Analysis of orbits arising in piecewise-smooth discontinuous maps

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Why piecewise-smooth discontinuous maps?

- Makes an appearance in various applications in electrical engineering and physics
Why piecewise-smooth discontinuous maps?

- Makes an appearance in various applications in electrical engineering and physics

Examples include:

- Controlled buck converter
- Boost converter in discontinuous mode
- Impact oscillators
1D linear piecewise smooth map

\[ x_{n+1} = f(x_n, a, b, \mu, \ell) = \begin{cases} 
ax_n + \mu & \text{for } x_n \leq 0 \\
bx_n + \mu + \ell & \text{for } x_n > 0 
\end{cases} \]

- The jump discontinuity is at \( x = 0 \)
- \( a \) and \( b \) are the slopes of the affine maps on either side of discontinuity
- \( \ell \) is the height of the "jump"
- \( \mu \) is the parameter to be varied
Our interest

- Do (periodic) orbits exist for such systems?
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- Do (periodic) orbits exist for such systems?
- If yes, then can these orbits be characterized, classified · · ·
The settings

Assumption: Let $0 < a < 1$ and $0 < b < 1$
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Equilibrium point in the left half $x_L = \frac{\mu}{1-a}$  
Equilibrium point in the right half $x_R = \frac{\mu+\ell}{1-b}$
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If the “jump” $\ell > 0$, then

No chance of a periodic orbit !!
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Assumption: Let $0 < a < 1$ and $0 < b < 1$

If the “jump” $\ell < 0$, then

Orbits can exist if $0 < \mu < -\ell$

Set $\ell = -1$ and therefore $0 < \mu < 1$
Some definitions

- Let $f$ be a map from $\mathbb{R}$ to $\mathbb{R}$. $p$ is a periodic point of order $k$ if $f^k(p) = p$, where $k$ is the smallest such positive integer.
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- Given a particular sequence of points $\{x_n\}_{n\geq 0}$ through which the system evolves, one can code this sequence into a sequence of $\mathcal{L}$s and $\mathcal{R}$s

- A periodic orbit has a string of $\mathcal{L}$s and $\mathcal{R}$s that keeps repeating. This repeating string is a **pattern** and denoted by $\sigma$
Some more definitions

- Length of the string $\sigma$ is denoted by $|\sigma|$ and gives the period of the orbit
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- A pattern consisting of a string of $L$'s followed by a string of $R$'s is called a prime pattern

- $LLLRRR$, $L^nR$, $LR^n$ are prime patterns
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- A pattern consisting of a string of $L$'s followed by a string of $R$'s is called a prime pattern.

- $L^n R$ is a $L$-prime pattern.

- $L R^n$ is a $R$-prime pattern.
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- $P_\sigma$ denotes the interval of parameter $\mu$ for which an orbit with pattern $\sigma$ exists

- A pattern consisting of a string of $L$s followed by a string of $R$s is called a **prime pattern**

- $L^nR$ is a $L$-prime pattern

- $L R^n$ is a $R$-prime pattern

- A pattern made up of two or more prime patterns is a **composite pattern**
Prime patterns

Theorem

For $a, b \in (0, 1)$, $L$-prime and $R$-prime patterns of any length are admissible.
Prime patterns

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For $a, b \in (0, 1)$, $L$-prime and $R$-prime patterns of any length are admissible

Consider the pattern $L^n R$. The length of this pattern is $n + 1$. From the map, one gets the following inequalities:

\[
\begin{align*}
x_0 & \leq 0, \\
x_1 &= ax_0 + \mu \leq 0, \\
x_2 &= ax_1 + \mu \leq 0, \\
&= a^2 x_0 + (a + 1)\mu \leq 0, \\
&\vdots \\
x_{n-1} &= a^{n-1} x_0 + \mu S_{n-2}^a \leq 0, \\
x_n &= a^n x_0 + \mu S_{n-1}^a > 0, \\
x_{n+1} &= x_0 = a^n bx_0 + (b S_{n-1}^a + 1)\mu - 1 \leq 0.
\end{align*}
\]
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For $a, b \in (0, 1)$, $L$-prime and $R$-prime patterns of any length are admissible

Therefore, $x_0 = \frac{(bS_n^a-1+1)\mu-1}{1-anb}$
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Substituting this $x_0$ into the inequalities give us inequalities that $\mu$ should satisfy
Prime patterns

Theorem

For \( a, b \in (0, 1) \), \( \mathcal{L} \)-prime and \( \mathcal{R} \)-prime patterns of any length are admissible.

Therefore, \( x_0 = \frac{(bS_n^{a-1}+1)\mu - 1}{1-a^n b} \)

Substituting this \( x_0 \) into the inequalities give us inequalities that \( \mu \) should satisfy.

Every \( \mathcal{L} \) in the pattern gives an upper bound for \( \mu \)

Every \( \mathcal{R} \) in the pattern gives a lower bound for \( \mu \)
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Therefore, $x_0 = \frac{(bS_{n-1}^a + 1)\mu - 1}{1 - anb}$

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Every $\mathcal{L}$ in the pattern gives an upper bound for $\mu$

Every $\mathcal{R}$ in the pattern gives a lower bound for $\mu$

$$P_{\mathcal{L}^n \mathcal{R}} = \begin{pmatrix} \frac{a^n}{S_n^a} & \frac{a^{n-1}}{a^{n-1}b + S_{n-1}^a} \end{pmatrix}$$
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Theorem

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Therefore, $x_0 = \frac{(bS_{n-1}^a + 1)\mu - 1}{1 - a^n b}$

Substituting this $x_0$ into the inequalities give us inequalities that $\mu$ should satisfy.

Every $\mathcal{L}$ in the pattern gives an upper bound for $\mu$.

Every $\mathcal{R}$ in the pattern gives a lower bound for $\mu$.

$$P_{\mathcal{L}^n \mathcal{R}} = \left[ \frac{a^n}{S_n^a}, \frac{a^{n-1}}{a^{n-1}b + S_{n-1}^a} \right]$$

Showing that $P_{\mathcal{L}^n \mathcal{R}} \neq \emptyset$ does the job.
Some more questions

- Are $L$-prime patterns and $R$-prime patterns the only prime patterns that are admissible?
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- Are prime patterns the only kind of patterns? For example, can there be a pattern like $LLLRRLLRLLLR$?
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- For a given $n$, how many distinct patterns exist with period $n$?
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- Are prime patterns the only kind of patterns? For example, can there be a pattern like $LLLRRRLRRLLRR$?
- Can we characterize all the possible types of admissible patterns?
- For a given $n$, how many distinct patterns exist with period $n$?
- Is there an algorithm that generates only the possible admissible patterns of period $n$?
Composite patterns

Theorem

For \( a, b \in (0, 1) \), no admissible pattern can contain consecutive \( \mathcal{L} \)s and consecutive \( \mathcal{R} \)s simultaneously.
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For \( a, b \in (0, 1) \), no admissible pattern can contain consecutive \( \mathcal{L} \)s and consecutive \( \mathcal{R} \)s simultaneously.

- For \( \mu < \frac{1}{b+1} \), every \( \mathcal{R} \) is immediately followed by \( \mathcal{L} \)
Composite patterns

Theorem

For $a, b \in (0, 1)$, no admissible pattern can contain consecutive $L$s and consecutive $R$s simultaneously.

- For $\mu < \frac{1}{b+1}$, every $R$ is immediately followed by $L$
- For $\mu > \frac{a}{a+1}$, every $L$ is immediately followed by $R$
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For $a, b \in (0, 1)$, no admissible pattern can contain consecutive $\mathcal{L}$s and consecutive $\mathcal{R}$s simultaneously.

- For $\mu < \frac{1}{b+1}$, every $\mathcal{R}$ is immediately followed by $\mathcal{L}$
- For $\mu > \frac{a}{a+1}$, every $\mathcal{L}$ is immediately followed by $\mathcal{R}$
- For $a, b \in (0, 1)$, $\frac{a}{a+1} < \frac{1}{b+1}$

QED
Composite patterns

Theorem

For $a, b \in (0, 1)$, no admissible pattern can contain consecutive $L$'s and consecutive $R$'s simultaneously.

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QED

Similar limits can be found for runs of $n$ symbols
Composite patterns

Lemma

For \(a, b \in (0, 1)\), all the admissible composite patterns are made up of either \(L\)-prime patterns or \(R\)-prime patterns but not both. Every composite pattern is a combination of exactly two prime patterns of successive lengths.
Lemma

For $a, b \in (0, 1)$, all the admissible composite patterns are made up of either $\mathcal{L}$-prime patterns or $\mathcal{R}$-prime patterns but not both. Every composite pattern is a combination of exactly two prime patterns of successive lengths.
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Theorem

For $a, b \in (0, 1)$, and any $n$, there exists $\phi(n)$ distinct admissible patterns of cardinality $n$, where $\phi$ is the Euler’s totient function.
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For $a, b \in (0, 1)$, and any $n$, there exists $\phi(n)$ distinct admissible patterns of cardinality $n$, where $\phi$ is the Euler’s totient function.

- $\phi(18) = 6 - 1, 5, 7, 11, 13, 17$
Composite patterns

Theorem

For $a, b \in (0, 1)$, and any $n$, there exists $\phi(n)$ distinct admissible patterns of cardinality $n$, where $\phi$ is the Euler’s totient function.

- $\phi(18) = 6 - 1, 5, 7, 11, 13, 17$
- Thus there are patterns of length 18 with $1, 5, 7, 11, 13, 17$ $\mathcal{L}$s in them
Calculation of $P_\sigma$

Given a pattern $\sigma$ which is admissible, how to calculate the interval $P_{\sigma \text{ma}}$
Calculation of $P_\sigma$

Given a pattern $\sigma$ which is admissible, how to calculate the interval $P_{\sigma}$

Consider the pattern

$$RLRLRLRLRLRLRLRLRLRLRLRLRLRL$$
Calculation of $P_\sigma$

Given a pattern $\sigma$ which is admissible, how to calculate the interval $P_{\sigma}$

Consider the pattern

\[ RLRLRLRLLRLLRLRLLRLRLLRLRLLRLL \]

Substitute 0 for $L$ and 1 for $R$
Calculation of $P_\sigma$

Given a pattern $\sigma$ which is admissible, how to calculate the interval $P_{\sigma}$

Consider the pattern
$$RLRLRLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLLRLL
Calculation of $P_\sigma$

Given a pattern $\sigma$ which is admissible, how to calculate the interval $P_{\sigma}$

Consider the pattern

$RLRLRLLRLLRLRLLRLLRLLRLLRLLRLLRLLRLLRLL$

Substitute 0 for $L$ and 1 for $R$

\[
\begin{align*}
RLRLRRLRLRRRLLRLRRLRLRRLRLRRLRL\quad &\mu_2 \\
101001010010010100100100100100\quad &\mu_2 \\
\end{align*}
\]

\[
\begin{align*}
LRLRLRLRLRLRLRLRLRLRLLL\quad &\mu_1 \\
0010010100100101001010010100101\quad &\mu_1 \\
\end{align*}
\]
Other cases

Assumption: Let $1 < a < \infty$ and $1 < b < \infty$
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Assumption: Let $1 < a < \infty$ and $1 < b < \infty$
If the “jump” $\ell > 0$, then
Other cases

**Assumption:** Let $1 < a < \infty$ and $1 < b < \infty$.

If the “jump” $\ell > 0$, then

No chance of a periodic orbit!!
Other cases

Assumption: Let $1 < a < \infty$ and $1 < b < \infty$

If the “jump” $\ell < 0$, then

- $\mu < 0$
- $-\ell > \mu \geq 0$
- $\mu \geq -\ell$
Other cases

**Assumption:** Let $1 < a < \infty$ and $1 < b < \infty$

If the “jump” $\ell < 0$, then

Orbits can exist if $0 < \mu < -\ell$

Set $\ell = -1$ and therefore $0 < \mu < 1$
Other cases – results

Assumption: $a, b > 1$

- Orbits are unstable
Other cases – results

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- Orbits are unstable
- $\mathcal{L}$-prime patterns and $\mathcal{R}$-prime patterns always present
Other cases – results

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- $L$-prime patterns and $R$-prime patterns always present
- The pattern $LLRRR$ always present
Other cases – results

Assumption: $a, b > 1$

- Orbits are unstable
- $\mathcal{L}$-prime patterns and $\mathcal{R}$-prime patterns always present
- The pattern $\mathcal{L} \mathcal{L} \mathcal{R} \mathcal{R}$ always present
- If pattern $\mathcal{L}^p \mathcal{R}^q$ is present, then $\mathcal{L}^{p_1} \mathcal{R}^{q_1}$ is also present where $p_1 < p$ and $q_1 < q$
Other cases – results

**Assumption:** $a, b > 1$

- Orbits are unstable
- $L$-prime patterns and $R$-prime patterns always present
- The pattern $LLRR$ always present
- If pattern $L^pR^q$ is present, then $L^{p_1}R^{q_1}$ is also present where $p_1 < p$ and $q_1 < q$
- Co-existence of patterns, multiple orbits exist
Other cases – results

**Assumption:** \( a, b > 1 \)

- Orbits are unstable
- \( L \)-prime patterns and \( R \)-prime patterns always present
- The pattern \( LLRR \) always present
- If pattern \( L^p R^q \) is present, then \( L^{p_1} R^{q_1} \) is also present where \( p_1 < p \) and \( q_1 < q \)
- Co-existence of patterns, multiple orbits exist
- Chaotic orbits exist !!
Chaotic orbits

**Assumption:** $a, b > 1$

Why?
Chaotic orbits

Assumption: \( a, b > 1 \)
Capture range for \( \mu \) is \( \left( \frac{a-1}{a}, \frac{1}{b} \right) \)

Only for values of \( a, b \) in blue – chaotic orbits
\( 1 < b < \frac{a}{a-1} \) and \( 1 < a < \frac{b}{b-1} \)
Chaotic orbits

Assumption: \( a, b > 1 \)
Some pictures
For \( a = 1.01, b = 1.01 \)
Chaotic orbits

Assumption: $a, b > 1$

Some pictures

For $a = 1.1, b = 1.1$
Other cases – results

Assumption: $0 < a < 1$ and $b > 1$

For $\ell < 0$
Other cases – results

**Assumption:** $0 < a < 1$ and $b > 1$

For $\ell < 0$

No orbits!!
Other cases – results

Assumption: $0 < a < 1$ and $b > 1$
For $\ell > 0$
Other cases – results

Assumption: $0 < a < 1$ and $b > 1$

For $\ell > 0$

Orbits possible...
Other cases – results

**Assumption:** $0 < a < 1$ and $b > 1$

Some pictures

For $a = 0.1$ and $b = 1.1$
Other cases – results

**Assumption:** $0 < a < 1$ and $b > 1$

Some pictures

For $a = 0.5$ and $b = 1.1$
Other cases – results

**Assumption:** $0 < a < 1$ and $b > 1$

Some pictures

For $a = 0.9$ and $b = 1.1$
Other cases – results

Assumption: $0 < a < 1$ and $b > 1$

Some pictures

For $a = 0.5$ and $b = 8$
Boundary cases

By pictures

For $a = 1$ and $b = 1.1$
Boundary cases

Some pictures
For $a = 0.5$ and $b = 1$
Boundary cases

Some pictures

For $a = 1$ and $b = 1$
Thank you very much