On Persistency of Excitation
[stability of adaptive systems]

Antonio Loría  Elena Panteley

IIT Bombay, Feb. 2018
Outline

- **Introduction**
  - Preliminaries: motivations, definitions
  - Recall on (basic) adaptive control

- **Linear time-varying systems**
  - Uniform persistency of excitation
  - A result on convergence rates
  - MRAC-type systems (SPR)

- **Nonlinear systems**
  - Uniform $\delta$-persistency of excitation
  - Necessary and sufficient conditions for stability

- **General conclusions**
Theorem (KYP) Let $Z(s) = C^\top [sI - A]^{-1}B$ be a $p \times p$ transfer function s.t.:
- the pair $(A, B)$ is completely controllable;
- the pair $(A, C)$ is completely observable.

Then, $Z(\cdot)$ is strictly positive real if and only if there exists a positive definite matrix $P$ such that

$$PA + A^\top P = -Q$$
$$PB = C^\top.$$ 

Theorem The matrix $A$ is Hurwitz (its eigen-values have strictly negative real parts) if and only if for any $Q = Q^\top$, positive definite, there exists $P = P^\top > 0$ s.t.

$$PA + A^\top P = -Q$$
[ Preliminaries ]

Definition 1 (Persistency of excitation)

A locally integrable function \( \Phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m \times n} \) is said to be persistently exciting if there exist \( T \) and \( \mu > 0 \) such that

\[
\int_{t}^{t+T} \Phi(s)\Phi(s)^\top ds \geq \mu \quad \forall t \geq 0
\]  

(1)

Remarks

- \( \Phi \), in the definition, is a function of time, only
- Typically, \( m \geq n \) hence, \( \Phi(t)\Phi(t)^\top \) is rank deficient for each \( t \geq 0 \) however, (1) may still hold; it is a lowerbound on the “average” of \( \Phi(t)\Phi(t)^\top \)

In dynamical systems:

e.g., \( \dot{x} = Ax \) “is GES” if \( A \) is Hurwitz (full rank and \( \lambda_{iR}(A) < 0 \))

\( \dot{x} = -\Phi(t)\Phi(t)^\top x \) is still GES iff \( \Phi \) is PE, even if \( \lambda_{iR}(-\Phi(t)\Phi(t)^\top) \not< 0 \)
Illustration of persistency of excitation

Consider the system $\dot{x} = -a(t)x$. Seemingly, $\exists \mu, \mu > 0 : \int_{t}^{t+3} a(s)^2 ds \geq \mu$
[ Preliminaries ]

**Fact.** Consider the system

\[ \dot{x} = -a(t)^2 x, \]

with \( a(t), \dot{a}(t) \) bounded.

The origin is globally exponentially stable iff there exist \( \mu, T > 0 \) such that

\[ \int_{t}^{t+T} a(s)^2 ds \geq \mu \quad \forall t \geq 0. \]

**Gradient systems.** Consider the system

\[ \dot{x} = -\Phi(t)\Phi(t)^T x, \quad \Phi(t) \in \mathbb{R}^{m \times n}, \quad m \geq n \]

with \( \Phi(t), \dot{\Phi}(t) \) bounded.

The origin is globally exponentially stable iff \( \Phi \) is persistently exciting.

—see e.g., [Anderson et al; Narendra & Annaswamy; Sastry & Bodson; ...]

**Rmk.** Convergence rates: [Sukumar et al; Loria & Panteley; Brocket ...]
Lemma 1 (linear MRAC). Consider the linear time-varying (LTV) system

\[
\begin{bmatrix}
\dot{e} \\
\dot{\tilde{\theta}}
\end{bmatrix} =
\begin{bmatrix}
A & B\phi(t)^T \\
-\phi(t)C & 0
\end{bmatrix}
\begin{bmatrix}
e \\
\tilde{\theta}
\end{bmatrix},
\]

- \( e \in \mathbb{R}^n \) is the tracking error
- \( \tilde{\theta} \in \mathbb{R}^m \) is the parameter estimation error
- \( \phi : \mathbb{R} \rightarrow \mathbb{R}^m \) is the regressor function.

Assume that:

- The triple \((A, B, C)\) is strictly positive real (satisfies the KYP lemma):
  \[
  V := z^T P z > 0 \implies \dot{V} = -|e|^2 \leq 0;
  \]

- \( \phi \) is absolutely continuous; \( \phi \) and \( \dot{\phi} \) are bounded almost everywhere;

Then, the origin is uniformly globally exponentially stable if and only if \( \phi \) is PE.
Introduction
[Basics on adaptive control]

Consider the linear autonomous system

\[ \dot{x} = Ax + Bu \]
\[ y =Cx \]

in canonical form.

- Let \((A,B)\) be controllable and \((A,C)\) be observable.
- Because \((A,B)\) is controllable, we can perform pole placement:
  
  \[ \text{there exists (a row vector) } K \text{ such that } (A - BK) \text{ is Hurwitz} \]
- However, if there is uncertainty in \(A\) we cannot compute the appropriate \(K\)
- Let \(u = -\hat{K}x\) where \(K\) is an estimate of (the ideal) \(K\);
  
  let \(\tilde{K} := \hat{K} - K\) then,

\[ \dot{x} = (A - BK)x - B\tilde{K}x \]
\[ y = Cx \]
Analysis.

- Let $A := A - BK$. By design, this matrix is Hurwitz.
- Also, the pair $(A, C)$ is controllable and $PB = C^\top$ therefore, let
  \[ V = \frac{1}{2}x^\top Px + \frac{1}{2\gamma}\tilde{K}\tilde{K}^\top \]
  \[ \implies \dot{V} = -x^\top [A^\top P + PA] x - x^\top PB x^\top \tilde{K}^\top + \frac{1}{\gamma}\dot{\tilde{K}}\tilde{K}^\top \]
- We use the (passivity-based) update law: $\dot{\tilde{K}} = \gamma x^\top C^\top x^\top$

Then:
  \[ \dot{V} = -x^\top Qx \]

Claim. [after adaptive control texts]: $x \to 0$ and $\tilde{K}$ is bounded.

Proof: After ch. III-Lemma 1, if a once continuously differentiable function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfies
  \[ \varphi, \dot{\varphi} \in \mathcal{L}_\infty, \quad \varphi \in \mathcal{L}_2. \]
Then, necessarily $\lim_{t \to \infty} \varphi(t) = 0$.

Rmk. Does $\tilde{K} \to 0$?
[Basics on adaptive control]

**Fact:** Adaptive control systems are, in general, **nonlinear time-varying**

The closed-loop system has the (familiar) form

\[
\dot{x} = Ax + B(t)\tilde{\theta}, \quad B(t) := -Bx(t)^\top \in \mathbb{R}^{n \times n}
\]

\[
\dot{\tilde{\theta}} = -\gamma C(t)x, \quad C(t) := -x(t)B^\top P \in \mathbb{R}^{n \times n}
\]

\[
A := (A - BK)
\]

We have: \(\tilde{\theta} \in L_\infty, \ x \to 0\)

\[
\tilde{\theta} = \tilde{K}^\top
\]

**Rmk.** The notations on the right are convenient, but, at best, ambiguous!

- For a start, the matrix \(B(t)\) depends on state trajectories hence, on the initial conditions (uniformity ...)

- If we the goal is to stir \(x(t) \to 0\), how to pretend to use persistency of excitation? \((x \equiv 0 \implies B \equiv 0\) convergence of \(\tilde{\theta}\)...)  

**Problem:** How do we ensure (uniform) stability and convergence?
Consider now the tracking control problem, to stir $x \rightarrow x^*$, for a pair of systems:

**Plant:**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= \Phi(x)\top \theta + g(x)u
\end{align*}
\]

**Reference model:**

\[
\begin{align*}
\dot{x}_1^* &= x_2^* \\
\vdots \\
\dot{x}_{n-1}^* &= x_n^* \\
\dot{x}_n^* &= f(x^*)
\end{align*}
\]

Let $u := g(x)^{-1}[f(x^*) - \Phi(x)\top \hat{\theta} - K(\cdot)e]$ and $\hat{\theta} = \gamma \Phi(x)e_n$

Then, define the error $e := x - x^*$. Its dynamics corresponds to

\[
\begin{align*}
\begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots \\ \vdots & \ddots & \ddots \\ -k_1 & \cdots & -k_n \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \Phi(x)\top \hat{\theta} \\
\dot{\hat{\theta}} &= \gamma \Phi(x)[0 \cdots 1]e
\end{align*}
\]
Common mistake.

Such closed-loop system, is commonly written in the compact form:

\[
\begin{bmatrix}
\dot{e} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
A & B\Phi^\top \\
-\Phi C & 0
\end{bmatrix}
\begin{bmatrix}
e \\
\tilde{\theta}
\end{bmatrix}, \quad z :=
\begin{bmatrix}
e \\
\tilde{\theta}
\end{bmatrix}
\]

Then, global exponential stability is some times claimed invoking Lemma 1; converse theorems are used to establish statements on robust stability, . . . !

Rmk. The function $\Phi$ depends on $x$ and, since $x := e + x^*(t)$, the system dynamics is, actually, nonlinear:

\[
\begin{bmatrix}
\dot{e} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
A & B\phi(t, z)^\top \\
-\phi(t, z)C & 0
\end{bmatrix}
\begin{bmatrix}
e \\
\tilde{\theta}
\end{bmatrix}, \quad \phi(t, z) := \Phi(e + x^*(t))
\]

while the system in Lemma 1 is \textit{linear}!!
Problem statement

How do we infer the (asymptotic) stability of the origin of

\[
\begin{bmatrix}
\dot{e} \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
A & B\phi(t, z)^T \\
-\phi(t, z)C & 0
\end{bmatrix}\begin{bmatrix}
e \\
\theta
\end{bmatrix},
\quad x := \begin{bmatrix}
e \\
\theta
\end{bmatrix}
\]

with $A$ Hurwitz, $(A, B)$ controllable, and $(A, C)$ observable?

What is more, how to guarantee the stability of the origin for

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
A(\cdot) & B(\cdot) \\
C(\cdot) & 0
\end{bmatrix}\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

where $A$, $B$ and $C$ are, generally speaking, functions of time and the states but have “certain structural properties”?

**Rmk.** We do not want to assume that $B(\cdot)$ is full rank
Consider the case-study:

\[
\begin{bmatrix}
\dot{e} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
A & B\phi(t, z)^\top \\
-\phi(t, z)C & 0
\end{bmatrix}
\begin{bmatrix}
e \\
\theta
\end{bmatrix}, \quad z := \begin{bmatrix}
e \\
\theta
\end{bmatrix}
\]

and assume that we know \( P \) such that, defining,

\[
V := e^\top Pe + \frac{1}{2}|\theta|^2 > 0,
\]

we obtain

\[
\dot{V} = -|e|^2 \leq 0.
\]

Inspired by Lemma 1, can we conjecture that some \textit{boundedness} conditions on \( \phi(t, z) \) in addition to \textit{persistency of excitation} should suffice for UGAS (UGES?).

\textbf{Problem:} What does PE mean for the \textit{state-dependent} function \( \phi(t, z) \)?

Some authors use:

\[
\int_t^{t+T} \phi(\tau, z(\tau, t_\circ, z_\circ))\phi(\tau, z(\tau, t_\circ, z_\circ))^\top d\tau \geq \mu I \quad \forall t \geq t_\circ.
\]
The solutions are bounded (UGS). Hence, we (re)consider the system as parameterized linear time-varying:

\[
\begin{bmatrix}
\dot{\bar{e}} \\
\dot{\bar{\theta}}
\end{bmatrix} =
\begin{bmatrix}
A & B\phi(t, z(t, t_o, z_o))^T \\
-\phi(t, z(t, t_o, z_o))C & 0
\end{bmatrix}
\begin{bmatrix}
\bar{e} \\
\bar{\theta}
\end{bmatrix}
\]

with i.c.: \((t_*, \bar{z}_*)\)

\(z(t)\) are solutions of the original NL system

Then, we observe the following:

- If we assume that \(\phi(t, z(t, t_o, z_o))\) is persistently exciting, \(i.e.,\)

\[
\int_{t}^{t+T} \phi(\tau, z(\tau, t_o, z_o))\phi(\tau, z(\tau, t_o, z_o))^T d\tau \geq \mu I \quad \forall t \geq t_o.
\]

(and if it is also bounded with bounded derivative) then, the origin is globally exponentially stable uniformly in the initial conditions \((t_*, \bar{z}_*)\).

- Iff the initial conditions \((t_*, \bar{z}_*) = (t_o, z_o)\) then, \(\bar{z}(t, t_*, \bar{z}_*) = z(t, t_o, z_o),\)
The solutions are bounded (UGS). Hence, we (re)consider the system as parameterized linear time-varying:

\[
\begin{bmatrix}
\dot{e} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
A & B\phi(t, z(t, t_0, z_0))^	op \\
-\phi(t, z(t, t_0, z_0))C & 0
\end{bmatrix}
\begin{bmatrix}
e \\
\theta
\end{bmatrix}
\]

with i.c.: \((t_*, z_*)\) \(z(t)\) are solutions of the original **NL** system

However, in

\[
\int_{t_0}^{t+T} \phi(\tau, z(\tau, t_0, z_0))\phi(\tau, z(\tau, t_0, z_0))^	op d\tau \geq \mu I \quad \forall t \geq t_0,
\]

**[Q1]** \(\mu,\) and \(T\) depend on the initial conditions that generate the trajectories of the original **nonlinear system** hence, we loose uniformity in \((t_0, z_0)\)

**[Q2]** What if \(\phi(t, 0) \equiv 0\) ? \ldots \quad the **PE** property is **lost** near the origin!

**Rmk.** We cannot claim global exponential stability for the nonlinear system
Linear parameterised time-varying systems
[ Q1: Problem statement ]

Let $\mathcal{D}$ be a closed set and let $\lambda \in \mathcal{D}$ be a parameter (e.g. $\lambda := (t_0, z_0)$, $\mathcal{D} := \mathbb{R}_{\geq 0} \times \mathbb{R}^n$)

We shall study systems of the form

$$
\begin{bmatrix}
\dot{e} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
A(t, \lambda) & B(t, \lambda)^\top \\
-C(t, \lambda) & 0
\end{bmatrix}
\begin{bmatrix}
e \\
\theta
\end{bmatrix},
\quad
z :=
\begin{bmatrix}
e \\
\theta
\end{bmatrix}
$$

(LTV)

where $e \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$, $A(t, \lambda) \in \mathbb{R}^{n \times n}$, $B(t, \lambda) \in \mathbb{R}^{n \times p}$, $C(t, \lambda) \in \mathbb{R}^{n \times p}$ are uniformly bounded.

We aim at establishing uniform exponential stability of the origin, i.e., that there exist $r$, $k$ and $\gamma > 0$ such that for all $t \geq t_0$, all $t_0 \geq 0$ and all $\lambda \in \mathcal{D}$,

$$
|z_0| < r \Rightarrow |z(t, \lambda, t_0, z_0)| \leq k |z_0| e^{-\gamma (t-t_0)}.
$$
Definition 2 (\(\lambda\)-uniform persistency of excitation) Let \(\phi : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n \times m}\), \(\phi(t, \lambda)\) be absolutely continuous in both arguments. We say that \(\phi(t, \lambda)\) is \(\lambda\)-uniformly persistently exciting (\(\lambda\)-uPE) if there exist \(\mu\) and \(T > 0\) such that

\[
\int_{t}^{t+T} \phi(\tau, \lambda)\phi(\tau, \lambda)^{\top} d\tau \geq \mu I, \quad \forall \ t \geq 0, \lambda \in \mathcal{D}.
\]

Lemma 2 (Measure Lemma) Consider a function \(\phi : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}\). Assume that there exists \(\phi_M\) such that \(|\phi(t, \lambda)| \leq \phi_M\) for all \(t \geq 0\) and all \(\lambda \in \mathcal{D}\). Assume further that \(\phi(\cdot, \cdot)\) is \(\lambda\)-uPE. Then, for any \(t \geq 0\) the measure of the set

\[
I_{\mu,t} := \left\{ \tau \in [t, t+T] : |\phi(\tau, \lambda)| \geq \frac{\mu}{2T\phi_M} \right\}
\]

satisfies

\[
\text{meas}[I_{\mu,t}] \geq \sigma_{\mu} := \frac{T\mu}{2T\phi_M^2} - \mu.
\]
Linear parameterised time-varying systems

[Example]

Claim. The origin of $\dot{x} = -\phi(t, \lambda)^2 x$ is uniformly globally exponentially stable

Idea: Let $V(x) := \frac{1}{2} |x|^2$ so that

$$\dot{V} = -\phi(t, \lambda)^2 x^2 \leq 0 \quad (\Rightarrow \text{UGS}).$$

Rmk. On each window $[t, t + T]$ there is a collection of intervals $I_{\mu,t}$ during which $\phi(t, \lambda)^2 \geq 0.5$ and $V(x(t))$ takes a “good” decrease
Linear parameterised time-varying systems

[ The essential tools ]

Lemma 3 (Integration lemma for UGES) Assume that there exist constants $r, c, p > 0$ such that the solution $x(\cdot; \lambda, t_\circ, x_\circ)$ of $\dot{x} = f(t, \lambda, x)$ satisfies

$$\max\left\{ |x|_\infty, |x|_p \right\} \leq c |x_\circ|$$

(3)

for all $x_\circ \in B_r$ and all $t_\circ \geq 0$. Then, the system is $\lambda$-ULES with $k_\lambda := ce^{1/p}$ and $\gamma_\lambda := [p c^p]^{-1}$. Moreover, if $c > 0$ exists for all $x_\circ \in \mathbb{R}^n$, the system $\lambda$-UGES (GES unif. in the i.c. and in $\lambda$).

Lemma 4 (Output injection) Let $A : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n \times n}$, $C : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{m \times n}$, and $K : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n \times m}$ be continuous and bounded on their domains.

- Assume that the origin of the system $\dot{x} = A(t, \lambda)x$ is $\lambda$-UGES.
- Then, the system $\dot{x} = A(t, \lambda)x + K(t, \lambda)y$ where $y := C(t, \lambda)x$, is $\lambda$-UGES if there exists $c > 0$ such that

$$\int_{t_\circ}^{\infty} |y(s)|^2 \, ds \leq c^2 |x_\circ|^2 \quad \forall (t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n.$$  (4)
Lemma 5 (Speed-gradient systems) For the system

\[ \dot{x} = -\phi(t, \lambda)\phi(t, \lambda)^\top x, \quad \phi(t, \lambda) \in \mathbb{R}^{m \times n} \]

assume that \( \phi(t, \lambda) \) is \( \lambda \)-uPE with parameters \( T \) and \( \mu > 0 \) and there exists a constant \( \phi_M > 0 \) such that, for almost all \( t \geq 0 \) and all \( \lambda \in \mathcal{D} \)

\[
\max \left\{ |\phi(t, \lambda)|, \left| \frac{\partial \phi(t, \lambda)}{\partial t} \right| \right\} \leq \phi_M. \tag{5}
\]

Then the system is \( \lambda \)-UGES with

\[
k = 1, \quad \gamma \geq \frac{\mu}{e^2 T[1 + \phi_M^4 T^2]}
\]

That is,

\[
|x(t)| \leq k|x_0|e^{-\lambda(t-t_0)} \quad \forall t \geq t_0, \ t_0 \geq 0, \ \lambda \in \mathcal{D}
\]
Theorem 1 (UGES of LTV) The origin of the system
\[
\begin{bmatrix}
\dot{e} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
A(t, \lambda) & B(t, \lambda)^\top \\
-C(t, \lambda) & 0
\end{bmatrix}
\begin{bmatrix}
e \\
\theta
\end{bmatrix},
z :=
\begin{bmatrix}
e \\
\theta
\end{bmatrix},
\]
(6)

under Assumptions 1 and 2, is $\lambda$-UGES if and only if $B(t, \lambda)$ is $\lambda$-uPE.

Assumption 1 there exists $b_M > 0$ such that, for almost all $t \geq 0$ and all $\lambda \in \mathcal{D}$
\[
\max \left\{ |A(t, \lambda)|, |B(t, \lambda)|, \left| \frac{\partial B(t, \lambda)}{\partial t} \right| \right\} \leq b_M.
\]
Assumption 2 There exist symmetric matrices $P(t, \lambda)$ and $Q(t, \lambda)$ such that
\[
P(t, \lambda)B(t, \lambda)^\top = C(t, \lambda)^\top
\]
\[-Q(t, \lambda) := A(t, \lambda)^\top P(t, \lambda) + P(t, \lambda)A(t, \lambda) + \dot{P}(t, \lambda)
\]
There exist $p_m, q_m, p_M, \text{ and } q_M > 0$ such that, for all $(t, \lambda) \in \mathbb{R}_{\geq 0} \times \mathcal{D}$,
\[
p_m I \leq P(t, \lambda) \leq p_M I,
q_m I \leq Q(t, \lambda) \leq q_M I
\]
Proof of Theorem 1. We split the system and use output injection:

First, consider the globally invertible change of coordinates:

\[
\begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
-B(t, \lambda) & I
\end{bmatrix}
\begin{bmatrix}
e \\
\theta
\end{bmatrix}
\]

so \( \{ z = 0 \} \) is \( \lambda \)-UGES for (6) if and only if so is \( \{ \xi = 0 \} \) for the system

\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{bmatrix} = \begin{bmatrix}
A(t, \lambda) & B(t, \lambda)^\top \\
-R_1(t, \lambda) & -B(t, \lambda)B(t, \lambda)^\top
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix} + \begin{bmatrix}
B(t, \lambda)^\top B(t, \lambda) \\
R_1(t, \lambda) - R_2(t, \lambda)
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
K(t, \lambda)
\end{bmatrix},
\]

We establish that:

1) the origin of \( \dot{\xi} = A(t, \lambda) \) is \( \lambda \)-UGES,
2) the solutions \( \xi(t, \lambda) \) are uniformly bounded,
3) \( \xi_1 \) is square integrable (uniformly in \( \lambda \)), and
4) \( K(t, \lambda) \) is bounded.
Corollary. The solutions satisfy the bound:

\[ |x(t, \lambda)| \leq t_M t^{\text{inv}}_M \left( \frac{\pi e}{\rho} \right)^{1/2} |x_\infty| e^{-\frac{\rho}{2\pi}(t - t_\infty)} \quad \forall t \geq t_\infty. \]

where:

\[ \pi := c_{32} + (c_* t^{\text{inv}}_M)^2 \left[ \frac{(c_{32} k_M)^2}{4(1 - \rho)} \right], \quad 0 < \rho \leq \min \left\{ p_m, \frac{1}{2k^2_M} \right\} \]

\[ c_{32} := \max \left\{ p_M, \frac{1}{2\gamma_x} \right\}, \quad \gamma_x := \frac{\mu}{T(1 + b^2_M T)} \]

- \( \gamma_x \) is the convergence rate for \( x(t, \lambda) \) in \( \dot{x}(t, \lambda) = -B(t, \lambda)B(t, \lambda)^\top x(t, \lambda) \)
- \( c_* \) is a bound on \( |e|_2 = \left( \int_{t_\infty}^{\infty} |e(t, \lambda)|^2 \right)^{1/2} \)
- \( t_M, t^{\text{inv}}_M \) are bounds on coordinates transformations
- \( k_M \) is a bound on an output injection term
- \( b_M \) is the bound on \( B(t, \lambda) \) and its derivative
Problem statement
[Model-Reference-Adaptive-Control]

“Since the solutions are bounded (UGS) one can consider the LTV system”:

\[
\begin{bmatrix}
\dot{\bar{e}} \\
\dot{\bar{\theta}}
\end{bmatrix}
= \begin{bmatrix}
A & B\phi(t, z(t, t_o, z_o))^\top \\
-\phi(t, z(t, t_o, z_o))C^\top & 0
\end{bmatrix}
\begin{bmatrix}
\bar{e} \\
\bar{\theta}
\end{bmatrix}
\]

\[
z(t) = [e(t)^\top, \theta(t)^\top]^\top
\]

(solutions of the original NL system)

However, in

\[
\int_{t}^{t+T} \phi(\tau, z(\tau, t_o, z_o))\phi(\tau, z(\tau, t_o, z_o))^\top d\tau \geq \mu I \quad \forall t \geq t_o.
\]

[Q1] \(\mu\), and \(T\) depend on the initial conditions that generate the trajectories of the original nonlinear system hence, we loose uniformity in \((t_o, z_o)\)

[Q2] What if \(\phi(t, 0) \equiv 0\) ? ... the PE property is lost near the origin!
Persistency of excitation for nonlinear systems

[ Q2: what if $\phi(t, 0 \equiv 0)$? ]

Example 1  Consider the system  
\[ \dot{z} = -\sin(t)^2 z^3 \quad \text{or, equivalently,} \]
\[ \dot{x} = -\sin(t)^2 z(t, \lambda)^2 x, \quad x_0 = z_0, \quad t^x = t^z := t_0 \]

• Assume that, given any $\delta > 0$, $\exists I_\delta \subset \mathbb{R}_{\geq 0}$, such that 
\[ |z(t, \lambda)| \geq \delta \quad \forall t \in I_\delta \]
then, defining $v(t) := \frac{1}{2} x(t)^2$, we have 
\[ \dot{v}(t) = -\delta^2 \sin(t)^2 v(t) \quad \forall t \in I_\delta \]

On the other hand,  
\[ \int_t^{t+\pi} \sin(\tau)^2 \delta^2 d\tau = \frac{\pi}{2} \delta^2 \]
that is, $\dot{v}(t) = -\varphi(t)^2 v(t)$, where $\varphi(t) := \sin(\tau)\delta$ is PE.

• We conclude that:  
\[ |z(t, \lambda)| \geq \delta \implies |z(t, \lambda)| \to 0 \quad \text{exponentially fast!} \]

• If this holds for any $\delta > 0$ we recover uniform attractivity
Persistency of excitation for nonlinear systems

[ Rationale ]

- The origin is UGS, i.e.
  \[
  \exists \gamma \in \mathcal{K}_{\infty} : \sup_{t \geq t_0} |z(t)| \leq \gamma (|z(t_0)|)
  \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE
Persistency of excitation for nonlinear systems

[ Rationale ]

- The origin is UGS, i.e.,

\[ \exists \gamma \in \mathcal{K}_\infty : \sup_{t \geq t_0} |z(t)| \leq \gamma (|z(t_0)|) \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE hence, exponential convergence to zero
- \( \delta \)-close to the origin, PE is lost
Persistency of excitation for nonlinear systems

[Rationale]

- The origin is UGS, i.e.,
  \[ \exists \gamma \in \mathcal{K}_\infty : \sup_{t \geq t_0} |z(t)| \leq \gamma (|z(t_0)|) \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE hence, exponential convergence to zero

- \( \delta \)-close to the origin, PE is lost

- Attractivity:
  - For each \( \varepsilon > 0 \) and \( r > 0 \), \( \exists T > 0 \) s.t.
    \[ |z(t_0)| \leq r \implies |z(t)| \leq \varepsilon \quad \forall t \geq t_0 + T \]
  - For each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) \) s.t.
    \[ |z(t'_0)| \leq \delta \implies |z(t)| \leq \varepsilon \quad \forall t \geq t'_0 \]
Persistency of excitation for nonlinear systems

[ Rationale ]

- The origin is UGS, i.e.,

\[ \exists \gamma \in \mathcal{K}_\infty : \sup_{t \geq t_0} |z(t)| \leq \gamma(|z(t_0)|) \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE

hence, exponential convergence to zero

- \( \delta \)-close to the origin, PE is lost

- Attractivity:

For each \( \varepsilon > 0 \) and \( r > 0 \), \( \exists T > 0 \) s.t.

\[ |z(t_0)| \leq r \implies |z(t)| \leq \varepsilon \quad \forall t \geq t_0 + T \]

- For each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) \) s.t.

\[ |z(t'_0)| \leq \delta \implies |z(t)| \leq \varepsilon \quad \forall t \geq t'_0 \]
Persistency of excitation for nonlinear systems

[Rationale]

- The origin is UGS, *i.e.*, \[ \exists \gamma \in \mathcal{K}_\infty : \sup_{t \geq t_0} |z(t)| \leq \gamma (|z(t_0)|) \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE hence, exponential convergence to zero

- \( \delta \)-close to the origin, PE is lost

Attractivity:
For each \( \epsilon > 0 \) and \( r > 0 \), \( \exists \ T > 0 \) s.t.
\[ |z(t_0)| \leq r \implies |z(t)| \leq \epsilon \quad \forall t \geq t_0 + T \]

For each \( \epsilon > 0 \) there exists \( \delta(\epsilon) \) s.t.
\[ |z(t'_0)| \leq \delta \implies |z(t)| \leq \epsilon \quad \forall t \geq t'_0 \]
Persistency of excitation for nonlinear systems

[ Rationale ]

- The origin is UGS, i.e.,
  \[ \exists \gamma \in \mathcal{K}_\infty : \sup_{t \geq t_o} |z(t)| \leq \gamma (|z(t_0)|) \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE hence, exponential convergence to zero
- \( \delta \)-close to the origin, PE is lost

- Attractivity:
  - For each \( \varepsilon > 0 \) and \( r > 0 \), \( \exists T > 0 \) s.t.
  \[ |z(t_0)| \leq r \implies |z(t)| \leq \varepsilon \quad \forall t \geq t_0 + T \]
  - For each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) \) s.t.
  \[ |z(t'_0)| \leq \delta \implies |z(t)| \leq \varepsilon \quad \forall t \geq t'_0 \]
Persistency of excitation for nonlinear systems

[Rationale]

- The origin is UGS, i.e.,
  \[ \exists \gamma \in \mathcal{K}_\infty : \sup_{t \geq t_0} |z(t)| \leq \gamma (|z(t_0)|) \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE hence, exponential convergence to zero
- \( \delta \)-close to the origin, PE is lost

- Attractivity:
  For each \( \varepsilon > 0 \) and \( r > 0 \), \( \exists T > 0 \) s.t.
  \[ |z(t_0)| \leq r \implies |z(t)| \leq \varepsilon \quad \forall t \geq t_0 + T \]
  For each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) \) s.t.
  \[ |z(t'_0)| \leq \delta \implies |z(t)| \leq \varepsilon \quad \forall t \geq t'_0 \]
Persistency of excitation for nonlinear systems

[Rationale]

- The origin is UGS, \( i.e. , \)
  \[ \exists \gamma \in \mathcal{K}_\infty : \sup_{t \geq t_o} |z(t)| \leq \gamma (|z(t_o)|) \]

- Trajectories \( \delta \)-far from the origin \( \Rightarrow \) PE hence, exponential convergence to zero
- \( \delta \)-close to the origin, PE is lost

- Attractivity:
  For each \( \varepsilon > 0 \) and \( r > 0 \), \( \exists \ T > 0 \) s.t.
  \[ |z(t_o)| \leq r \implies |z(t)| \leq \varepsilon \quad \forall \ t \geq t_o + T \]

- For each \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) \) s.t.
  \[ |z(t'_o)| \leq \delta \implies |z(t)| \leq \varepsilon \quad \forall \ t \geq t'_o \]
Persistency of excitation for nonlinear systems

[Uniform \( \delta \)-Persistency of excitation]

Consider nonlinear time-varying systems:

\[
\dot{x} = F(t, x)
\]

where \( F(\cdot, \cdot) \) is such that solutions exist (locally) and are unique.

Let \( x \in \mathbb{R}^n \) be partitioned into \( x := [x_1^T, x_2^T]^T \) where \( x_1 \in \mathbb{R}^{n_1} \) and \( x_2 \in \mathbb{R}^{n_2} \). Define the column vector function \( \phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m \) to be such that \( \phi(\cdot, x) \) is locally integrable for each \( x \in \mathbb{R}^n \).

**Definition 3 (U\( \delta \)-PE)** A function \( \phi(\cdot, \cdot) \) is said to be uniformly \( \delta \)-persistently exciting with respect to \( x_1 \) if for each \( x \in (\mathbb{R}^{n_1} \setminus \{0\}) \times \mathbb{R}^{n_2} \) there exist \( \delta > 0, T > 0 \) and \( \mu > 0 \) s.t.

\[
|z - x| \leq \delta \quad \Rightarrow \quad \int_t^{t+T} |\phi(\tau, z)| \, d\tau \geq \mu \quad \forall t \in \mathbb{R}. \tag{8}
\]
Lemma 6  The function \( \phi(\cdot, \cdot) \) is U\( \delta \)-PE with respect to \( x_1 \) if and only if

(A) for each \( \delta > 0 \) and \( \Delta \geq \delta \) there exist \( T > 0 \) and \( \mu > 0 \) such that, for all \( t \in \mathbb{R} \),

\[
| x_1 | \in [\delta, \Delta], \ | x_2 | \in [0, \Delta] \implies \int_t^{t+T} | \phi(\tau, x) | d\tau \geq \mu \quad \forall \ t \in \mathbb{R}.
\]

Example 2  Remember the system \( \dot{x} = -\sin(t)^2 x^3 \); for the function \( \phi(t, x) := \sin(t)^2 x^2 \) we have:

\[
x \in [\delta, \Delta] \implies \int_t^{t+\pi} \sin(\tau)^2 \delta^2 d\tau = \frac{\pi}{2} \delta^2
\]

That is, \( \phi(t, x) := \sin(t)^2 x^2 \) is U\( \delta \)-PE.
**Lemma 7** If \( (t, x) \mapsto \phi \) is continuous in \( x \) uniformly in \( t \) then \( \phi(\cdot, \cdot) \) is \( \text{U}_\delta \text{-PE} \) with respect to \( x_1 \) if and only if

(B) for each \( x \) such that \( x_1 \neq 0 \) there exist \( T > 0 \) and \( \mu > 0 \) such that,

\[
\int_{t}^{t+T} |\phi(\tau, x)| \, d\tau \geq \mu \quad \forall t \in \mathbb{R}
\]


**Example 3** Let \( \phi(t, x) := \Phi(t)^\top x \). Then, \( \phi(t, x) \) is \( \text{U}_\delta \text{-PE} \) with respect to \( x \) if and only if there exist \( T \) and \( \mu > 0 \) such that

\[
\int_{t}^{t+T} \Phi(\tau)\Phi(\tau)^\top \, d\tau \geq \mu I \quad \forall t \in \mathbb{R}.
\]
Example 4 Consider once more the function $\phi(t, x) := \sin(t)^2 x^2$ which is uniformly continuous. We see that

$$x \neq 0 \implies \int_t^{t+\pi} \sin(\tau)^2 x^2 d\tau = \frac{\pi}{2} x^2 =: \mu(x)$$

Actually, in general, we also have the following:

Lemma 8 The function $\phi(\cdot, \cdot)$ is $U\delta$-PE with respect to $x_1$ if and only if

(C) for each $\Delta > 0$ there exist $\mu_\Delta \in K$ and $\theta_\Delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ continuous strictly decreasing such that, for all $t \in \mathbb{R}$,

$$\left\{ |x_1| \neq 0, |x_2| \in [0, \Delta] \right\} \implies \int_t^{t+\theta_\Delta(|x_1|)} |\phi(\tau, x)| d\tau \geq \mu_\Delta(|x_1|).$$

Rmk. It is clear that, in general, for nonlinear functions, the “PE bound” depends on the “parameter” $x$. 
Persistency of excitation for nonlinear systems

[\text{U}_\delta\text{-PE: A sufficient and necessary condition}]

\textbf{Theorem 2 (UGAS }\Rightarrow\text{ U}_\delta\text{-PE)} \quad \text{The origin of the system}
\begin{equation*}
\dot{x} = F(t, x)
\end{equation*}
where \( F(\cdot, \cdot) \) is Lipschitz in \( x \) uniformly in \( t \), is UGAS only if \( F(\cdot, \cdot) \) is U\( \delta \)\text{-PE with respect to } x \in \mathbb{R}^n. \quad \bullet
\end{equation*}

\textbf{Rmk.} Sufficiency also holds under extra conditions.

\textbf{Proposition 1} \quad \text{The origin of the system}
\begin{equation*}
\dot{z} = -v(t)^2z^3
\end{equation*}
is UGAS if and only if \( v(t) \) is persistently exciting (in the usual sense). \quad \bullet
\end{equation*}

\textbf{Sketch of proof:} The origin is UGS because \( V = |z|^2 \) yields \( \dot{V} = -v(t)^2z^4 \leq 0 \).
The function \( \phi(t, z) = v(t)^2z^3 \) is U\( \delta \)\text{-PE:}
\begin{equation*}
\int_t^{t+T} v(\tau)^2d\tau \geq \mu \quad \forall t \geq 0, \; z \neq 0 \quad \Rightarrow \quad \int_t^{t+T} |v(\tau)^2z^3| \, d\tau \geq \mu |z|^3
\end{equation*}
Theorem 3 [11] Consider the system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
A(t, x) & B(t, x) \\
C(t, x) & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

under the following assumptions:

- We have a Lyapunov function \( V \) such that
  \[
  \alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|)
  \]
  \[
  \dot{V}(t, x) \leq -\alpha_3(|x_1|) \quad \text{a.e.}
  \]

- The functions \( A, B \) and \( C \) are locally Lipschitz in \( x \) uniformly in \( t \), uniformly bounded in \( t \), \( B \) is once differentiable with partial derivatives in \( t \), and
  \( A(t, x)|_{x_1=0} = C(t, x)|_{x_1=0} = 0 \).

The origin is UGAS if and only if \( B(t, x)x_2|_{x_1=0} \) is \( U\delta\)-PE with respect to \( x_2 \).
Persistency of excitation for nonlinear systems
[Slotine & Li adaptive controller]

Proposition 2 Consider the lossless Lagrangian system (without friction)
\[ D_\theta(q)\ddot{q} + C_\theta(q, \dot{q})\dot{q} + g_\theta(q) = u \]
in closed loop with the certainty-equivalence controller
\[ u = D_\dot{\theta}(q)\dot{q}_r + C_\dot{\theta}(q, \dot{q})\dot{q}_r + g_\dot{\theta}(q) - k_ds \]
\[ \dot{\hat{\theta}} = -\Gamma\Phi(t, s, \tilde{q})^T s \]
\[ \dot{\tilde{q}} := \dot{q}_d - \lambda\tilde{q}, \quad s := \dot{q} - \dot{q}_r \]

Then, the origin if uniformly globally asymptotically stable for any \( \lambda, \ k_d > 0 \)
if and only if \( \Phi_o(t) := \Phi(t, 0, 0) \) is persistently exciting, that is
\[ \int_{t}^{t+T} \Phi_o(\tau)\Phi_o(\tau)^T d\tau \geq \mu, \ \forall t \geq 0. \]

Rmk. Note that \( \Phi_o(t) \) is such that
\[ \Phi_o(t)^T \theta = D_\theta(q_d(t))\ddot{q}_d(t) + C_\theta(q_d(t), \dot{q}_d(t))\dot{q}_d(t) + g_\theta(q_d(t)) \]
Analysis of the closed-loop system.

The closed-loop dynamics, for which we have $x_1 \rightarrow 0$, is

\[
\begin{bmatrix}
\dot{\tilde{q}} \\
\dot{s}
\end{bmatrix} = \begin{bmatrix}
-\lambda I & I \\
0 & -D_{\theta}^{-1}(\cdot)[C_{\theta}(\cdot) + k_d I]
\end{bmatrix} \begin{bmatrix}
\tilde{q} \\
s
\end{bmatrix} + \begin{bmatrix}
0 \\
D_{\theta}^{-1}(\cdot)\Phi(t, \tilde{q}, s)^{\top}
\end{bmatrix} \dot{\theta}
\]

\[
\dot{\theta} = -\Gamma^{-1} \begin{bmatrix}
0 & \Phi(t, \tilde{q}, s)D_{\theta}(\cdot)
\end{bmatrix} \begin{bmatrix}
\lambda k_d I & 0 \\
0 & D_{\theta}^{-1}(\cdot)
\end{bmatrix} \begin{bmatrix}
\tilde{q} \\
s
\end{bmatrix}
\]

The result follows, directly, from Theorem 3, by recognizing that system has the interconnected passive-systems form

\[
\begin{align*}
\dot{x}_1 &= A(t, x_1)x_1 + B(t, x_1)x_2 \\
\dot{x}_2 &= -B(t, x_1)^{\top}P x_1
\end{align*}
\]
Consider the system:

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_1 x_3 \\
\dot{x}_3 &= u_2
\end{align*}
\]

**Rmk.** [Brocket] The origin is not stabilizable via smooth autonomous state-feedback

**Proposition:** Let \( u_2(t, x) := -ax_3 - u_1(t, x)x_2 \) then,

\[
\begin{align*}
\dot{x}_1 &= u_1(t, x) \\
\dot{x}_2 &= u_1(t, x)x_3 \\
\dot{x}_3 &= -ax_3 - u_1(t, x)x_2
\end{align*}
\]

**Rmk.** The general \( n \)-dimensional case is also solvable similarly
Consider the system:

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_1 x_3 \\
\dot{x}_3 &= u_2
\end{align*}
\]

**Rmk.** [Brocket] The origin is not stabilizable via smooth autonomous state-feedback

**Proposition:** Let \( u_2(t, x) := -ax_3 - u_1(t, x)x_2 \) then,

\[
\begin{align*}
\dot{x}_1 &= u_1(t, x) \\
\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} &= \begin{bmatrix}
-a & -u(t, x) \\
u(t, x) & 0
\end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \end{bmatrix}
\end{align*}
\]

**Rmk.** We need \( u_1 \) to stabilize the \( x_1 \)-equation and to excite the \( x_2, x_3 \)-equations

We use: \( u_1 := -k_1 x_1 + h(t, x_2, x_3) \). For instance

\[
u_1 := -k_1 x_1 + \sin(t) \left[ |x_2|^2 + |x_3|^2 \right] \]
Bibliographical remarks

- Persistency of excitation was originally introduced by K. J. Åström from LTH, Sweden, in [1], in a discrete-time context.

- It has been thoroughly developed by authors that include: Narendra, Anderson, Annaswamy, Iannou, to mention a few:
  - In [9] the authors give a very detailed account of persistency of excitation and uniform asymptotic stability. See also [7, 8]
  - “Classical” theory of (linear) adaptive control systems is extensively explained in [2]; in particular, output injection.

- The material on linear parameterized systems is taken from [3]

- The concept of uniform δ-PE was originally introduced in [6]. More elaborated definitions and tools appeared in [4, 10, 5]