Maximum hands-off control and discrete-valued control

Masaaki Nagahara[1,2]

[1] The University of Kitakyushu
nagahara@kitakyu-u.ac.jp
[2] Visiting Faculty at IITB
(SysCon, 1st floor, visiting faculty room)

1 March 2017, IIT Bombay

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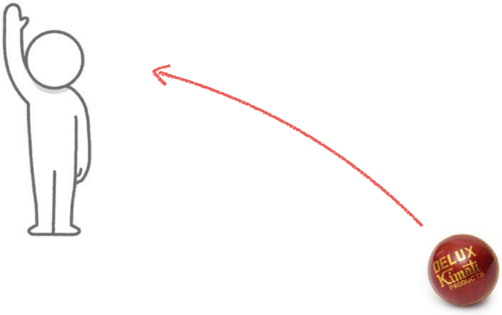

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A simple example

Plant: $G(s) = 1/s^2$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Feasible control

Fix $T > 0$. Find a feasible control $u(t)$, $t \in [0, T]$ that drives the state from $x(0)$ to $x(T) = [0, 0]^\top$ that satisfies

$$|u(t)| \leq 1, \quad \forall\, t \in [0, T].$$

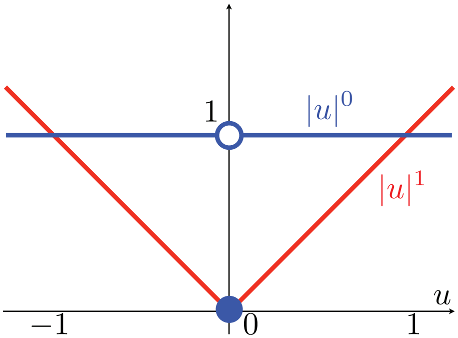
Maximum hands-off control problem

Find a feasible control that minimizes the L^0 norm of u:

$$J_0(u) = \mu(\mathrm{supp}(u)) = \int_0^T |u(t)|^0 dt \quad \text{(the length of the support)}$$

L^0 norm and L^1 norm

$$J_0(u) = \mu(\mathrm{supp}(u)) = \int_0^T |u(t)|^0 dt$$



$$J_1(u) = \int_0^T |u(t)| dt,$$

A simple example

Plant: $G(s) = 1/s^2$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

L^1-optimal control

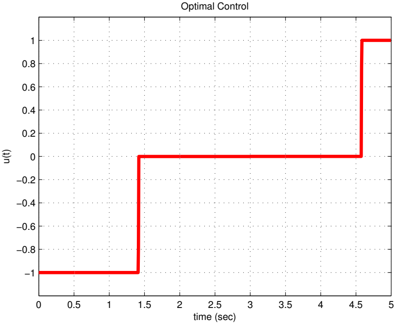
Fix $T > 0$. Find a feasible control $u(t)$, $t \in [0, T]$ that drives the state from $x(0)$ to $x(T) = [0, 0]^\top$, that satisfies $|u(t)| \leq 1$, $\forall\, t \in [0, T]$, and that minimizes the L^1 norm of u:

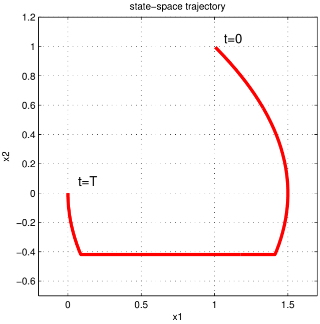
$$J_1(u) = \int_0^T |u(t)| dt.$$

- Also known as *fuel optimal control*.
- *A convex optimization problem!*

A simple example

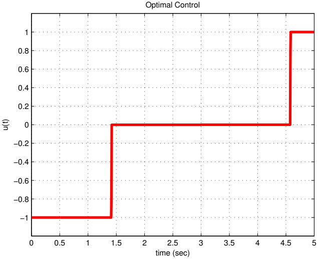
L^1-optimal control $u^*(t)$ and trajectory $x^*(t)$ [Athans and Falb, 1966]





- $u^*(t) \equiv 0$ over $[3-\sqrt{10}/2, 3+\sqrt{10}/2] \approx [1.4, 4.6]$
- $u^*(t)$ is *sparse* ($\|u^*\|_0 = |\mathrm{supp}(u^*)| \approx 1.84 < 5 = T$)
- In fact, it is *the sparsest* (i.e., *maximum hands-off control*).

Why hands-off control is *green*?



- Reduced fuel and electric power consumption
- Reduced CO_2, noise, and vibration
- Data compression
  - Sparse signals can be effectively compressed;
    see e.g. [Nagahara, Quevedo, Østergaard, IEEE Trans. AC 2014]

Maximum hands-off control and L^1 optimality

Plant

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad t \geq 0, \quad x(0) = x_0$$

$$x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}$$

Theorem

Assume that the L^1-optimal control problem is *normal* [a] (or *non singular*) and has at least one solution. Then

$$\{L^0 \text{ optimal controls}\} = \{L^1 \text{ optimal controls}\}$$

[a] When the optimal control is *uniquely determined almost everywhere* from the minimum principle, the control problem is called *normal*.

A maximum hands-off control problem (non convex optimization) can be solved via a related L^1 optimal control problem (convex)!

Sufficient condition for normality

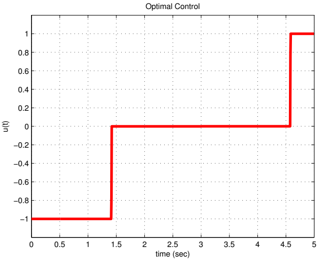
Lemma [Athans & Falb, 1966]

Assume the plant is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0.$$

If the plant is *controllable* and *A is non singular*, then for any initial state $x(0) \in \mathbb{R}^n$, the L^1-optimal control problem is normal.

L^1/L^2-optimal control for continuous control



- Maximum-hands off control is discontinuous
  - the "bang-off-bang" property
- Smoothing by adding L^2 norm:

$$J_{12} = \|u\|_1 + \frac{1}{2} r \|u\|_2^2$$

- L^1/L^2-optimal control is *continuous in t*.

L^1/L^2-optimal control for continuous control

L^1/L^2-optimal control

Plant: $\dot{x}(t) = f(x) + g(x)u$

Assumption: f, g, $\frac{df}{dx}$, $\frac{dg}{dx}$ are continuous in x.

Constraints: $x(0) = x_0$; $x(T) = 0$; $|u(t)| \leq 1 \ \forall t \in [0, T]$

Cost function: $J_{12} = \|u\|_1 + \frac{1}{2} r \|u\|_2^2$

Proposition

The L^1/L^2-optimal control $u_{12}^*(t)$ is continuous in t over $[0, T]$.

Proposition

Assume the L^1-optimal control problem is normal and its solution exists. Then

$$u_{12}^*(t) \to u_1^*(t) = u_0^*(t), \quad \text{a.a. } t \in [0, T],$$

as $r \to 0$.

CLOT

CLOT (Combined L-One and Two)-optimal control

Plant: $\dot{x}(t) = f(x) + g(x)u$

Assumption: f, g, $\frac{df}{dx}$, $\frac{dg}{dx}$ are continuous in x.

Constraints: $x(0) = x_0$; $x(T) = 0$; $|u(t)| \leq 1 \ \forall t \in [0, T]$

Cost function: $J_{\mathrm{CLOT}} = \|u\|_1 + r\|u\|_2$ (cf. $J_{12} = \|u\|_1 + \frac{1}{2}r\|u\|_2^2$)

- Motivated by CLOT in signal processing [Ahsen, Challapalli, & Vidyasagar 2016]
- The CLOT optimization may give much sparser but still continuous control [Challapalli, Nagahara, & Vidyasagar 2016] (submitted to IFAC2017)

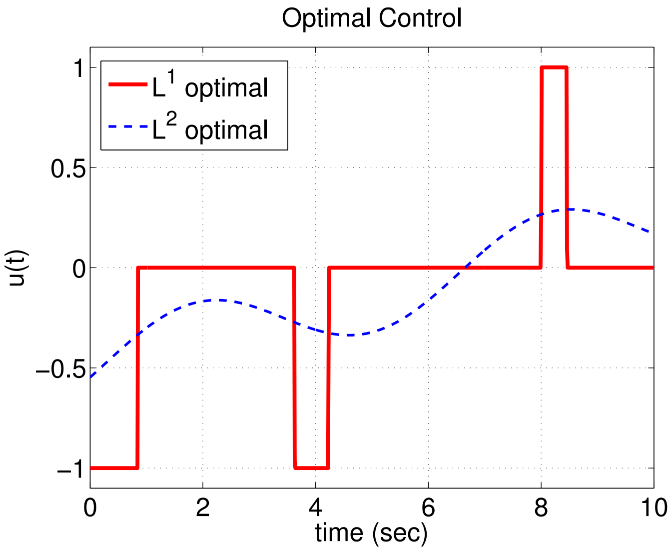
Example: control problem

- Plant: $P(s) = \frac{1}{s^2(s^2+1)}$

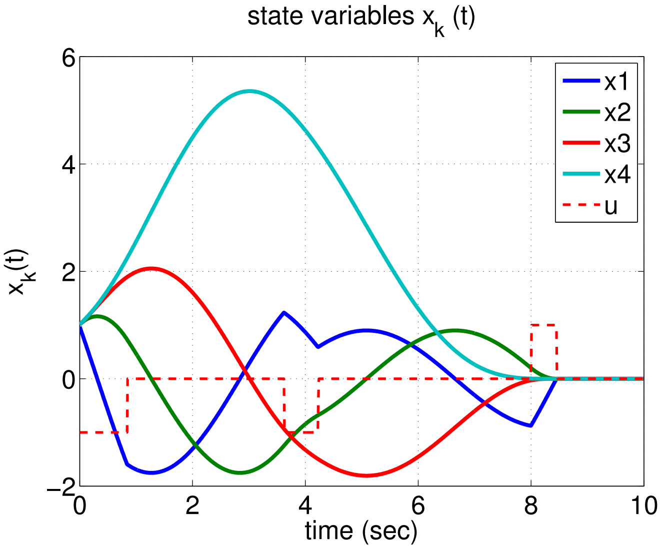
$$\frac{d\boldsymbol{x}(t)}{dt} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t).$$

- Final time: $T = 10$.
- State Constraints: $x(0) = [1, 1, 1, 1]^\top$ and $x(10) = 0$
- Control constraint: $|u(t)| \leq 1, \quad \forall t \in [0, 10]$

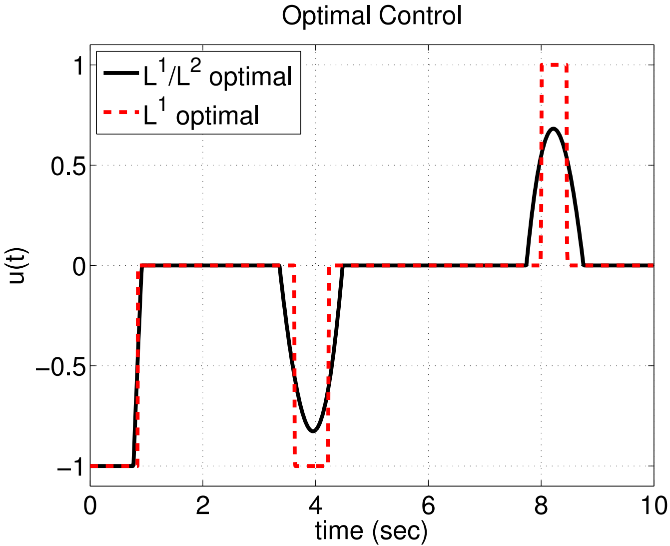
Examples: optimal controls



Examples: states with maximum hands-off control



Examples: L^1/L^2-optimal control



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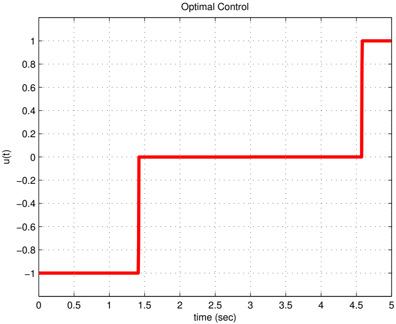
Discrete-valued control



- The L^1 optimal control takes values ± 1 and 0.
- This is discrete valued.
- Discrete-valued control has merits of
  - discretization (quantization) of control
  - data compression
  - simple actuation

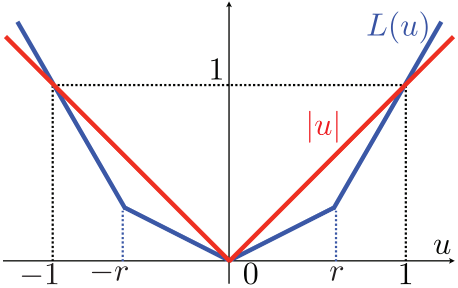
Discrete-valued control

- L^1 optimal control:

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad \int_0^T |u| dt \quad \longrightarrow \quad u(t) \text{ takes } \pm 1, 0$$

- Let us consider

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad \int_0^T L(u) dt \quad \longrightarrow \quad u(t) \text{ takes } ?\, \pm 1, \pm r, 0$$



Problem formulation

Plant

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad t \geq 0, \quad x(0) = x_0$$

Feasible control

Fix $T > 0$. Find a discrete-valued control $u(t)$, $t \in [0, T]$ that drives the state from $x(0)$ to $x(T) = 0$ and satisfies $U_{\min} \leq u(t) \leq U_{\max}$, $\forall t \in [0, T]$.

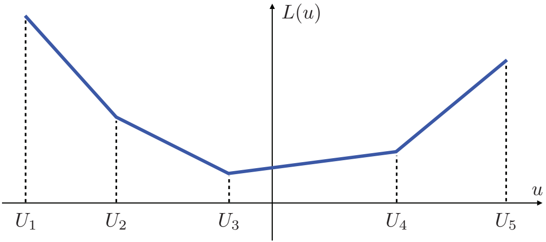
Discrete-valued control

Find a feasible control that satisfies

$$u(t) \in \{U_1, U_2, \ldots, U_N\}$$

where $U_{\min} = U_1 < U_2 < \cdots < U_N = U_{\max}$.

Cost function



Sum of absolute values (SOAV):

$$L(u) = \sum_{i=1}^{N} w_i |u - U_i|$$

SOAV optimal control

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad \int_0^T L(u(t)) dt \to u(t) \in \{U_1, U_2, \dots, U_N\}$$

Discrete-valued control

SOAV optimal control:

$$\underset{u}{\text{minimize}} \int_0^T \underbrace{\sum_{i=1}^{N} w_i |u(t) - U_i|}_{=L(u(t))} \, dt = \sum_{i=1}^{n} w_i \|u - U_i\|_1$$

subject to

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0, \quad x(T) = 0$$

$$U_{\min} \leq u(t) \leq U_{\max}$$

Theorem

If the SOAV optimal control is normal (or nonsingular), then the optimal solution satisfies

$$u(t) \in \{U_1, U_2, \dots, U_N\}, \quad \text{a.a. } t \in [0, T].$$

Discrete-valued control for linear plants

SOAV optimal control:

$$\underset{u}{\text{minimize}} \int_0^T \sum_{i=1}^N w_i |u(t) - U_i| dt = \sum_{i=1}^n w_i \|u - U_i\|_1$$

$$\text{subject to}$$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad x(T) = 0$$

$$U_{\min} \leq u(t) \leq U_{\max}$$

Theorem

If (A, B) is controllable, A is nonsingular, and

$$a_k \triangleq \sum_{i=1}^k - \sum_{i=k+1}^N \neq 0, \quad \forall k = 1, 2, \ldots, N-1,$$

then the optimal solution satisfies $u(t) \in \{U_1, U_2, \ldots, U_N\}$, a.a. $t \in [0, T]$.

Discrete-valued control for linear plants

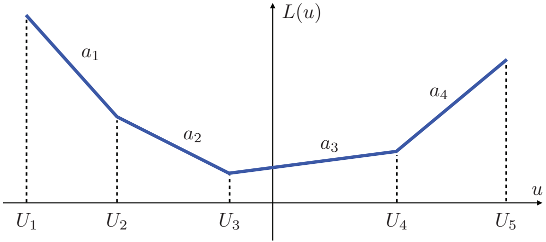
Theorem

If (A, B) is controllable, A is nonsingular, and

$$a_k \triangleq \sum_{i=1}^{k} w_i - \sum_{i=k+1}^{N} w_i \neq 0, \quad \forall k = 1, 2, \dots, N-1,$$

then the optimal solution satisfies $u(t) \in \{U_1, U_2, \dots, U_N\}$, a.a. $t \in [0, T]$.

a_k is the slope of the line between U_k and U_{k+1}



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Maximum hands-off control

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Discrete-valued control

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Conclusion

- Maximum hands-off control is *green control*.
  - uses less fuel and electric power
  - reduces CO2, noise, and vibration
  - gives effective data compression for networked control systems
- L^0 optimality $=$ L^1 optimality
  - under the assumption of normality.
- Continuous control by L^1/L^2-optimal control
- Discrete-valued control via SOAV optimization