Recent Results on Parametric Robust Control for LTI Systems

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Contents

Contents

Contents

Contents

Contents

Contents

Contents

1 Introduction

Summary

- The traditional concepts in control theory solved part of the problems of parametric robust control but there were still others to solve.

- Then it was necessary to try different points of view on the subject.

- First we will give a brief review of some traditional concepts and then we will discuss some different points of view.

Linear Time Invariant Systems

$\dot{x} = Ax + Bu$

$y = Cx + Du$

Properties

Stability

Controllability

Observability

1 Introduction

Stability

$p(s) = |sI - A|$
$p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n$
$p(s) = (s - z_1)(s - z_2) \cdots (s - z_n)$

Stability $\longleftrightarrow R(z_i) \leq 0$

Criterions

1856	Hermite-Biehler
1875	Routh
1895	Hurwitz
1914	Liénard-Chipart
2001	César Elizondo

Controllability

$$U = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Controllability $\longleftrightarrow$ $\mathrm{Rank}(U) = n$

Parametric Uncertainty

$$\dot{x} = A(q)x + B(q)u$$
$$y = C(q)x + D(q)u$$

$$q = [q_1 \; q_2 \; \ldots \; q_\ell]^T$$
$$q_i \in \left[q_i^-, \; q_i^+\right]$$
$$Q = \left\{q = [q_1 \; q_2 \; \ldots \; q_\ell]^T \mid q_i \in [q_i^-, \; q_i^+], \; i = 1, 2, \ldots, \ell\right\}$$

$$p(s, \; q) = c_0(q) + c_1(q)s + c_2(q)s^2 + \cdots + c_n(q)s^n$$
$$U(q) = \left[B(q) \;\; A(q)B(q) \;\; A(q)^2B(q) \;\; \cdots \;\; A^{n-1}(q)B(q)\right]$$

2 Stability of Real Fixed Polynomials

Hermite-Biehler

$p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n$
$p(s) = p_e(s) + s p_o(s)$

ω_{ei} is the root i of the even part $p_e(s)$
ω_{oi} is the root i of the odd part $p_o(s)$

Interlacing Property

$$0 < \omega_{e,1} < \omega_{o,1} < \omega_{e,2} < \omega_{i,2} \cdots$$

2 Stability of Real Fixed Polynomials

Routh Theorem

$$p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n$$

$$\begin{array}{ccccc} s^n & c_n & c_{n-2} & c_{n-4} & \cdots \\ s^{n-1} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\ s^{n-2} & a_{3,1} & a_{3,2} & \cdots & \\ s^{n-3} & a_{4,1} & a_{4,2} & \cdots & \\ \vdots & \vdots & \vdots & & \end{array}$$

$$a_{i,j} = \frac{(a_{i-1,1}a_{i-2,j+1} - a_{i-2,1}a_{i-1,j+1})}{a_{i-1,1}} \ \forall i \geq 3$$

$$a_{i,j} = c_{n+1-i-2(j-1)} \ \forall i \leq 2$$

The number of $R(z_i) > 0$ = number of variations of sign in the first column

Hurwitz Theorem

A Polynomial $p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n$ with positive coefficients, is stable iff and only if the set of principal minors Δ_i of matrix H are positive.

$$H = \begin{bmatrix} c_{n-1} & c_{n-3} & \cdots & & & \\ c_n & c_{n-2} & \cdots & & & \\ 0 & c_{n-1} & c_{n-3} & \cdots & & \\ 0 & c_n & c_{n-2} & \cdots & & \\ & & \vdots & \vdots & & \\ & & & & \cdots & c_0 \end{bmatrix}^{n \times n}$$

$$\Delta_1 = c^{n-1}, \quad \Delta_2 = \begin{vmatrix} c_{n-1} & c_{n-3} \\ c_n & c_{n-2} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} c_{n-1} & c_{n-3} & c_{n-5} \\ c_n & c_{n-2} & c_{n-4} \\ 0 & c_{n-1} & c_{n-3} \end{vmatrix}, \ldots$$

$$\Delta_k = c^{n-1} a_{3,1} a_{4,1} \cdots a_{k+1,1}, \qquad a_{k,1} = \frac{\Delta_{k-1}}{\Delta_{k-2}}, \ k > 2$$

2 Stability of Real Fixed Polynomials

Liénard-Chipart

$p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n$

$$H = \begin{bmatrix} c_{n-1} & c_{n-3} & \cdots & & & \\ c_n & c_{n-2} & \cdots & & & \\ 0 & c_{n-1} & c_{n-3} & \cdots & & \\ 0 & c_n & c_{n-2} & \cdots & & \\ & & \vdots & \vdots & & \\ & & & & \cdots & c_0 \end{bmatrix}^{n \times n}$$

$\Delta_{even} > 0 \longleftrightarrow \Delta_{odd} > 0$

Advantages and Disadvantages

1856	Hermite-Biehler	Get roots
1875	Routh	Use division
1895	Hurwitz	Many arithmetic operations
1914	Liénard-Chipart	Many arithmetic operations
2001	César Elizondo	

A Recent Stability Theorem, C. Elizondo-González 2001

$p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_{n-1} s^{n-1} + c_n s^n$

The number of $R(z_i) > 0$ = number of variations of sign in the σ column

σ_1	c_n	c_{n-2}	c_{n-4}	$\cdots$
σ_2	c_{n-1}	c_{n-3}	c_{n-5}	$\cdots$
σ_3	$e_{3,1}$	$e_{3,2}$	$\cdots$	
σ_4	$e_{4,1}$	$e_{4,2}$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$		
$\sigma_{(n-1)}$	$e_{(n-1),1}$	$e_{(n-1),2}$		
σ_n	$e_{n,1}$			
$\sigma_{(n+1)}$	$e_{(n+1),1}$			

Where:

3 A Recent Stability Theorem

A Recent Stability Theorem, C. Elizondo-González 2001

$$e_{i,j} = (e_{i-1,1}e_{i-2,j+1} - e_{i-2,1}e_{i-1,j+1}), \forall 3 \leq i \leq \ n+1$$
$$e_{i,j} = c_{n+1-i-2(j-1)} \ \forall i \leq 2$$

$$\sigma_i = Sign(e_{i,1}) \forall i \leq 2$$
$$\sigma_i = Sign(e_{i,1}) \prod_{j=1}^{(i+1-m)/2} Sign(e_{m+2(j-1),1}) \forall i \geq 3$$
$m = 3$ para i par, $\quad m = 2$ para i non

3 A Recent Stability Theorem

$p(s) = s^5 + s^4 + s^3 + 3s^2 + 2s + 1$

Recent Stability Theorem

Recent Table

$\sigma_1 = +$	1	1	2
$\sigma_2 = +$	1	3	1
$\sigma_3 = -$	-2	1	
$\sigma_4 = +$	-7	-2	
$\sigma_5 = +$	-11		
$\sigma_6 = +$	22		

The polynomial has two roots in the right half of the complex plane

Routh Theorem

Routh Table

s^5	1	1	2
s^4	1	3	1
s^3	-2	1	
s^2	7/2	1	
s^1	$11/2$		
s^0	1		

A Recent Stability Theorem, C. Elizondo-González 2001

Comparison: Hurwitz Theorem, Recent Theorem

degree	Hurwitz Theorem		Recent Theorem	
n	$\times$	$+ o -$	$\times$	$+ o -$
3	4	1	2	1
4	9	2	5	2
5	66	18	9	4
6	193	45	14	6
7	780	156	20	9

4 Stability of Polynomials With Parametric Uncertainty

Parametric Uncertainty

- $p(s,q) = c_0(q) + c_1(q)s + c_2(q)s^2 + \cdots + c_n(q)s^n$ is a polynomial depending on $q \in Q$.
- $P(s,Q) = \{p(s,q) \mid q \in Q\}$ is a **Family of Polynomials**.

Stability of Polynomials With Parametric Uncertainty

Interval	$c_i(q) = q_i$
Affine	$c_i(q) = 3q_1 + 4q_2 + 6q_4$
Multilinear	$c_i(q) = 3q_1q_2q_4 + 2q_1$
Polynomic	$c_i(q) = 3q_1^2q_2^4q_4^2 + 2q_1$

Interval	1978	Kharitonov	Convexity, Testing Subset
Affin	1988	Bartlet, Hollot and Huang	Convexity, Testing Subset
Multilinear			
Polynomic			

4 Stability of Polynomials With Parametric Uncertainty

Parametric Uncertainty

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Multilinear			
Polynomic			

5 Sign Decomposition

Parametric Uncertainty

Let $\hat{q} = [\hat{q}_1\ \hat{q}_2\ ...\ \hat{q}_\ell]^T \mid \hat{q}_i^- \leq \hat{q}_i \leq \hat{q}_i^+$ be a parametric vector with uncertainty.

$$\hat{q}_i = \hat{q}_i^- + \left[\frac{\hat{q}_i^+ - \hat{q}_i^-}{q_i^+ - q_i^-}\right](q_i - q_i^-) \mid\ q_i^+ > q_i^- > 0$$

Let $Q \subset P = \left\{q = [q_1\ q_2\ ...\ q_\ell]^T \mid 0 < q_i^- \leq q_i \leq q_i^+\ \ \forall i = 1, 2, \cdots \ell\right\}$ be an uncertainty parametric box.

Robust Positivity Problem

Consider the example $f(q) = 4 + q_1 - q_2 + 8q_1^2 q_2 - 9q_1 q_2^2$

$$¿\, f(q) > 0\ \forall q \in Q\ ?$$

Parametric Uncertainty

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Robust Positivity Problem

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Nondecreasing Functions

Let $f(q), \;\; f : \Re^{\ell} \rightarrow \Re$ be a multivariable polynomic function such that $q \in Q \subset P$, then all its expressions are nondecreasing functions.

Example $f(q) = 4 + q_1 - q_2 + 8q_1^2 q_2 - 9q_1 q_2^2$

$$f(q) = \left(4 + q_1 + 8q_1^2 q_2\right) - \left(q_2 + 9q_1 q_2^2\right)$$

Definition of Sign Decomposition

Let $f : \Re^{\ell} \to \Re$ be a continuous function defined in $Q \subset P \subset \Re^{\ell}$

Is said that $f(\cdot)$ has **Sign Decomposition** in Q if there exist two bounded continuous nondecreasing functions $f_n(\cdot) \geq 0$ and $f_p(\cdot) \geq 0$, such that $f(q) = f_p(q) - f_n(q) \; \forall \; q \in Q$

Then they are called:
The **Positive Part** $f_p(\cdot)$ of the function.
The **Negative Part** $f_n(\cdot)$ of the function.

Example

$f(q) = \left(4 + q_1 + 8q_1^2 q_2\right) - \left(q_2 + 9q_1 q_2^2\right)$ such that $q \in Q \subset P$ then:

$f_p(q) = 4 + q_1 + 8q_1^2 q_2$
$f_n(q) = q_2 + 9q_1 q_2^2$

5 Sign Decomposition

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$f_n(q) = q_2 + 9q_1 q_2^2$

5 Sign Decomposition

Elementary Properties

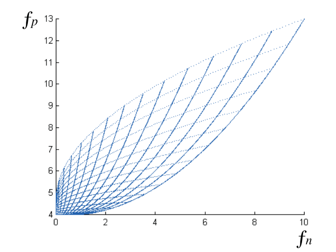
1) All the polynomic functions have sign decomposition.
2) $f_p(\cdot)$ and $f_n(\cdot)$ are independent functions, therefore $[f_n(\cdot)\, f_p(\cdot)]$ is a basis in $\Re^2$
3) $(f_n(\cdot),\, f_p(\cdot))$ can be considerate as a representation of the function in $\Re^2$

(f_n, f_p) Plane

All function $f : \Re^\ell \to \Re$ with sign decomposition in $Q \subset P \subset \Re^\ell$, has graphic representation on $\Re^2$ by means of its negative and positive parts $f_n(\cdot)$ and $f_p(\cdot)$ with graphical representation in the (f_n, f_p) plane.

Example

$$f(q) = (4 + q_1 + 8q_1^2 q_2) - (q_2 + 9q_1 q_2^2)$$



Properties of $f(q)$ on the $(f_n,\, f_p)$ Plane

1) Given a vector $q^* \in Q \subset P$, a function $f(q)$ with sign decomposition in Q is represented as the point $(f_n(q^*),\, f_p(q^*))$ on the $(f_n,\, f_p)$ plane.

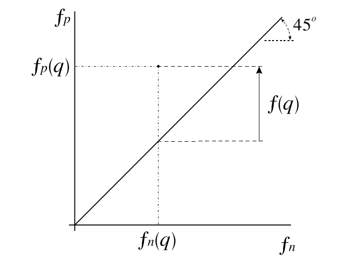
2) $f(q) = 0$ is is represented as 45^o line on $(f_n,\, f_p)$ plane.

3) A function $f(q)$ with sign decomposition in Q is positive iff the point $(f_n(q),\, f_p(q))$ is above the 45^o line on $(f_n,\, f_p)$ plane.

4) A function $f(q)$ with sign decomposition in Q is negative iff the point $(f_n(q),\, f_p(q))$ is below the 45^o line on $(f_n,\, f_p)$ plane.

5 Sign Decomposition

(f_n, f_p) Plane



Bounding

Since each expression of $f(q)$ with sign decomposition in Q is a nondecreasing function, then the negative and positive parts are bounded as follow.

$$f_p(v^{min}) \leq f_p(q) \leq f_p(v^{max}) \; \forall q \in Q \quad \text{and} \quad f_n(v^{min}) \leq f_n(q) \leq f_n(v^{max}) \; \forall q \in Q$$

where $v^{min} = \left[q_1^- \; q_2^- \; \cdots \; q_\ell^- \right]^T$ and $v^{max} = \left[q_1^+ \; q_2^+ \; \cdots \; q_\ell^+ \right]^T$

Rectangle Theorem

Let $f : \Re^{\ell} \to \Re$ be a continuous function with sign decomposition in a box $Q \subset P \subset \Re^{\ell}$ with minimum and maximum Euclidean vertices $v^{\min}$, $v^{\max}$, then:

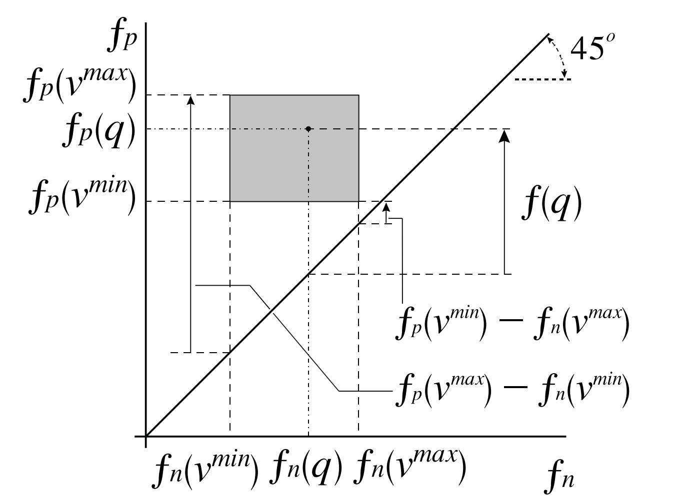
a) $f(q)$ is lower and upper bounded by $f_p(v^{\min}) - f_n(v^{\max})$ and $f_p(v^{\max}) - f_n(v^{\min})$ respectively;

b) The graphical representation of the function $f(q)$, $\forall q \in Q$ in (f_n, f_p) plane, it is contained in the rectangle with vertices $(f_n(v^{\min}),\ f_p(v^{\min}))$, $(f_n(v^{\max}),\ f_p(v^{\max}))$, $(f_n(v^{\min}),\ f_p(v^{\max}))$ and $(f_n(v^{\max}),\ f_p(v^{\min}))$

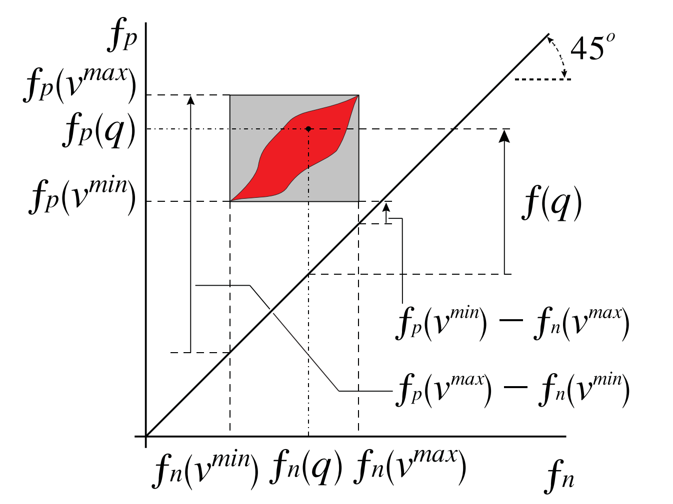
c) If the lower right vertex $(f_n(v^{\max}),\ f_p(v^{\min}))$ is over 45^o line then $f(q) > 0\ \forall q \in Q$

d) If the upper left vertex $(f_n(v^{\max}),\ f_p(v^{\min}))$ is below 45^o line then $f(q) < 0\ \forall q \in Q$

Rectangle Theorem

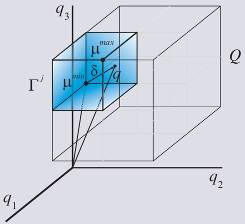


Rectangle Theorem



5 Sign Decomposition

Γ Boxes



Linear, Nonlinear and Independent Parts

Since $q = \mu_0 + \delta$ then $f(q) = f(\mu_0 + \delta)$

Where $\Delta = \left\{\delta \mid \delta_i \in [0, \delta_i^{max}],\ \delta_i^{max} = \mu_i^{max} - \mu_i^{min}\right\}$

$$f(q) = f(\mu^{min}) + \sum_{i=1}^{\ell} \left.\frac{\partial f(q)}{\partial q_i}\right|_{\mu_i} \delta_i + \Phi(\delta)$$

$$f(q) = f^{min} + f_L(\delta) + f_N(\delta) \mid \delta \in \Delta$$

$f^{min} \triangleq$ Independent Part $= f(\mu^{min})$

$f_L(\delta) \triangleq$ Linear Part $= \nabla f(q)|_{\mu^{min}} \cdot \delta$

$f_N(\delta) \triangleq$ Nonlinear Part

$= f(\mu^{min} + \delta) - f^{min} - f_L(\delta) \forall \delta \in \Delta$

5 Sign Decomposition

Polygon Theorem

Let $f : \Re^{\ell} \to \Re$ be a continuous function with sign decomposition in Q, let q, δ be two vectors such that $q = \mu_0 + \delta,\ \forall\delta \in \Delta$.

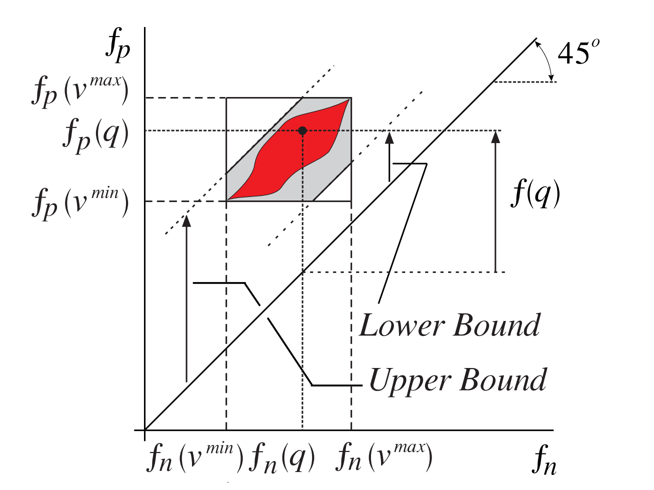
a) The lower and upper bounds of the function $f(q)$ are:
$Lower\ Bound = f^{min} + f_{L\,\min} - f_{Nn}(\delta^{max})$ and
$Upper\ Bound = f^{min} + f_{L\,\max} + f_{Np}(\delta^{max})\ \forall q \in Q$

b) The bounds of incise "a", are contained in the interval defined by the bounds of the rectangle theorem
$f_p(\mu^{min}) - f_n(\mu^{max}) \leq Lower\ Bound \leq Upper\ Bound \leq f_p(\mu^{max}) - f_n(\mu^{min})$

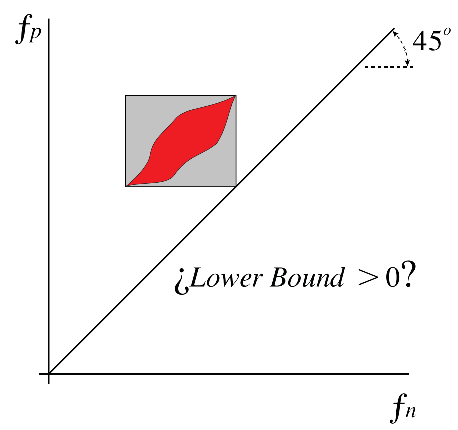
c) The graphical representation of the function $f(q)$, $\forall q \in \Gamma$ in the $(f_n,\ f_p)$ plane, it is contained in the polygon defined by the intersection of rectangle of rectangle theorem and the space between two 45^o lines separates of the origin by the $Lower\ Bound = f^{\min} + f_{L\,\min} - f_{Nn}(\delta^{max})$ and
$Upper\ Bound\ f^{min} + f_{Lmax} + f_{Np}(\delta^{max})$

5 Sign Decomposition

Polygon Theorem



Problem: Â¿Does The Lower Bound is Positive?



$(\alpha,\ \beta)$ Representation

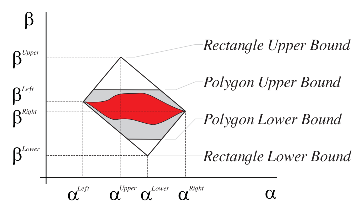
Let $T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ be a linear transformation and let $(\alpha(\cdot),\ \beta(\cdot))$ be a representation of the function in $\Re^2$: $\begin{bmatrix} \alpha(\cdot) \\ \beta(\cdot) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f_n(\cdot) \\ f_p(\cdot) \end{bmatrix}$

$$(\alpha(\cdot),\ \beta(\cdot)) = T\,(f_n(\cdot), f_p(\cdot))$$
$$\alpha(\cdot) = f_n(\cdot) + f_p(\cdot)$$
$$\beta(\cdot) = f_p(\cdot) - f_n(\cdot)$$

$$T^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$(f_n(\cdot), f_p(\cdot)) = T^{-1}\,(\alpha(\cdot),\ \beta(\cdot))$$

$$f_n(\cdot) = \tfrac{1}{2}\,(\alpha(\cdot) + \beta(\cdot))$$
$$f_p(\cdot) = \tfrac{1}{2}\,(\beta(\cdot) - \alpha(\cdot))$$

Rectangle and Polygon Theorems in $(\alpha,\ \beta)$ Representation



$$
\begin{aligned}
Rectangle\ \alpha^{Lower\ Bound} &= f_p(v^{min}) + f_n(v^{max}) \\
Rectangle\ \beta^{Lower\ Bound} &= f_p(v^{min}) - f_n(v^{max}) \\
Rectangle\ \alpha^{Upper\ Bound} &= f_p(v^{max}) + f_n(v^{min}) \\
Rectangle\ \beta^{Upper\ Bound} &= f_p(v^{max}) - f_n(v^{min})
\end{aligned}
$$

$$
\begin{aligned}
Polygon\ \alpha_{Lower\ Bound} &= \alpha^{min} + \alpha_{Lmin} + \tfrac{1}{2}(\alpha_N(\delta^{max}) - \beta_N(\delta^{max})) \\
Polygon\ \beta_{Lower\ Bound} &= \beta^{min} + \beta_{Lmin} - \tfrac{1}{2}(\alpha_N(\delta^{max}) - \beta_N(\delta^{max})) \\
Polygon\ \alpha_{Upper\ Bound} &= \alpha^{min} + \alpha_{Lmax} + \tfrac{1}{2}(\alpha_N(\delta^{max}) + \beta_N(\delta^{max})) \\
Polygon\ \beta_{Upper\ Bound} &= \beta^{min} + \beta_{Lmax} + \tfrac{1}{2}(\alpha_N(\delta^{max}) + \beta_N(\delta^{max}))
\end{aligned}
$$

Some Properties of the $(\alpha,\ \beta)$ Representation

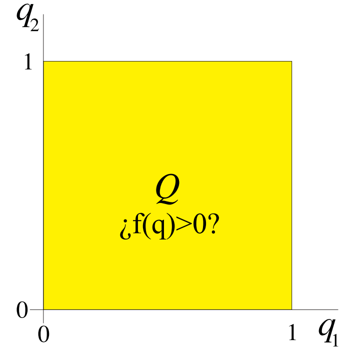
Addition	$f_1(q) + f_2(q)$	$\alpha(q) = \alpha_1(q) + \alpha_2(q)$	$\beta(q) = \beta_1(q) + \beta_2(q)$
Subtraction	$f_1(q) - f_2(q)$	$\alpha(q) = \alpha_1(q) + \alpha_2(q)$	$\beta(q) = \beta_1(q) - \beta_2(q)$
Product	$f_1(q)f_2(q)$	$\alpha(q) = \alpha_1(q)\alpha_2(q)$	$\beta(q) = \beta_1(q)\beta_2(q)$

Box Partition Theorem

Let $f : \Re^{\ell} \rightarrow \Re$ be a continuous function with sign decomposition in Q such that $Q \subset P \subset \Re^{\ell}$ is a box with minimum and maximum Euclidean vertexes $v^{\min},\ v^{\max}$. Then The function $f(q)$ is positive (negative) in Q if and only if exist a Γ boxes set, such that $Q = \bigcup_j \Gamma^j$ and *Lower Bound* $\geq c > 0$ for each one Γ^j box (*Upper Bound* $\leq c < 0$ for each one Γ^j box)

Example. Analytical Method

$$f(q) = \left(4 + q_1 + 8q_1^2 q_2\right) - \left(q_2 + 9q_1 q_2^2\right) \; > 0 \; \forall q \in Q \subset P?$$



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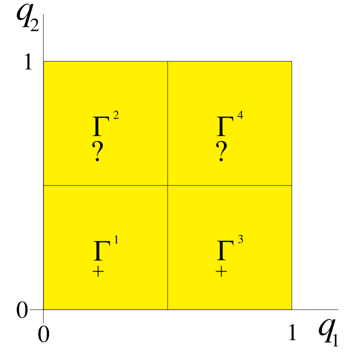

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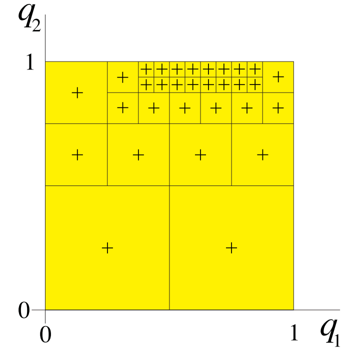
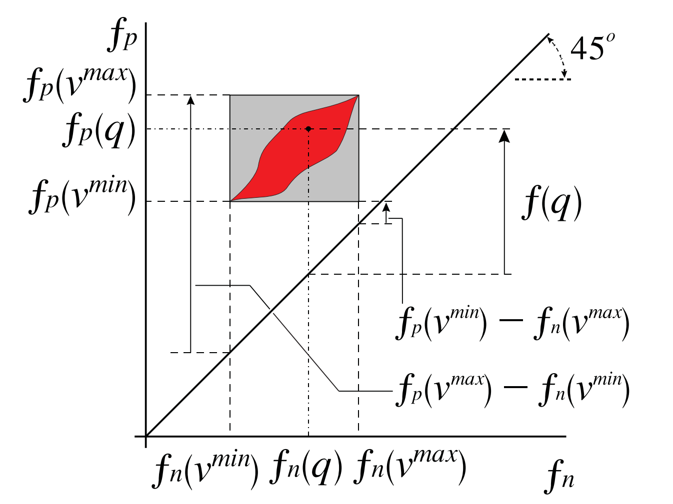
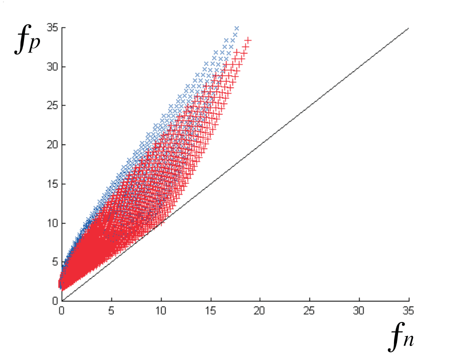


Rectangle Theorem



Example. Graphical Way

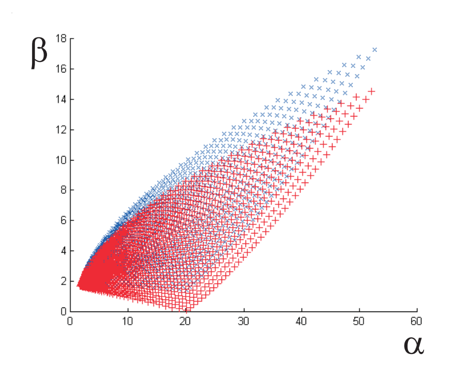
$$f(q) = \left(1.7 + q_1 + 8q_1^2 q_2\right) - \left(q_2 + 9q_1 q_2^2\right) \; > 0 \; \forall q \in Q \subset P?$$



5 Sign Decomposition

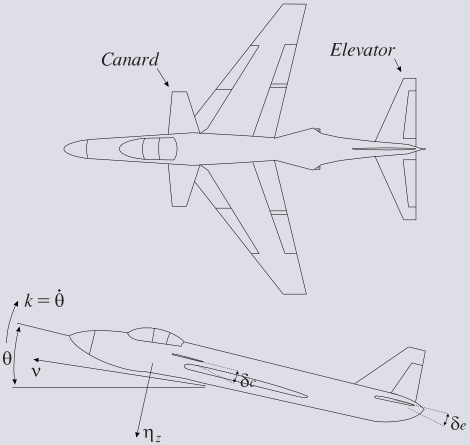
Example. Graphical Way

$$f(q) = \left(1.7 + q_1 + 8q_1^2 q_2\right) - \left(q_2 + 9q_1q_2^2\right) \ > 0 \ \forall q \in Q \subset P$$



Sign Decomposition Applied to The Recent stability Theorem

Example. Aircraft F4-E.



Model (J. Ackermann)

$$
\begin{aligned}
x_1 &= n_z && \text{normal aceleration} \\
x_2 &= k && \text{pitch rate} \\
x_3 &= \delta_e && \text{elevator angle}
\end{aligned}
$$

$x = [x_1\ x_2\ x_3]^T$

$\dot{x} = Ax + Bu$

$\delta_{ecom} = u$ and $\delta_{ccom} = -0.7u$.

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & -14 \end{bmatrix}
$$

$$
B = \begin{bmatrix} b_1 \\ 0 \\ 14 \end{bmatrix}
$$

Sign Decomposition Applied to The Recent stability Theorem

	I	II
$Mach$	0.9	1.5
$Altitude$	35000	35000
a_{11}	-0.667	-0.5162
a_{12}	18.11	26.96
a_{21}	0.08201	-0.6896
a_{22}	-0.6587	-1.225

$p_I(s) = -14 \cdot 6319 + 17 \cdot 514702s + 15 \cdot 3255s^2 + 1s^3$

Unstable

$p_{II}(s) = 269 \cdot 14 + 43 \cdot 601s + 15 \cdot 741s^2 + 1s^3$

Stable

6 Application to Robust Stability

Sign Decomposition Applied to The Recent stability Theorem

$a_{ij}(q) = a_{ijII} + (a_{ijI} - a_{ijII})q_k$
$q = [q_1\ q_2\ q_3\ q_4]^T \mid q_k \in [0,\ 1]$

$$\begin{aligned} a_{11}(q) &= -0.5162 - 0.1508q_1 \\ a_{12}(q) &= 26.96 - 8.8500q_2 \\ a_{21}(q) &= -0.6896 + 0.7716q_3 \\ a_{22}(q) &= -1.225 + 0.5663q_4 \end{aligned}$$

$$A = \begin{bmatrix} a_{11}(q) & a_{12}(q) & a_{13} \\ a_{21}(q) & a_{22}(q) & a_{23} \\ 0 & 0 & -14 \end{bmatrix}$$

- The characteristic polynomial is independent of a_{13} and a_{23}
- The system will be analyzed in a box such that $q_k \in [0,\ 0.8]$

Sign Decomposition Applied to The Recent stability Theorem

$$p(s,q) = c_0(q) + c_1(q)s + c_2(q)s^2 + c_3(q)s^3$$

$$c_0(q) = 269.14 - 85.441q_2 - 291.23q_3 + 95.601q_3q_2 - 4.0925q_4 \\ +2.5862q_1 - 1.1956q_1q_4$$

$$c_1(q) = 43.601 - 6.103q_2 - 20.802q_3 + 6.8287q_3q_2 - 8.2205q_4 \\ +2.2959q_1 - 8.5398 \times 10^{-2}q_1q_4$$

$$c_2(q) = 15.741 + 0.1508q_1 - 0.5663q_4$$

$$c_3(q) = 1$$

Sign Decomposition Applied to The Recent stability Theorem

Recent Table

1	$c_1(q)$
$c_2(q)$	$c_0(q)$
$c_2(q)c_1(q) - c_0(q)$	
$e_{4,1}(q)$	

Sign Decomposition Applied to The Recent stability Theorem

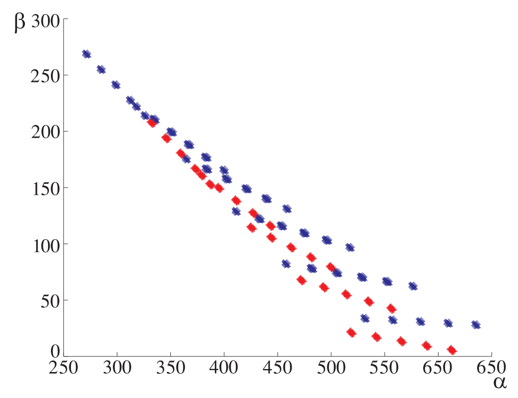


Figure : Robust Positivity of the Coefficient $e_{2,2}(q)$

Sign Decomposition Applied to The Recent stability Theorem

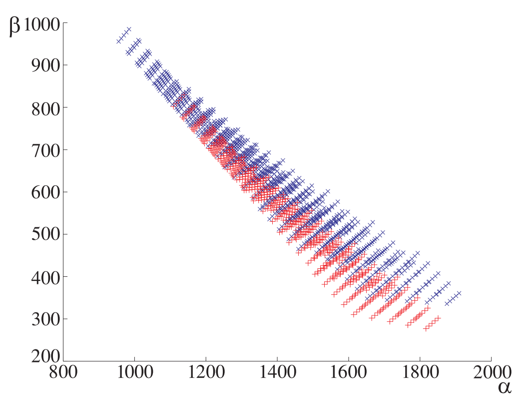


Figure : Robust Positivity of the Coefficient $e_{3,1}(q)$

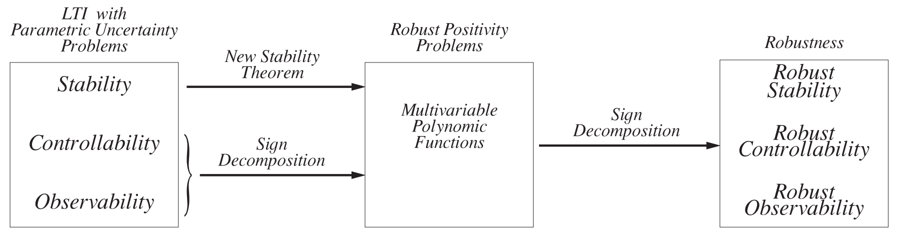
7 Conclusions

Conclusions 1

• In this talk a brief historical background of stability of LTI systems with fixed parameters and parametric uncertainty was presented.
• Traditional theorems for LTI systems with fixed parameters where analyzed.
• A Recent stability theorem for LTI system, including an example, was analyzed.
• The Recent stability theorem for LTI system was compared with Hurwitz theorem, the Recent theorem possess advantages with respect to Hurwitz theorem.
• The mathematical tool Sign Decomposition was presented.
• The robust stability of the aircraft F4-E was analyzed by means of the Recent stability theorem and sign decomposition.

7 Conclusions

Global Conclusion



Thank you very much for your attention