu_e}.$$

In particular find numerically the coeffecient A discussed in the chapter. $m_{\pi^-} = 139MeV/c^2$. $m_{\pi^0} = 135MeV/c^2$.

In above take weak fine structure constant $\alpha_w = \frac{1}{29}$.

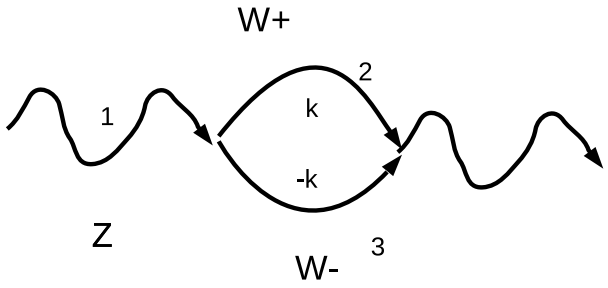


Fig. 7.11 Fig. shows a Z photon transit to $W+$ and $W-$ photon, which recombine to give back Z boson.

Chapter 8
Scattering in Quantum Chromodynamics

8.1 Quarks, color and gluons

The nucleus of a atom is made of protons and neutrons. Protons and neutrons are themselves divisible and composed of elementary particles quarks. Quarks can have charge $\frac{2}{3}e$ as in quarks u, c, t or charge $-\frac{1}{3}e$ as in quarks d, s, b. As $electron, neutrino$ forms a doublet $\begin{pmatrix} e \\ \nu \end{pmatrix}$ which interacts through weak force we have the doublets $\begin{pmatrix} u \\ d \end{pmatrix}$, $\begin{pmatrix} c \\ s \end{pmatrix}$ and $\begin{pmatrix} t \\ b \end{pmatrix}$ forms a doublet which interacts through weak force.

Baryons are composed of three quarks, like proton is uud and neutron udd. Similarly mesons are composed of a quark and anti-quark like pion π^+ is $u\bar{d}$ and π^- is $d\bar{u}$, kaon K^+ is $u\bar{s}$ and K^- is $\bar{u}s$.

Quarks also posses like charge another property called color. Quarks come in three color r, g, b or $red, green, blue$, we write the state of a quarks as

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.1}$$

The color state are states in three dimension space. Transition between then is mediated by six gluons, which can be represented as $SU(3)$ generators , which are $su(3)$ matrices, the eight dimensional space consisting of matrices. The extra two diagonal generators gives two extra gluons making in all 8 gluons.

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

λ_j is what we call Gauge potential, the gluon that transits b to g written as $\bar{b}g$ takes the following form which can be written as a Gauge potential,

Consider the matrix

$$\begin{pmatrix} 0 & \exp(i(kx-\omega t)) & 0 \\ \exp(-i(kx-\omega t)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \cos(kx-\omega t)\lambda_1 + \sin(kx-\omega t)\lambda_2$$

This corresponds to a gluon $\bar{b}r$ with momentum k that makes a transition from b to r by absorbing the gluon or we can transit from r to b by emitting this gluon. In first case the momentum increases by k and in second case decreases by k.

Consider the matrix

$$\cos(kx-\omega t)\lambda_4 + \sin(kx-\omega t)\lambda_5 = \begin{pmatrix} 0 & 0 & \exp(i(kx-\omega t)) \\ 0 & 0 & 0 \\ \exp(-i(kx-\omega t)) & 0 & 0 \end{pmatrix}$$

This corresponds to a quark $\bar{g}r$ with momentum k that makes a transition from g to b by absorbing the gluon or we can transit from r to g by emitting this gluon. In first case the momentum increases by k and in second case decreases by k.

How does this transition take place. The state of the g quark is a spinor $u_1 \exp(ipx)$ with momentum k and let $q = p + k$, then we make transition to red spinor $u_2 \exp(iqx)$, with amplitude that is

$$\mathcal{M} = \frac{C}{\sqrt{2E_k}} \bar{u}_2 \epsilon_\mu \gamma^\mu u_1 \tag{8.2}$$

where $C = \frac{\hbar c g_s}{\sqrt{V}}$ with g_s strong coupling constant and ϵ_μ gluon polarization.

8.2 Quark Quark interaction and Color factor

8.2.1 quark-antiquark interaction

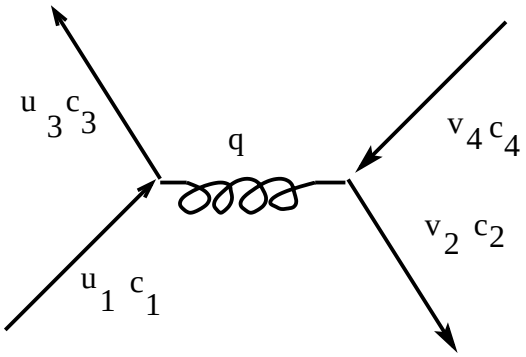


Fig. 8.1 Fig. shows quark-antiquark interaction by gluon exchange.

Let us consider a meson with quark and antiquark pair. Let us hypothesize the color state be say $r\bar{g}$. Fig. 8.1 shows the Feynman diagram for quark-antiquark interaction. c_i are the colors on quarks with $c_1 = c_3 = r$ and $c_2 = c_4 = \bar{b}$. Red scatters to red and blue to blue. This can be accounted by two set of gluons as in λ^3 and λ^8. The resulting amplitude is as in electron-electron scattering

$$\mathcal{M} = C^2 \frac{\bar{u}_3 \gamma^\mu u_1 \bar{v}_4 \gamma_\mu v_2}{q^2} \left(c_1' \lambda^3 c_1 c_2' \lambda^3 c_2 + c_1' \lambda^8 c_1 c_2' \lambda^8 c_2 \right) \tag{8.3}$$

$$f = \left(c_1' \lambda^3 c_1 c_2' \lambda^3 c_2 + c_1' \lambda^8 c_1 c_2' \lambda^8 c_2 \right) = \frac{-1}{3} \tag{8.4}$$

f is called the color factor. This leads to interaction potential

$$V = \frac{-1}{3} \frac{\alpha_s \hbar c}{r} \tag{8.5}$$

Observe the potential is repulsive. The quarks in a meson are infact in the singlet state $\frac{r\bar{r}+g\bar{g}+b\bar{b}}{\sqrt{3}}$. Let us calculate the color factor for this the singlet state. Now we get a color factor we have to calculate $r\bar{r}$ scattering to $r\bar{r}$ and $r\bar{r}$ scattering to $b\bar{b}$ and $g\bar{g}$. The resulting factor is $\frac{-8}{3}$.

This leads to interaction potential

$$V = -\frac{8}{3} \frac{\alpha_s \hbar c}{r} \tag{8.6}$$

Observe the potential is attractive explaining why mesons are in singlet state

8.2.2 triplet state

Let the quark-anti-quark pair be in the triplet state $\frac{r\bar{r}+b\bar{b}-2g\bar{g}}{\sqrt{6}}$, let us calculate the color factor for this the singlet state. Now we get a color factor we have to calculate $r\bar{r}$ scattering to $r\bar{r}$ and $r\bar{r}$ scattering to $b\bar{b}$ and $g\bar{g}$. The resulting factor is $\frac{-}{3}$.

This leads to interaction potential

$$V = -\frac{1}{3}\,\frac{\alpha_s \hbar c}{r} \tag{8.7}$$

8.2.3 quark-quark interaction

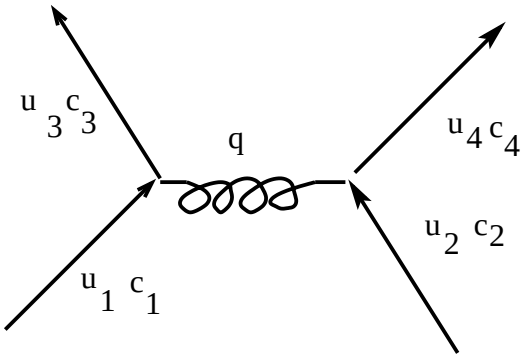


Fig. 8.2 Fig. shows quark-quark interaction with gluon exchange

For Baryons like proton we have the color singlet state

$$\frac{rgb - grb + gbr - bgr + brg - rbg}{\sqrt{6}}$$

. We now have quark quark interaction as shown in Fig. 8.2. Again interaction potential is negative.

8.3 Proton Collisions and Reactions: Pions and Kaons

Protons with enough energy when collide can produce pions. They exchange momentum $p+k$ with gluons, not directly, the gluons create an anti-quark-quark pair with momentum p and k respectively, the quark further exchanges momentum with second proton, resulting in a pion and protons with reduced energy. This is shown

in 8.3. The color of the proton quark changes which can be restored by exchanging a third gluon shown in dotted 8.3.

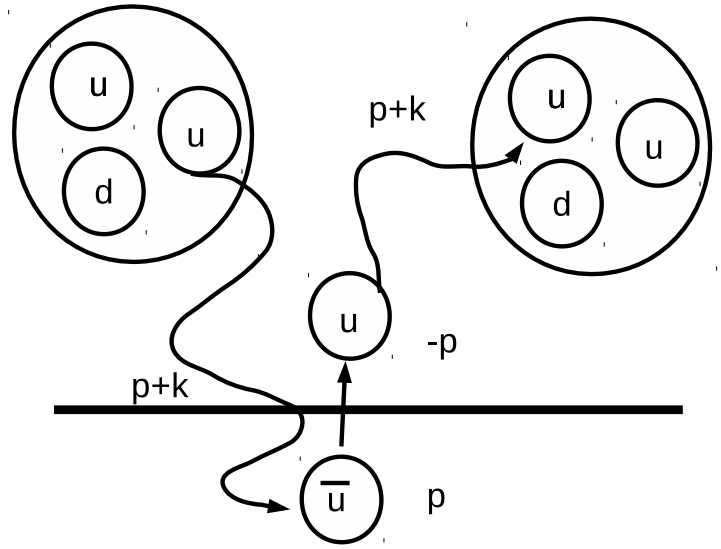


Fig. 8.3 Fig. shows how two protons collide with sufficient energy, exchange a $p+k$ momentum photon, not directly, but by exciting a quark-antiquark pair, the neutral pion, the energy of quark pair is paid by change in momentum of protons.

Protons with enough energy when collide can produce positve-negative pion pairs. They exchange momentum $p+k$ with gluons, not directly, the gluons create an antiquark-quark pair with momentum p and k respectively, the quark further exchanges momentum with second proton, we can repeat the process resulting in two pairs $u\bar{u}$ and $d\bar{d}$ which reassemble to give positive pion π^+ ($u\bar{d}$) and negative pion π^+ ($u\bar{d}$). This is shown in 8.4.

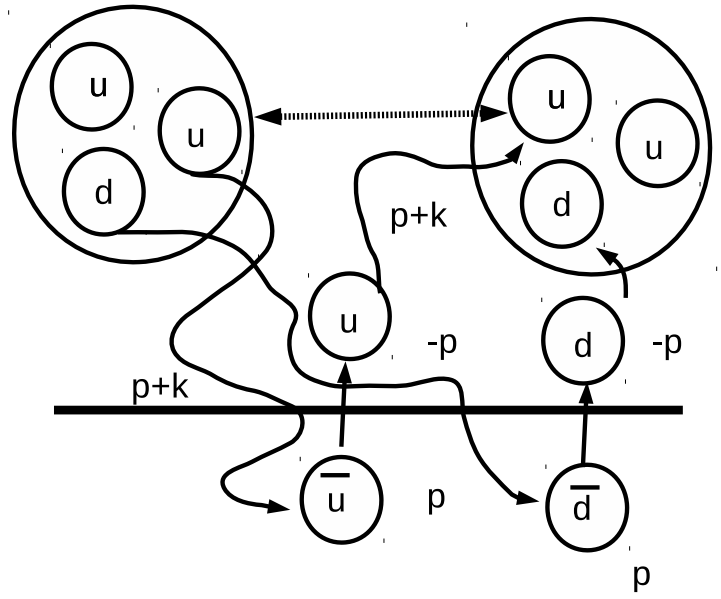


Fig. 8.4 Fig. shows how two protons collide with sufficient energy, exchange a $p+k$ momentum photon, not directly, but by exciting a two quark-antiquark pair, the positive and negative pion, the energy of quark pair is paid by change in momentum of protons.

Protons with enough energy when collide can produce positive-negative pion pairs. They exchange momentum $p+k$ with gluons, not directly, the gluons create an antiquark-quark pair with momentum p and k respectively, the quark further exchanges momentum with second proton, we can repeat the process resulting in two pairs $u\bar{u}$ and $d\bar{d}$ which reassemble to give positive pion K^+ ($u\bar{s}$) and negative kaon K^- ($\bar{u}s$). This is shown in 8.5.

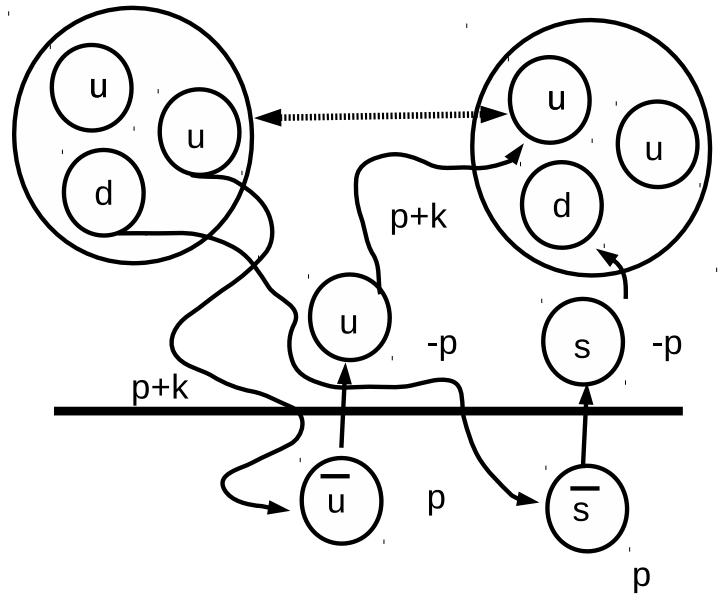


Fig. 8.5 Fig. shows how two protons collide with sufficient energy, exchange a $p+k$ momentum photon, not directly, but by exciting a two quark-antiquark pair, the positive and negative kaon, the energy of quark pair is paid by change in momentum of protons.

8.4 Strong Nuclear Force

Neighboring Protons have quarks which can talk trough gluons, but proton is color neutral so there is no net potential due to this. But by exchanging gluons we can create a pion which can exchange momentum between protons. Shown in 8.7, is how a u blue quark emits a $p+k$ gluon which excites a antiquark-quark pair with momentum p and k and color $\bar{g}$ and b. Color of the emitting proton quark changes to green but the produced quark b can exchange again with other proton quark to make it blue as shown in Fig. 8.7 to produce a quark-antiquark pair that is $g\bar{g}$.

The amplitude of pion exchange $\mathcal{M} \propto \frac{1}{q^2}$, where q is the exchange momentum , which is

$$\mathcal{M} \propto \frac{1}{|k|^2 + m_0^2}$$

, this gives a potential

$$V \propto \frac{exp(-\frac{r}{r_0})}{r},$$

where $r_0 = \frac{\hbar}{m_0 c}$ is the pion Compton wavelength $\sim fm$, then the potential is very short range around fm scale. At around $2fm$ it is $30MeV$. It is attractive.

8.5 Pair production

Pion is $\pi_0 = \frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$, what does it mean ? The quark pair $u\bar{u}$ can annihilate to give a gluon which creates pair $d\bar{d}$, called pair production. The two quark-antiquark states are almost degenerate so we can have a superposition $\pi_0 = \frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$, with smaller energy.

8.6 Asymptotic freedom

In Vacuum polarization in QED, we saw how energy of a exchange photon is modified. Same happens to exchange gluon, in QCD. There are now two kind of vertices as shown in Fig. 8.8, where exchange gluon splits into two gluons (Boson vertex) or quark-antiquark pair (Fermion vertex). We already studies Fermion vertex in QED in chapter 3 and Boson vertex in Gauge potential in weak interaction in chapter 4. We saw amplitude of Vertex B in Fig. 8.8 is negative, QED correction to photon energy is negative and interactions become stronger. We saw amplitude of Vertex A in Fig. 8.8 is positve, QCD correction to photon energy is positive and interactions become weaker. The boson veretx interaction scales $\frac{E_1^2}{E_2}E_3$, if we look at small distances, E_1 is large and hence we have very weak interactions termed asymptotic freedom.

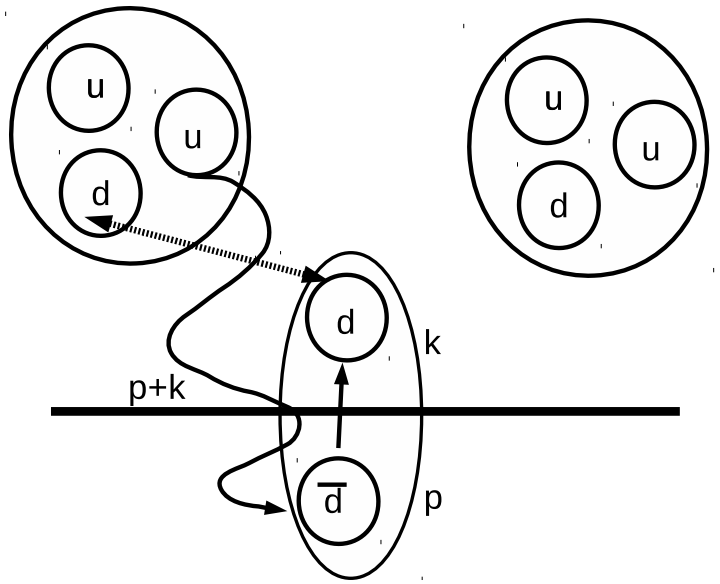


Fig. 8.6 Fig. shows how two protons exchange momentum $p+k$, by emitting and absorbing a pion that mediates a nuclear force.

1. What is quark-antiquark potential at distance of 1 fm is they are in singlet state ?
2. What is quark-antiquark potential at distance of 1 fm is they are in triplet state ?
3. What is energy of outgoing protons if a they collide at 2 GeV each to make a π_0 at rest ?

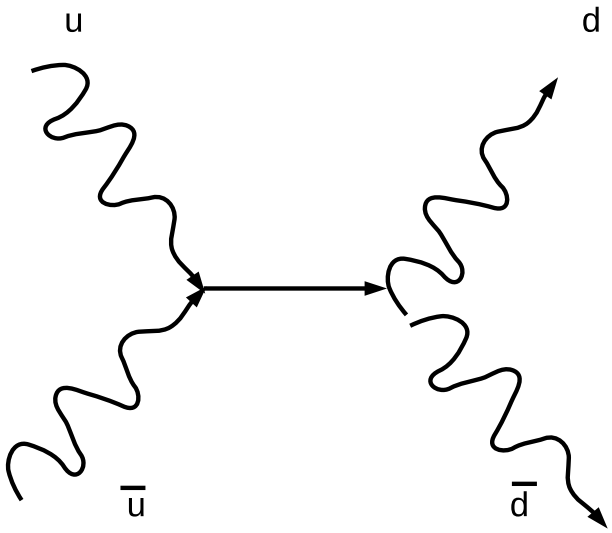
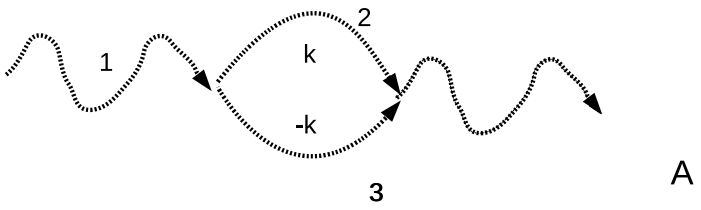


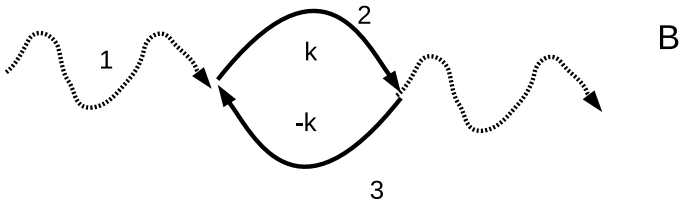
Fig. 8.7 Fig. shows how quark-antiquark $u\bar{u}$ annhilate to create the pair $d\bar{d}$.

4. What is their outgoing momentum in above ?

5. What is energy of outgoing protons if a they collide at 2 GeV each to make a π^+ and π^- at rest ?



Boson Vertex A



Fermion Vertex B

Fig. 8.8 Fig. shows the Boson and the Fermion veretx for energy correction of a Gluon

Chapter 9
Collisions and Scattering: Electron-proton, proton-antiproton, proton-proton

9.1 Electron Proton Scattering

Beautiful electron proton scattering experiments were carried out by Robert Hofstadter in 1950's [62]. These were Electron scattering experiments can be elastic or inelastic (where we excite internal modes of nucleus). Cross-section of scattering sheds light on spatial charge and magnetic moment distribution of the proton (any other nucleus in general).

9.1.1 Elastic scattering

lets first discuss elastic scattering [63] where nucleus internal modes are not excited, so that its mass stays same. This is at electron energies $\ll Mc^2$, (in MeV). Fig. 9.1 depicts how electron scatters of nucleus at certain choice of θ. Let E_0 and E_1 be incident energy of electron and m mass of electron, M mass of nucleus which is at rest, then conserving energy momentum gives

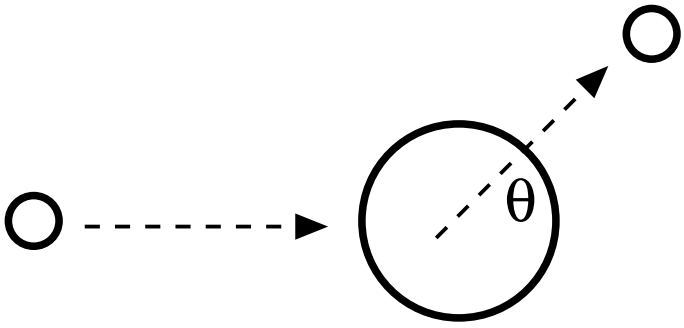


Fig. 9.1 Fig. depicts how electron scatters of nucleus at certain choice of θ.

$$\frac{1}{E_1} - \frac{1}{E_0} = \frac{1-\cos\theta}{Mc^2}, \tag{9.1}$$

when electron energies are relativistic with $E_0 = \frac{hc}{\lambda_0}$ and $E_1 = \frac{hc}{\lambda_1}$, with λ_i de-broglie wavelength's, we have

$$\lambda_1 - \lambda_0 = \frac{h(1-\cos\theta)}{Mc}, \tag{9.2}$$

Proton is three quarks as an approximation equal masses, theta x_1, x_2, x_3 be their coordinates and let $X_1 = \frac{x_1+x_2+x_3}{3}$, $X_2 = x_1 - x_2$ and $X_3 = \frac{x_1+x_2}{2} - x_3$ be Center of Mass and two relative coordinates.

Using $\sum_i k_i x_i = \sum_i K_i X_i$, we have $K_1 = k_1 + k_2 + k_3$, $K_2 = \frac{k_1-k_2}{2}$ and $K_3 = \frac{k_1+k_2-2k_3}{3}$.

When electron transfer momentum q to k_1 we have proton wavefunction ϕ_0 change by

$$\phi_0' = \exp(iqx_1)\phi_0 = \exp(iqX_1)\exp(i\frac{q}{2}X_2)\exp(i\frac{q}{3}X_3)\phi_0$$

The amplitude for this momentum transfer is $\mathcal{M}_0 \propto \frac{ee_1}{q^2\epsilon_0 V}$, where V is electron column and e_1 quark charge. Overlap of new wavefunction with old one is simply

$$\Omega = \langle\phi_0'|\phi_0\rangle = \int \cos(\frac{q}{2}X_2)\phi_0^2 dX_2 \int \cos(\frac{q}{3}X_3)\phi_0^2 dX_3 \sim \frac{1}{q^2}$$

Cross section of scattering $\propto \Omega^2 \propto q^{-4}$, that's it, this is what we find in the experiments, the elastic cross section dies as $\frac{1}{q^4}$ which means there are **three quarks** ! else it will die as $\frac{1}{q^{2n}}$ for n quarks, that's it.

Now lets calculate $\mathcal{M}_0$ in three different regimes when energy E of the incident electron satisfies $E \ll m_e c^2$, this is the limit of elastic scattering.

In this limit, straightforward computation shows for p momentum of the electron, the cross-section

$$l \sim \frac{\alpha\hbar}{p\sin^2\frac{\theta}{2}}, \tag{9.3}$$

its scaled de-Broglie wavelength. More precisely, we get

$$l \sim \frac{\alpha\hbar\sqrt{1+(\frac{v}{c}\cos\frac{\theta}{2})^2}}{p\sin^2\frac{\theta}{2}}, \tag{9.4}$$

For incident and scattered electrons at energy E_1 and E_3 and $E_1 \gg m_e c^2$, we have,

$$l \sim \frac{\alpha\hbar c f(\theta)}{E_1\sin^2\frac{\theta}{2}}\frac{E_3}{E_1}, \tag{9.5}$$

Details of $f(\theta)$ is a spinor excercise.

9.1.2 Deep Inelastic Scattering

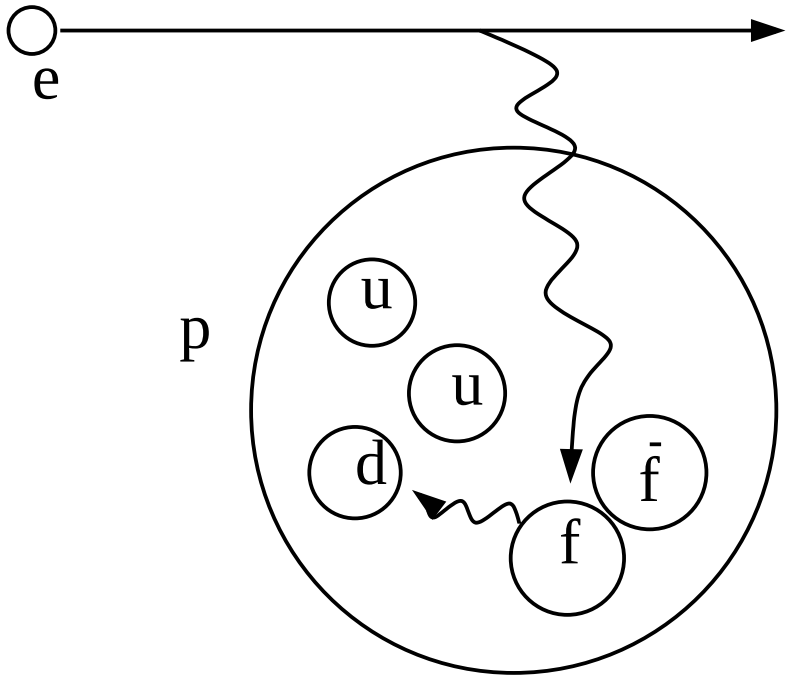


Fig. 9.2 Fig. depicts deep inelastic scattering of electron and proton.

We talked about elastic scattering of electrons and protons. But at high energies we can have in-elastic scattering [64], where by we can excite the internal modes off the proton, such that its internal energy or mass rises from M to W. Since q needed to create new mass is big comparable r_0^{-1} (radius of proton), cross section will be every small, and will die more with increasing q. But we donot need to talk to proton directly. We can first exchange momentum and create a quark-antiquark $f\bar{f}$ pair (meson) as shown in 9.2, this is creating energy , but we cannot just do it in thin air, because we cannot balance energy , we do this by further exchanging momentum with the proton and burning this energy in the heavy mass of proton. Scattering amplitude of $f\bar{f}$ has no q dependence, we are just scattering a free particle, so inelastic cross-section is independent of q for a gives x, where

$$x = \frac{q^2}{q^2 + W^2 - M^2}$$

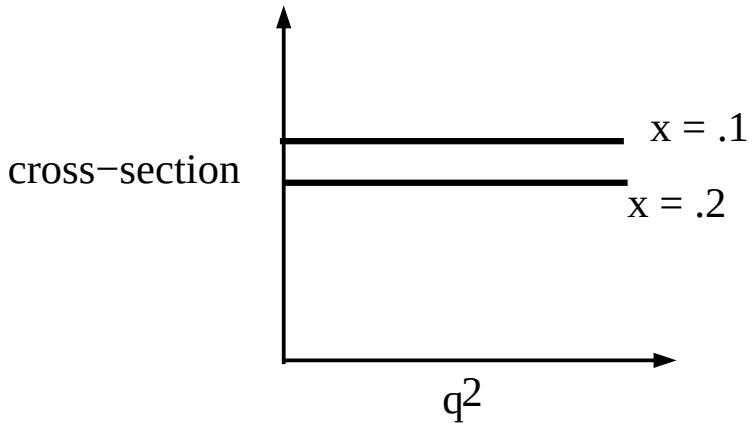


Fig. 9.3 Fig. depicts inelastic cross section as a function of q^2 for given x.

9.2 Discovery of W-Z Bosons

The $W-Z$ Boson predicted by Weinberg, Salam and Glashgow (1968) were discovered in CERN lab in (1983) by group of Carlo Rubia. They collided prodons (uud) and antiprotons ($\bar{u}\bar{u}\bar{d}$) to give $u\bar{d} \rightarrow W^+$ and $\bar{u}d \rightarrow W^-$ Bosons, which decayed to positrons and electrons respectively, which were detected. Masses were estimated to around 80 GeV. The $u\bar{u} \rightarrow Z$ was also detected by its decay to electron-positron pair, with Mass estimated to 90 GeV. Clearly the Center of Mass energy for these measurements is around $90 \times 3 \sim 270$ GeV.

9.3 Discovery of Higgs Boson

Higgs boson was discovered at CERN in 2013 in two big projects ATLAS and CIMMS. Protons at Teravolt energies collide to generate top quark-antiquark pairs which recombine to give Higgs boson as in 9.4. Fig. 9.4 shows how each proton produces a Higgs pair and the two opposite momentum Higgs from colliding protons can then give W^+ and W^- Bosons respectively which can combine to form Z boson which readily decays intro electron-positron pair.

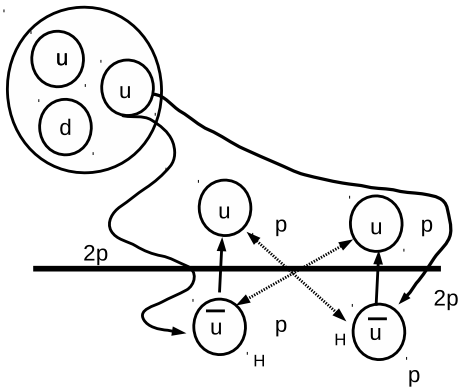


Fig. 9.4 Fig. shows Higgs production from Proton collisions.

9.4 Problems

1. Calculate the approximate (without spinor sum) differential cross-section of electron-positron collision at 1 GeV each.

2. Calculate the approximate differential cross-section of electron-positron collision at 1 GeV each to produce muon-antimuon pair.

3. Calculate the approximate differential cross-section of $\nu_e + \mu \rightarrow \nu_\mu + e$ collision at 1 GeV each.

4. Calculate the approximate differential cross-section of $\nu_e + \mu \rightarrow \nu_e + \mu$ collision at 1 GeV each.

5. Calculate scattering cross-section of electrons of protons when electron energy is 10 KeV.

References

1. T. D. Lee and C. N. Yang, Phys. Rev., 104 (1956) 254.
2. C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, Phys. Rev., 105 (1957) 1413.
3. Peter, Higgs, " Broken Symmetries and the Masses of Gauge bosons". Physical Review Letters, 13 (16) (1964).
4. Francois Englert, Robert Brout, " Broken Symmetry and the Mass of Gauge Vector Mesons". Physical Review Letters. 13 (9), 321-23 (1964).
5. P.W. Anderson, Phys. Rev. 130, 439 (1963).
6. A. Salam, J.C. Ward, " Weak and electromagnetic interactions". Nuovo Cimento, 11 (4): 568-577 (1959).
7. S. Weinberg " A Model of Leptons" Phys. Rev. Lett, 19: 1264-66 (1967).
8. W. A. Cottingham, and D.A. Greenwood, "An introduction to standard model of particle physics", Cambridge Univ. Press, (2007).
9. M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory", Westview Press, (1995).
10. D. Griffiths, "Introduction to Elementary Particles", Wiley-VCH, (2004).
11. Mark Thomson, "Modern particle Physics", Cambridge University Press (2013).
12. David J. Griffiths, "Introduction to Quantum Mechanics"
13. Matthew Schwartz, " Quantum Field Theory and the Standard Model", Cambridge University Press (2014)
14. Compton, Arthur H. (May 1923). "A Quantum Theory of the Scattering of X-Rays by Light Elements". Physical Review. 21 (5): 483¡V502.
15. Hofstadter, Robert, " Electron Scattering and Nuclear Structure", Rev. Mod. Phys. 28, 214–254 (1956).
16. Amsler, Claude (2014). " Deep inelastic electron-proton scattering". Nuclear and Particle Physics.
17. Max Planck, "On the Theory of the Energy Distribution Law of the Normal Spectrum", M. Planck, Verhandl. Dtsch. phys. Ges., 2, 237 (1900).
18. A. Einstein, "On the heuristic point of view of production and transformation of light" Annalen der Physik, vol. 17, 1905.
19. Bohr, Niels. "The Theory of Spectra and Atomic Constitution; three essays." Cambridge: Cambridge University Press (1922).
20. De Broglie, L., "A tentative theory of light quanta", Phil. Mag., 47 (1924), 446–458.
21. C. Davisson and L. H. Germer, " Diffraction of Electrons by a Crystal of Nickel", Phys. Rev. 30, 705 (1927).
22. W. Heisenberg, "Uber quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen,"Z. Phys. 33, 879–893 (1925).
23. M. Born and P. Jordan, "Zur Quantenmechanik," Z. Phys. 34, 858–888 (1925).
24. M. Born, W. Heisenberg, and P. Jordan, "Zur Quantenmechanik II," Z. Phys. 35, 557–615 (1926).
25. Erwin Schrödinger, "Quantisation as a Problem of Proper Values (Part I)", Annalen der Physik (4), vol. 79, 1926.
26. Erwin Schrödinger, "Quantisation as a Problem of Proper Values (Part II)", Annalen der Physik (4), vol. 79, 1926.
27. Erwin Schrödinger, "On the Relation between the Quantum Mechanics of Heisenberg, Born, and Jordan, and that of Schrodinger", Annalen der Physik (4), vol. 79, 1926.
28. Erwin Schrödinger, "The Continuous Transition from Micro- to Macro-Mechanics", Die Naturwissenschaften, 28, pp. 664-666, 1926.
29. P. A. M. Dirac, "The fundamental equations of quantum mechanics", Proc. R. Soc. London, Ser. A 109, 642–653 (1926).

30. Claude Cohen-Tannoudji, Bernard Diu, Frank Laloe, “Quantum Mechanics”, Vol. 1 and 2 , First Edition, Wiley (1991).
31. J. J. Sakurai and Jim Napolitano, “Modern Quantum Mechanics”, 3rd edition, Cambridge University Press (2020).
32. David J. Griffiths, “Introduction to Quantum Mechanics”, Pearson Prentice Hall (2004).
33. Claude Cohen-Tannoudji, Jacques Dupont-Roc, Gilbert Grynberg, “Atom Photon Interaction”, Wiley, 1992.
34. Marlan O. Scully, M. Suhail Zubairy, “Quantum Optics”, Cambridge University Press (1997).
35. Mark Thomson, “Modern particle Physics”, Cambridge University Press (2013).
36. David J. Griffiths, “Introduction to Quantum Mechanics”, Pearson Prentice Hall (2004).
37. Eberhard Haug; Werner Nakel (2004). The elementary process of bremsstrahlung. Scientific Lecture Notes in Physics. Vol. 73. River Edge, NJ: World Scientific.
38. Harold J. Metcalf and Peter van der Straten, “Laser cooling and trapping”, Springer (1999).
39. Young, Andrew T (1981). “Rayleigh scattering”. Applied Optics. 20 (4): 533–5.
40. C.V. Raman “The scattering of light by anisotropic molecules”, Nature, 1923, Vol.112, p165-166.
41. C.V. Raman, “The scattering of light by liquid and solid surfaces”, Nature, 1923, Vol.112, p281-282.
42. Strommen, Dennis P; Nakamoto, Kazuo (1977). “Resonance raman spectroscopy”. Journal of Chemical Education. 54 (8): 474.
43. Goeppert-Mayer M (1931). “Über Elementarakte mit zwei Quantensprüngen”. Annals of Physics. 9 (3): 273–95.
44. Kaiser, W.; Garrett, C. G. B. (1961). “Two-Photon Excitation in CaF2:Eu2+”. Physical Review Letters. 7 (6): 229.
45. Shaowei Li, Yanping Li, Rongxing Yi, Liwei Liu and Junle Qu, “Coherent Anti-Stokes Raman Scattering Microscopy and Its Applications”, Front. Phys., 17 December 2020.
46. Shen YR, Bloembergen N, “Theory of stimulated Brillouin and Raman scattering”, Phys. Rev. A. 1965 ; 137:1787–1799.
47. D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, “Quantum dynamics of single trapped ions” Rev. Mod. Phys. 75, 281 (2003).
48. Nicholas Bloembergen, “Nonlinear Optics”, World Scientific, 4 ed., (1996).
49. A. Einstein, “On the heuristic point of view of production and transformation of light” Annalen der Physik, vol. 17, 1905.
50. Compton, Arthur H. (May 1923). "A Quantum Theory of the Scattering of X-Rays by Light Elements". Physical Review. 21 (5): 483–502.
51. Jacques I. Pankove, “Optical Processes in Semiconductors”, Dover, (1976).
52. Franck, J.; Hertz, G. (1914). “ Über Zusammenstöße zwischen Elektronen und Molekülen des Quecksilberdampfes und die Ionisierungsspannung desselben”
53. Franck, James (1965). “ Transformation of Kinetic Energy of Free Electrons into Excitation Energy of Atoms by Impacts”. Nobel Lectures, Physics 1922–1941. Elsevier. Translation of Franck’s Nobel lecture that he gave December 11, 1926.
54. Eberhard Haug; Werner Nakel (2004). The elementary process of bremsstrahlung. Scientific Lecture Notes in Physics. Vol. 73. River Edge, NJ: World Scientific.
55. Donald A. Glaser “Some Effects of Ionizing Radiation on the Formation of Bubbles in Liquids”. Physical Review. 87 (4): 665 (1952).
56. Das Gupta, N. N.; Ghosh S. K. (1946). “ A Report on the Wilson Cloud Chamber and its Applications in Physics “. Reviews of Modern Physics. 18 (2): 225–365.
57. Anderson, C. D. (1933). “ The Positive Electron”. Physical Review. 43 (6): 491–494.
58. C. Davisson and L. H. Germer, “ Diffraction of Electrons by a Crystal of Nickel”, Phys. Rev. 30, 705 (1927).
59. Knoll, M.; Ruska, E. (1932). “ Beitrag zur geometrischen Elektronenoptik. I “. Annalen der Physik. 404 (5): 607–640.
60. Shull, Clifford G. (1995-10-01). “ Early development of neutron scattering “ . Reviews of Modern Physics. American Physical Society (APS). 67 (4): 753–757.

61. Seegar RJ. and Teller E. On the Inelastic Scattering of Neutrons by Crystal Lattices. Phys. Rev. 62:37 - 40, 1942.
62. Hofstadter, Robert, " Electron Scattering and Nuclear Structure", Rev. Mod. Phys. 28, 214–254 (1956).
63. Mark Thomson, "Modern Particle Physics", Cambridge University Press, 2013.
64. Amsler, Claude (2014). " Deep inelastic electron-proton scattering". Nuclear and Particle Physics.
65. C. Kittel, "Introduction to Solid State Physics", 8th Edition, John Wiley and Sons 2005.
66. N. W. Ashcroft and D. Mermin, "Solid State Physics", Harcourt College Publishers, 1976.
67. Steve Simon, "Oxford Solid State Basics", Oxford U. Press, 2013.
68. C. Kittel and H. Kroemer, "Thermal Physics", Freeman and Co., (2002).
69. M. Tinkham, "Introduction to Superconductivity", 2nd edition, McGraw Hill 1996.
70. P. G. De Gennes, "Superconductivity of metals and alloys", W.A. Benjamin, Inc. 1966.
71. Kamerlingh Onnes, H., "Further experiments with liquid helium. D. On the change of electric resistance of pure metals at very low temperatures, etc. V. The disappearance of the resistance of mercury." Comm. Phys. Lab. Univ. Leiden; No. 122b, 1911.
72. John Bardeen, Leon Cooper, J. R. Schriffer "Theory of Superconductivity", Physical Review. 108 (5): 1175 (1957).
73. G. Bednorz and K. A. Müller (1986). "Possible high Tc Superconductivity in the Ba-La-Cu-O System". Z. Phys. B. 64 (1): 189–193.
74. B. D. Josephson (1962). "Possible new effects in superconductive tunnelling". Phys. Lett. 1 (7): 251–253.
75. chulz, H. J. (March 1995). "Fermi liquids and non–Fermi liquids". In "proceedings of les Houches Summer School Lxi", ed. E. Akkermans, G. Montambaux, J. Pichard, et J. Zinn-Justin (Elsevier, Amsterdam. 1995 (533).
76. Levine, Raphael (2005). Molecular Reaction Dynamics. Cambridge: Cambridge University Press.
77. Dutton, Zachary, Ginsberg, Naomi S, Slowe Christopher, Hau, Lene Vestergaard (1 March 2004). "The art of taming light: ultra-slow and stopped light". Europhysics News. 35 (2): 33–39.
78. Sørensen, Anders; Mølmer, Klaus (March 1, 1999). "Quantum Computation with Ions in Thermal Motion". Physical Review Letters. 82 (9): 1971–1974.
79. "Continuous-wave room-temperature diamond maser", Jonathan D. Breeze, Enrico Salvadori, Juna Sathian, Neil McN. Alford Christopher W. M. Kay , Nature, volume 555, pages493–496 (2018).
80. Hell, S. W.; Wichmann, J. (1994). "Breaking the diffraction resolution limit by stimulated emission: Stimulated-emission-depletion fluorescence microscopy". Optics Letters. 19 (11): 780–782.
81. Mansuripur, Masud. "The Faraday Effect". Optics and Photonics News (10): 32–36. Retrieved 15 June 2014.
82. Malcolm Levitt, "Spin Dynamics: Basics of Magnetic Resonance", Wiley, 2008.
83. Braginsky, V. (1980). "Quantum Nondemolition Measurements". Science. 209 (4456): 547–557.