Analysis tools of sliding mode systems

Yury Orlov

CICESE Research Center
Electronics and Telecommunication Department
Carretera Ensenada-Tijuana 3918, Zona Playitas, B.C., México 22860.

Outline

1. Introduction
   - Variable Structure Systems (VSS')
   - What Sliding Modes (SM's) Are
   - First and Second Order SM's

Outline

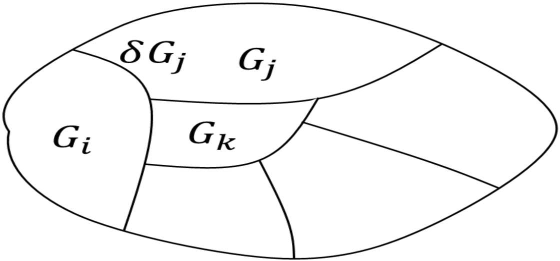
1. Introduction
   - Variable Structure Systems (VSS')
   - What Sliding Modes (SM's) Are
   - First and Second Order SM's
2. Mathematical Tools of VSS'
   - Filippov Solutions
   - Invariance Principle
   - Finite Time Stability Analysis of Homogeneous SM Systems.

Outline

1. Introduction
   - Variable Structure Systems (VSS')
   - What Sliding Modes (SM's) Are
   - First and Second Order SM's
2. Mathematical Tools of VSS'
   - Filippov Solutions
   - Invariance Principle
   - Finite Time Stability Analysis of Homogeneous SM Systems.
3. Concluding Remarks

Introduction

Variable Structure System (VSS)

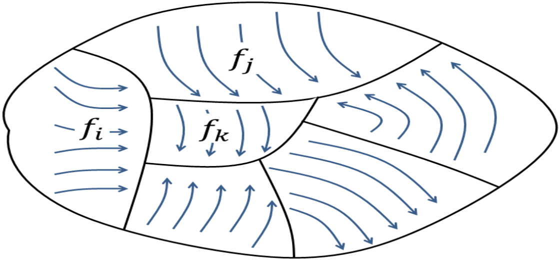


- **Operating domain with disjoint interiors**

$$G_j \subset \mathbf{R}^n, \; j = 1, \ldots, N$$

and boundaries ∂G_j of measure zero.

Variable Structure System (VSS) (cont'd)



- **Individual subsystem**

$$\dot{x} = f_j(x),\ \ x \in G_j,\ \ j = 1, \ldots, N$$

of class C^0 **with finite limit**

$$\lim_{x^* \to x} f_j(x^*) = f_j(x),\ \ x^* \in G_j,\ \ x \in \partial G_j$$

at the boundary.

- **Significantly different from the behavior of each individual subsystem.**

Variable Structure Dynamics

Utkin *Sliding Modes in Cnntrol and Optimization* Springer, 1992
Edwards, Spurgeon *Sliding Mode Control – Theory and Applications* CRS, 1998

- **Significantly different from the behavior of each individual subsystem.**
- **Sliding Modes (SM) along the boundaries ∂G_j, if any, to be defined**

- **Significantly different from the behavior of each individual subsystem.**
- **Sliding Modes (SM) along the boundaries ∂G_j, if any, to be defined**
- **May be possible to stabilize a system by varying its structure, even if all individual subsystems are unstable.**

Example: VSS puzzle

Controlled Plant

$$\ddot{x} = u(x, \dot{x}) \tag{1}$$

Two unstable structures $u = u_1$ **and** $u = u_2$**:**

$$u_1(x, \dot{x}) = 6\dot{x} + 16x \ \textbf{unstable saddle} \tag{2}$$
$$u_2(x, \dot{x}) = 6\dot{x} - 16x \ \textbf{unstable focus.} \tag{3}$$

Example: VSS puzzle (cont'd)

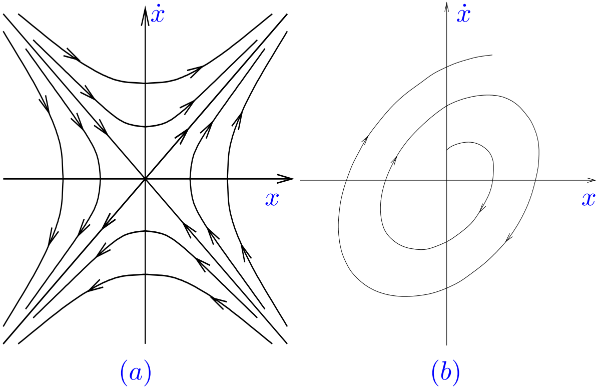


Figure: Phase portrait of unstable saddle (a); unstable focus (b)

Example: VSS puzzle (cont'd)

Switching rule

$$u(x,\dot{x}) = \begin{cases} 6\dot{x} + 16x & if \quad xs(x,\dot{x}) < 0 \\ 6\dot{x} - 16x & if \quad xs(x,\dot{x}) > 0 \end{cases} \tag{4}$$

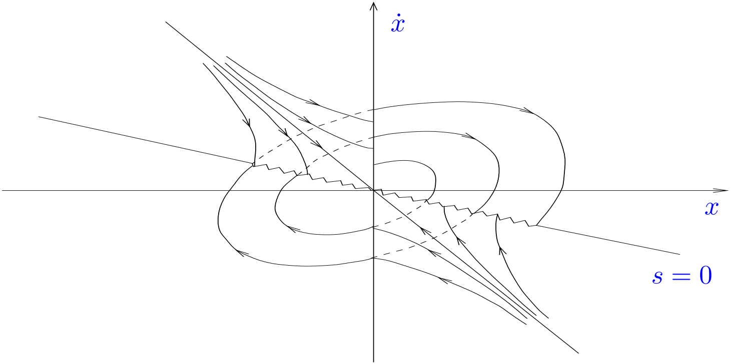
forcing the system structure to slide along the surface

$$s(x,\dot{x}) = \dot{x} + cx, \;\; c > 0 \tag{5}$$

results in asymptotical stability of the closed-loop system.

Example: VSS puzzle (cont'd)

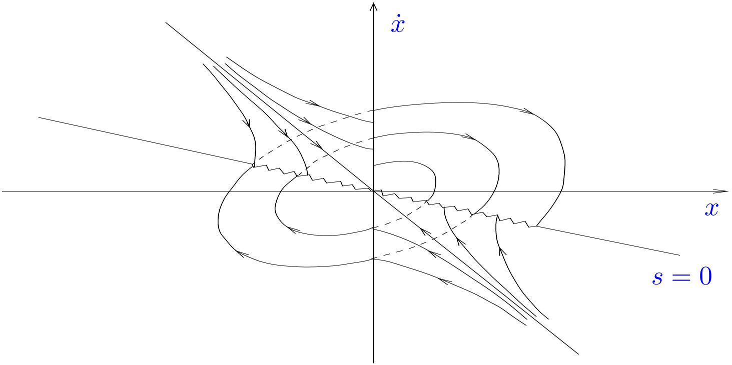
Phase portrait of the closed-loop VSS



- SM equation $s = \dot{x} + cx = 0$ is of reduced order

Example: VSS puzzle (cont'd)

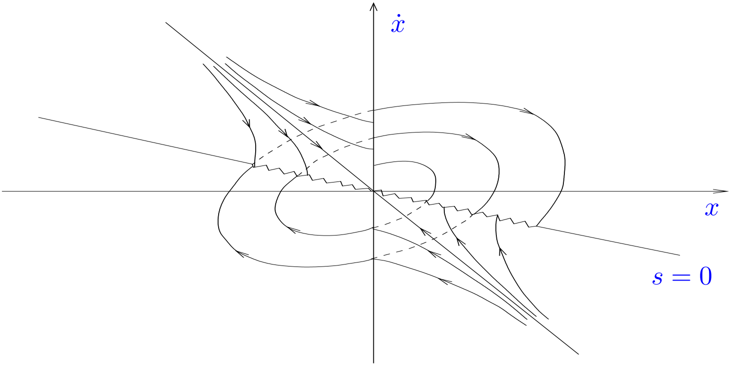
Phase portrait of the closed-loop VSS



- SM equation $s = \dot{x} + cx = 0$ is of reduced order
- SM does not depend on the plant dynamics

Example: VSS puzzle (cont'd)

Phase portrait of the closed-loop VSS



- SM equation $s = \dot{x} + cx = 0$ is of reduced order
- SM does not depend on the plant dynamics
- The gain c is at the designer's will

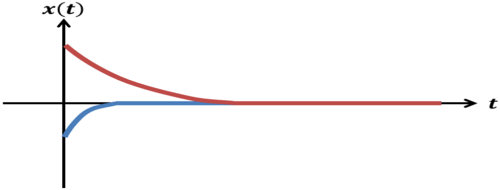
Scalar relay system

$$\dot{x} = f(t) - M sign\ x \quad \text{with} \quad \|f\|_\infty < M \tag{6}$$

Lyapunov function $V(x) = x^2$

$$\dot{V} = 2x\dot{x} = 2|x|[f(t)sign x - M] \leq -2(M - \|f\|_\infty) \leq -2\sqrt{V}(M - \|f\|_\infty)$$

$\dot{x}(t) = 0$ for all $t \geq T$ and some $T > 0 \Rightarrow M sign\ 0 \overset{???}{=} f(t)$



Sliding Mode in a Relay System

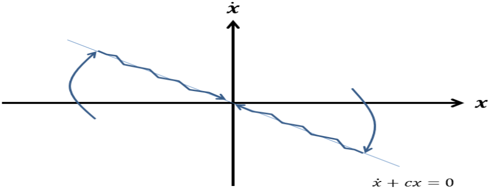
$$\ddot{x} + a_2\dot{x} + a_1 x = u + f(t)$$

$$u = -M signs, s = \dot{x} + cx$$

$$signs = \begin{cases} +1, & \text{if } s > 0 \\ -1, & \text{if } s < 0, \end{cases}$$

M, c, a_1, a_2 are constant parameters

$f(t)$ is a disturbance



ANTICIPATED SLIDING MODE FEATURES

ANTICIPATED SLIDING MODE FEATURES

- Counteracting non-vanishing disturbances and plant uncertainties.

ANTICIPATED SLIDING MODE FEATURES

- Counteracting non-vanishing disturbances and plant uncertainties.
- Synthesis decomposition: SM control is synthesized to steer the system to a switching manifold in finite time; after that the system slides along this manifold selected independently of the control law.

PRINCIPAL OPERATING MODES

ANTICIPATED SLIDING MODE FEATURES

- Counteracting non-vanishing disturbances and plant uncertainties.
- Synthesis decomposition: SM control is synthesized to steer the system to a switching manifold in finite time; after that the system slides along this manifold selected independently of the control law.

PRINCIPAL OPERATING MODES

- First order SMs occur on the manifold of co-dimension 1.
  Extensively developed from early 60s – Emel'yanov & Utkin school

ANTICIPATED SLIDING MODE FEATURES

- Counteracting non-vanishing disturbances and plant uncertainties.
- Synthesis decomposition: SM control is synthesized to steer the system to a switching manifold in finite time; after that the system slides along this manifold selected independently of the control law.

PRINCIPAL OPERATING MODES

- First order SMs occur on the manifold of co-dimension 1.
  Extensively developed from early 60s – Emel'yanov & Utkin school
- Alternatively, the state can be forced to avoid evolving on the switching manifolds, while steering to their intersections of higher co-dimension where HOSM (higher order sliding mode) occurs.
  Fuller phenomenon discovered in 1960
  Systematic study from late 80s – Emel'yanov, Korovin, Levantovskii

Example of SOSM: The Fuller phenomenon

Fuller, IFAC World Congress, Moscow, 1960

Optimal Control Problem

$$\int_0^{\infty} x^2(t)dt \to \min \tag{7}$$

subject to

$$\ddot{x} = u(x, \dot{x}) \tag{8}$$

under the input constraint

$$|u(t)| \leq 1 \; for \; all \; t \geq 0. \tag{9}$$

Minimum principle yields the optimal synthesis

$$u(x, \dot{x}) = \begin{cases} 1 & if \quad s(x, \dot{x}) < 0 \\ -1 & if \quad s(x, \dot{x}) > 0 \end{cases} . \tag{10}$$

with the switching curve

$$s(x, \dot{x}) = x + c\dot{x}^2 sign \; \dot{x} \tag{11}$$

for some constant c

Example: The Fuller phenomenon (cont'd)

Fuller, IFAC World Congress, Moscow, 1960

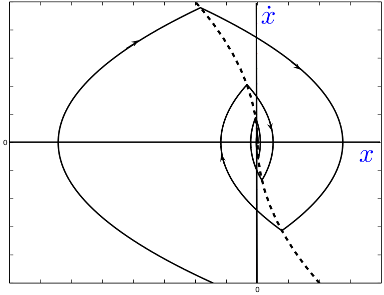


Figure: Fuller phenomenon (dotted line is for the switching curve, solid line is for an optimal trajectory).

Example: The Fuller phenomenon features

Fuller, IFAC World Congress, Moscow, 1960

- No sliding modes on the switching curve, the optimal trajectories cross it at countably many points.

Example: The Fuller phenomenon features

Fuller, IFAC World Congress, Moscow, 1960

- No sliding modes on the switching curve, the optimal trajectories cross it at countably many points.
- The switching times have a finite accumulation point (*Zeno* behavior).

Example: The Fuller phenomenon features

Fuller, IFAC World Congress, Moscow, 1960

- No sliding modes on the switching curve, the optimal trajectories cross it at countably many points.
- The switching times have a finite accumulation point (*Zeno* behavior).
- The closed-loop variable structure system is steered to the origin in finite time.

Example: The Fuller phenomenon features

Fuller, IFAC World Congress, Moscow, 1960

- No sliding modes on the switching curve, the optimal trajectories cross it at countably many points.
- The switching times have a finite accumulation point (*Zeno* behavior).
- The closed-loop variable structure system is steered to the origin in finite time.
- After that there appears a so-called SOSM (sliding mode of the second order).

Example: The Fuller phenomenon features

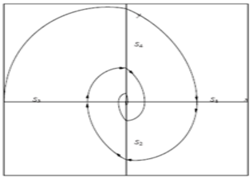
Fuller, IFAC World Congress, Moscow, 1960

- No sliding modes on the switching curve, the optimal trajectories cross it at countably many points.
- The switching times have a finite accumulation point (*Zeno* behavior).
- The closed-loop variable structure system is steered to the origin in finite time.
- After that there appears a so-called SOSM (sliding mode of the second order).
- The sliding manifold has codimension 2, i.e., it is confined to the origin only.

Twisting Algorithm

Dynamic state feedback (knowledge of both x and $\dot{x}$ is required)

$$\begin{aligned} \dot{x} = f + u, \quad \dot{u} &= -a\,sign\ x - b\,sign\ \dot{x}, \quad a > b > 0 \\ &\Downarrow \quad y = f + u \\ \dot{x} = y,\ \dot{y} &= \dot{f} - a\,sign\ x - b\,sign\ y, \quad |\dot{f}| < \min\{b, a-b\} \end{aligned}$$



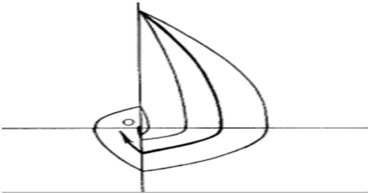
Discontinuities:
– in the controller dynamics, not in the plant!
Analysis tools of:
– robustness, finite time stability, settling time estimation, tuning rules ???

Supertwisting Algorithm

Dynamic position feedback (only knowledge of x is required)

$$\begin{aligned} \dot{x} &= f + u, \quad u = v - \mu\sqrt{|x|} sign\ x \\ \dot{v} &= -\nu sign\ x, \quad \mu, \nu > 0 \end{aligned}$$

$$y = f + v \Rightarrow \dot{x} = y - \mu\sqrt{|x|} sign\ x,\ \dot{y} = \dot{f} - \nu sign\ x, \quad |\dot{f}| < \min\left\{\mu, \frac{\mu\nu}{1+\mu}\right\}$$



Discontinuities:
– in the controller dynamics, not in the plant!

Analysis tools of:
– robustness, finite time stability, settling time estimation, tuning rules ???

Mathematical Tools of VSS

Non-autonomous VSS

$$\dot{x} = f(x,t), x \in \mathbf{R^n} \tag{12}$$

f is piece-wise continuous

Definition

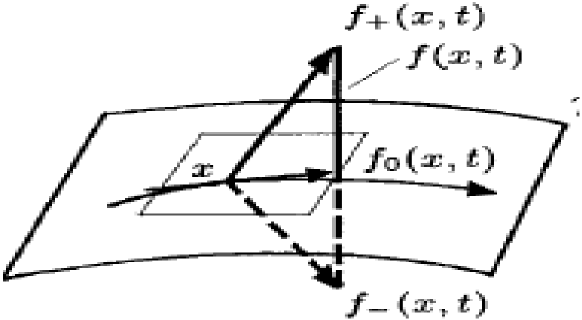
Let $F(x,t)$ be the smallest convex closed set that contains all the limit points of $f(x^*,t)$ as $x^* \to x$, $t = const$, and $(x^*,t) \in \mathbf{R}^{n+1} \setminus (\cup_{j=1}^{N} \partial G_j)$. An absolutely continuous function $x(\cdot)$ is a *Filippov solution* of (12) on an interval I if it satisfies the differential inclusion

$$\dot{x} \in F(x,t) \tag{13}$$

almost everywhere on I.

Geometrical Illustration of Filippov Convex Hull

$$\dot{x} = \begin{cases} f_+(x,t) & \text{if } s(x) > 0 \\ f_-(x,t) & \text{if } s(x) < 0 \end{cases}$$



Computation of Filippov Velocity f_0

- f_0 belongs to convex hull of f_+ and f_-

$$\Downarrow$$

$$f_0(x,t) = \mu(x,t) f_+(x,t) + [1 - \mu(x,t)] f_-(x,t), \ \ \mu(x,t) \in [0,1]$$

Computation of Filippov Velocity f_0

- f_0 belongs to convex hull of f_+ and f_-

$$\Downarrow$$

$$f_0(x,t) = \mu(x,t) f_+(x,t) + [1 - \mu(x,t)] f_-(x,t), \;\; \mu(x,t) \in [0,1]$$

- Once $f_0(x,t)$ belongs to tangential plane it is orthogonal to $grad\ s$, i.e.,

$$grad^T s(x) \; \{\mu(x,t) f^+(x,t) + [1 - \mu(x,t)] f^-(x,t)\} = 0$$

$$\Downarrow$$

$$\mu(x,t) = \frac{grad^T s(x) \; f^-(x,t)}{grad^T s(x) \; [f^-(x,t) - f^+(x,t)]}$$

- VSS trajectories are defined in the conventional sense beyond the discontinuity manifold $s = 0$

- VSS trajectories are defined in the conventional sense beyond the discontinuity manifold $s = 0$
- Sliding modes on $s = 0$ are subject to refining as they do not exist in the conventional sense

Solution Redefinition on a Discontinuity Manifold

Summary

- VSS trajectories are defined in the conventional sense beyond the discontinuity manifold $s = 0$
- Sliding modes on $s = 0$ are subject to refining as they do not exist in the conventional sense
- Regularization procedure should be invoked to provide a physical sense behind a sliding mode on $s = 0$.

Solution Redefinition on a Discontinuity Manifold

Summary

- VSS trajectories are defined in the conventional sense beyond the discontinuity manifold $s = 0$
- Sliding modes on $s = 0$ are subject to refining as they do not exist in the conventional sense
- Regularization procedure should be invoked to provide a physical sense behind a sliding mode on $s = 0$.
- Filippov solution concept for an ODE with discontinuous right-hand side is the most adequate solution redefinition as it covers existing regularizations such as hysteresis switching, delayed switching and many others

- VSS trajectories are defined in the conventional sense beyond the discontinuity manifold $s = 0$
- Sliding modes on $s = 0$ are subject to refining as they do not exist in the conventional sense
- Regularization procedure should be invoked to provide a physical sense behind a sliding mode on $s = 0$.
- Filippov solution concept for an ODE with discontinuous right-hand side is the most adequate solution redefinition as it covers existing regularizations such as hysteresis switching, delayed switching and many others
- For controlled VSS, the equivalent control method (EQM) is an alternative to the Filippov solutions, well-suited to unmodelled dynamics of the applied actuator(s).

Solution Redefinition on a Discontinuity Manifold

Summary

- VSS trajectories are defined in the conventional sense beyond the discontinuity manifold $s = 0$
- Sliding modes on $s = 0$ are subject to refining as they do not exist in the conventional sense
- Regularization procedure should be invoked to provide a physical sense behind a sliding mode on $s = 0$.
- Filippov solution concept for an ODE with discontinuous right-hand side is the most adequate solution redefinition as it covers existing regularizations such as hysteresis switching, delayed switching and many others
- For controlled VSS, the equivalent control method (EQM) is an alternative to the Filippov solutions, well-suited to unmodelled dynamics of the applied actuator(s).
- Filippov convexization and EQM result in the same provided the underlying VSS is affine (linear in control). Just in case all possible regularizations yield the same.

Stability analysis

Basic Definitions

Nonautonomous VSS

$$\dot{x} = \varphi(x, t) \tag{14}$$

- $x = (x_1, \ldots, x_n)^T$ is the state vector,

Basic Definitions

Nonautonomous VSS

$$\dot{x} = \varphi(x, t) \tag{14}$$

- $x = (x_1, \ldots, x_n)^T$ is the state vector,
- $t \in \mathbf{R}$ is the time variable,

Basic Definitions

Nonautonomous VSS

$$\dot{x} = \varphi(x,t) \tag{14}$$

- $x = (x_1, \ldots, x_n)^T$ is the state vector,
- $t \in \mathbf{R}$ is the time variable,
- $\varphi(x,t) = (\varphi_1(x,t), \ldots, \varphi_n(x,t))^T$ is a piece-wise continuous function,

Basic Definitions

Nonautonomous VSS

$$\dot{x} = \varphi(x,t) \tag{14}$$

- $x = (x_1, \dots, x_n)^T$ is the state vector,
- $t \in \mathbf{R}$ is the time variable,
- $\varphi(x,t) = (\varphi_1(x,t), \dots, \varphi_n(x,t))^T$ is a piece-wise continuous function,
- $\varphi(x,t)$ undergoes discontinuities on the boundary set $\mathcal{N} = \bigcup_{j=1}^{N} \partial G_j$,

Nonautonomous VSS

$$\dot{x} = \varphi(x,t) \tag{14}$$

- $x = (x_1, \ldots, x_n)^T$ is the state vector,
- $t \in \mathbf{R}$ is the time variable,
- $\varphi(x,t) = (\varphi_1(x,t), \ldots, \varphi_n(x,t))^T$ is a piece-wise continuous function,
- $\varphi(x,t)$ undergoes discontinuities on the boundary set $\mathcal{N} = \bigcup_{j=1}^{N} \partial G_j$,
- Boundaries ∂G_j of the disjoint continuity domains $G_j \subset \mathbf{R}^{n+1},\ j = 1, \ldots, N$ of $\varphi(x,t)$ are of zero measure.

The precise meaning of the differential equation $\dot{x} = \varphi(x,t)$ with a piece-wise continuous right-hand side is defined in the sense of *Filippov* as that of the differential inclusion

$$\dot{x} \in \Phi(x,t) \tag{15}$$

The precise meaning of the differential equation $\dot{x} = \varphi(x,t)$ with a piece-wise continuous right-hand side is defined in the sense of *Filippov* as that of the differential inclusion

$$\dot{x} \in \Phi(x,t) \tag{15}$$

- $\Phi(x,t)$ is the smallest convex closed set containing all the limit values of $\varphi(x^*,t)$ for $(x^*,t) \in \mathbf{R}^{n+1} \setminus \mathcal{N},\ x^* \to x,\ t = const$.

The precise meaning of the differential equation $\dot{x} = \varphi(x,t)$ with a piece-wise continuous right-hand side is defined in the sense of *Filippov* as that of the differential inclusion

$$\dot{x} \in \Phi(x,t) \tag{15}$$

- $\Phi(x,t)$ is the smallest convex closed set containing all the limit values of $\varphi(x^*,t)$ for $(x^*,t) \in \mathbf{R}^{n+1} \setminus \mathcal{N},\ x^* \to x,\ t = const$.
- *Filippov solution* of the underlying system exists for arbitrary initial conditions $x(t_0) = x^0 \in \mathbf{R}^n$.

The precise meaning of the differential equation $\dot{x} = \varphi(x,t)$ with a piece-wise continuous right-hand side is defined in the sense of *Filippov* as that of the differential inclusion

$$\dot{x} \in \Phi(x,t) \tag{15}$$

- $\Phi(x,t)$ is the smallest convex closed set containing all the limit values of $\varphi(x^*,t)$ for $(x^*,t) \in \mathbf{R}^{n+1} \setminus \mathcal{N},\ x^* \to x,\ t = const$.
- *Filippov solution* of the underlying system exists for arbitrary initial conditions $x(t_0) = x^0 \in \mathbf{R}^n$.
- Such a solution is locally defined on some time interval $[t_0, t_1)$, however, it is generally speaking non-unique.

Basic Definitions
Recall

The precise meaning of the differential equation $\dot{x} = \varphi(x, t)$ with a piece-wise continuous right-hand side is defined in the sense of *Filippov* as that of the differential inclusion

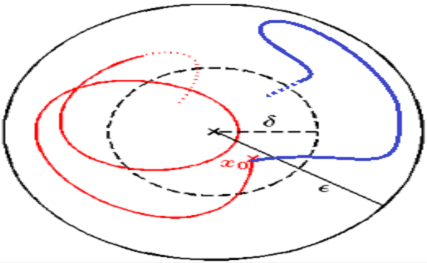
$$\dot{x} \in \Phi(x, t) \tag{15}$$

- $\Phi(x, t)$ is the smallest convex closed set containing all the limit values of $\varphi(x^*, t)$ for $(x^*, t) \in \mathbf{R}^{n+1} \setminus \mathcal{N},\ x^* \to x,\ t = const$.
- *Filippov solution* of the underlying system exists for arbitrary initial conditions $x(t_0) = x^0 \in \mathbf{R}^n$.
- Such a solution is locally defined on some time interval $[t_0, t_1)$, however, it is generally speaking non-unique.
- Example of an ambiguous behaviour

$$\dot{x} = sign\ x \Rightarrow \text{ solutions } x(t) = 0,\ x(t) = t,\ x(t) = -t.$$

The solution $x = 0$ of $\dot{x} \in \Phi(x,t)$ is stable (uniformly stable) iff for each $t_0 \in \mathbf{R}$, $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ (respectively, $\delta(\varepsilon)$ independent on t_0) such that each ***Filippov*** **solution $x(t, t_0, x^0)$ with the initial data $x(t_0) = x^0 \in B_\delta$ within the ball B_δ exists for all $t \geq t_0$ and satisfies the inequality**

$$\|x(t, t_0, x^0)\| < \varepsilon, \; t_0 \leq t < \infty.$$



- **The solution $x = 0$ of the underlying differential inclusion is (uniformly) asymptotically stable iff it is (uniformly) stable and**

$$lim_{t\to\infty}\|x(t,t_0,x^0)\| = 0 \tag{16}$$

holds for all solutions $x(t,t_0,x^0)$ initialized within some B_δ (uniformly in t_0 and x^0).

- **The solution $x = 0$ of the underlying differential inclusion is (uniformly) asymptotically stable iff it is (uniformly) stable and**

$$lim_{t\to\infty}\|x(t,t_0,x^0)\| = 0 \tag{16}$$

holds for all solutions $x(t,t_0,x^0)$ initialized within some B_δ (uniformly in t_0 and x^0).

- **If (16) holds true for all solutions $x(t,t_0,x^0)$ regardless of the choice of the initial data (and, respectively, it is uniform in t_0 and $x^0 \in B_\delta$ for each $\delta > 0$), the solution $x = 0$ is said to be globally (uniformly) asymptotically stable.**

Example

- $\dot{x} = sign\ x,\ \dot{y} = -2sign\ y$

Example

- $\dot{x} = sign\ x,\ \dot{y} = -2sign\ y$
- Lyapunov function $V = |x| + |y|$

Example

- $\dot{x} = sign\ x,\ \dot{y} = -2sign\ y$
- Lyapunov function $V = |x| + |y|$
- Time derivative beyond the manifold $xy = 0$ is as simple as $\dot{V} = \dot{x}sign\ x + \dot{y}sign\ y = 1 - 2 = -1$

Example

- $\dot{x} = sign\ x,\ \dot{y} = -2sign\ y$
- Lyapunov function $V = |x| + |y|$
- Time derivative beyond the manifold $xy = 0$ is as simple as $\dot{V} = \dot{x}sign\ x + \dot{y}sign\ y = 1 - 2 = -1$
- This is however insufficient to ensure the system stability

Example

- $\dot{x} = sign\ x,\ \dot{y} = -2sign\ y$
- Lyapunov function $V = |x| + |y|$
- Time derivative beyond the manifold $xy = 0$ is as simple as $\dot{V} = \dot{x}sign\ x + \dot{y}sign\ y = 1 - 2 = -1$
- This is however insufficient to ensure the system stability
- Indeed, the system generates unstable sliding modes on the x-axis:

$$\dot{x} = sign\ x,\ y = 0$$

Example

- $\dot{x} = sign\ x,\ \dot{y} = -2sign\ y$
- Lyapunov function $V = |x| + |y|$
- Time derivative beyond the manifold $xy = 0$ is as simple as $\dot{V} = \dot{x}sign\ x + \dot{y}sign\ y = 1 - 2 = -1$
- This is however insufficient to ensure the system stability
- Indeed, the system generates unstable sliding modes on the x-axis:

$$\dot{x} = sign\ x,\ y = 0$$

- The time derivative of $V = |x| + |y|$ on the sliding line $y = 0$ is positive definite $\dot{V} = \dot{x}sign\ x = 1$

Example

- $\dot{x} = sign\ x,\ \dot{y} = -2sign\ y$
- Lyapunov function $V = |x| + |y|$
- Time derivative beyond the manifold $xy = 0$ is as simple as $\dot{V} = \dot{x}sign\ x + \dot{y}sign\ y = 1 - 2 = -1$
- This is however insufficient to ensure the system stability
- Indeed, the system generates unstable sliding modes on the x-axis:

$$\dot{x} = sign\ x,\ y = 0$$

- The time derivative of $V = |x| + |y|$ on the sliding line $y = 0$ is positive definite $\dot{V} = \dot{x}sign\ x = 1$
- Thus, the system is unstable with the trajectories, escaping to infinity along the x-axis.

- Seeking for a positive (semi)definite Lipschitz-continuous Lyapunov function $V(x,t)$, nonincreasing along the system trajectories.

- Seeking for a positive (semi)definite Lipschitz-continuous Lyapunov function $V(x,t)$, nonincreasing along the system trajectories.
- Special attention to the behavior of the composed function $V(x(t),t)$ on sliding manifolds and on nondifferentiability sets of $V(x,t)$!!!

- Seeking for a positive (semi)definite Lipschitz-continuous Lyapunov function $V(x,t)$, nonincreasing along the system trajectories.
- Special attention to the behavior of the composed function $V(x(t),t)$ on sliding manifolds and on nondifferentiability sets of $V(x,t)$!!!
- All the system trajectories are concluded to be bounded and, due to Filippov, they prove to be globally defined, possibly non-uniquely, in the direction of increasing t.

- Seeking for a positive (semi)definite Lipschitz-continuous Lyapunov function $V(x,t)$, nonincreasing along the system trajectories.
- Special attention to the behavior of the composed function $V(x(t),t)$ on sliding manifolds and on nondifferentiability sets of $V(x,t)$!!!
- All the system trajectories are concluded to be bounded and, due to Filippov, they prove to be globally defined, possibly non-uniquely, in the direction of increasing t.
- By applying standard Lyapunov arguments, the system stability is guaranteed.

Stability and Nonsmooth Lyapunov Functions

The analysis to be presented

- Seeking for a positive (semi)definite Lipschitz-continuous Lyapunov function $V(x, t)$, nonincreasing along the system trajectories.
- Special attention to the behavior of the composed function $V(x(t), t)$ on sliding manifolds and on nondifferentiability sets of $V(x, t)$!!!
- All the system trajectories are concluded to be bounded and, due to Filippov, they prove to be globally defined, possibly non-uniquely, in the direction of increasing t.
- By applying standard Lyapunov arguments, the system stability is guaranteed.
- Asymptotic stability is additionally to be studied (Barbalat lemma, extended invariance principle...)

$V(x,t)$ is Lipschitz continuous, $x(t)$ is a solution of $\dot{x} = \varphi(x,t)$

$$\Downarrow$$

The composite function $V(x(t),t)$ is absolutely continuous and

$$\frac{d}{dt}V(x(t),t) = \left.\frac{d}{dh}V(x(t)+h\dot{x}(t),t+h)\right|_{h=0}$$

almost everywhere.

Theorem

There exists a Lipschitz-continuous, positive definite, decrescent function $V(x,t)$ *such that its time derivative*

$$\frac{d}{dt}V(x(t),t) = \frac{d}{dh}V(x(t)+h\dot{x}(t),t+h)\bigg|_{h=0} \leq 0 \tag{17}$$

for almost all t *and for all trajectories* $x(t)$ *of the VSS* $\dot{x}=\varphi(x,t)$, *initialized within some* B_δ.

$$\Downarrow$$

The VSS is uniformly stable.

Taking care just at the nondifferentiability set of V

Corollary: The stability of the VSS $\dot{x} = \varphi(x, t)$ remains in force if the time derivative $\frac{d}{dt}V(x(t), t)$ is nonpositive at the points of the nondifferentiability set $\mathcal{N}_V$ of $V(x, t)$ and in the continuity domain of the function $\varphi(x, t)$ where it is expressed in the standard form

$$\frac{d}{dt}V(x,t) = \frac{\partial V(x,t)}{\partial t} + grad\, V(x,t) \cdot \varphi(x,t), \; (x,t) \in \mathbf{R}^{n+1} \setminus (\mathcal{N} \cup \mathcal{N}_V) \quad (18)$$

Taking care just at the nondifferentiability set of V

Corollary: The stability of the VSS $\dot{x} = \varphi(x,t)$ remains in force if the time derivative $\frac{d}{dt}V\left(x\left(t\right),t\right)$ is nonpositive at the points of the nondifferentiability set $\mathcal{N}_V$ of $V\left(x,t\right)$ and in the continuity domain of the function $\varphi\left(x,t\right)$ where it is expressed in the standard form

$$\frac{d}{dt}V\left(x,t\right) = \frac{\partial V\left(x,t\right)}{\partial t} + grad\, V\left(x,t\right) \cdot \varphi\left(x,t\right),\ \left(x,t\right) \in \mathbf{R}^{n+1} \setminus \left(\mathcal{N} \cup \mathcal{N}_V\right) \quad (18)$$

- Indeed, at the discontinuity points $\left(x,t\right) \in \mathcal{N}$ of the function $\varphi\left(x,t\right)$, the right-hand side $\Phi\left(x,t\right)$ of the corresponding differential inclusion $\dot{x} \in \Phi(x,t)$ is obtained by closure of the graph of $\varphi\left(x,t\right)$ and by passing over to a convex hull.

$\Downarrow$

Taking care just at the nondifferentiability set of V

Corollary: The stability of the VSS $\dot{x} = \varphi(x,t)$ remains in force if the time derivative $\frac{d}{dt}V(x(t),t)$ is nonpositive at the points of the nondifferentiability set $\mathcal{N}_V$ of $V(x,t)$ and in the continuity domain of the function $\varphi(x,t)$ where it is expressed in the standard form

$$\frac{d}{dt}V(x,t) = \frac{\partial V(x,t)}{\partial t} + grad\, V(x,t) \cdot \varphi(x,t), \; (x,t) \in \mathbf{R}^{n+1} \setminus (\mathcal{N} \cup \mathcal{N}_V) \quad (18)$$

- Indeed, at the discontinuity points $(x,t) \in \mathcal{N}$ of the function $\varphi(x,t)$, the right-hand side $\Phi(x,t)$ of the corresponding differential inclusion $\dot{x} \in \Phi(x,t)$ is obtained by closure of the graph of $\varphi(x,t)$ and by passing over to a convex hull.

$$\Downarrow$$

- These procedures do not increase the upper value of (18) and hence the negative semidefiniteness of (18) guarantees the negative definiteness of $\frac{d}{dt}V(x,t)$ for all $(x,t) \in \mathcal{N}$.

Moreover

Simple hint

If any trajectory intersects $\mathcal{N}_V \setminus \{x=0\}$ just on a set of measure 0 then it suffices to verify the negative semidefinbiteness of the Lyapunov time derivative beyond sliding modes and nondifferentiability set.

Corollary: No trajectory of the VSS $\dot{x} = \varphi(x,t)$ stay in the nondifferentiablity set $\mathcal{N}_V \setminus \{x=0\}$ possibly except the origin for a finite time interval.

$$\Downarrow$$

The VSS is stable provided that $\frac{d}{dt}V(x,t) \leq 0$ for all $(x,t) \in \mathbf{R}^{n+1} \setminus (\mathcal{N} \cup \mathcal{N}_V)$.

Moreover

Simple hint

If any trajectory intersects $\mathcal{N}_V \setminus \{x = 0\}$ just on a set of measure 0 then it suffices to verify the negative semidefinbiteness of the Lyapunov time derivative beyond sliding modes and nondifferentiability set.

Corollary: No trajectory of the VSS $\dot{x} = \varphi(x, t)$ stay in the nondifferentiablity set $\mathcal{N}_V \setminus \{x = 0\}$ possibly except the origin for a finite time interval.

$$\Downarrow$$

The VSS is stable provided that $\frac{d}{dt}V\,(x, t) \leq 0$ for all $(x, t) \in \mathbf{R}^{n+1} \setminus (\mathcal{N} \cup \mathcal{N}_V)$.

- Indeed, any trajectory of the VSS is in $\left(\mathbf{R}^{n+1} \setminus \mathcal{N}_V\right) \cup \{x = 0\}$ almost always

$$\Downarrow$$

Moreover

Simple hint

If any trajectory intersects $\mathcal{N}_V \setminus \{x = 0\}$ just on a set of measure 0 then it suffices to verify the negative semidefinbiteness of the Lyapunov time derivative beyond sliding modes and nondifferentiability set.

Corollary: No trajectory of the VSS $\dot{x} = \varphi(x, t)$ stay in the nondifferentiablity set $\mathcal{N}_V \setminus \{x = 0\}$ possibly except the origin for a finite time interval.

$$\Downarrow$$

The VSS is stable provided that $\frac{d}{dt}V(x, t) \leq 0$ for all $(x, t) \in \mathbf{R}^{n+1} \setminus (\mathcal{N} \cup \mathcal{N}_V)$.

- Indeed, any trajectory of the VSS is in $\left(\mathbf{R}^{n+1} \setminus \mathcal{N}_V\right) \cup \{x = 0\}$ almost always

$$\Downarrow$$

- $\frac{d}{dt}V(x(t), t) \leq 0$ is satisfied almost everywhere.

Krasovskii–LaSalle Invariance Principle

- Ensures the convergence of the state trajectories $x(t)$ to the largest invariant subset M of the manifold where the time derivative of the Lyapunov function takes no value.

Krasovskii–LaSalle Invariance Principle

- Ensures the convergence of the state trajectories $x(t)$ to the largest invariant subset M of the manifold where the time derivative of the Lyapunov function takes no value.
- $M \subset \mathbf{R}^n$ is an invariant set of (14) if, for all $x_0 \in M$, the trajectories initialized at x_0 at some time t_0 remain in M for all $t > t_0$.

Krasovskii–LaSalle Invariance Principle

- Ensures the convergence of the state trajectories $x(t)$ to the largest invariant subset M of the manifold where the time derivative of the Lyapunov function takes no value.
- $M \subset \mathbf{R}^n$ is an invariant set of (14) if, for all $x_0 \in M$, the trajectories initialized at x_0 at some time t_0 remain in M for all $t > t_0$.
- Proven for autonomous continuous dynamic systems

Krasovskii–LaSalle Invariance Principle

- Ensures the convergence of the state trajectories $x(t)$ to the largest invariant subset M of the manifold where the time derivative of the Lyapunov function takes no value.
- $M \subset \mathbf{R}^n$ is an invariant set of (14) if, for all $x_0 \in M$, the trajectories initialized at x_0 at some time t_0 remain in M for all $t > t_0$.
- Proven for autonomous continuous dynamic systems
- In general, not valid for non-autonomous systems

Krasovskii–LaSalle Invariance Principle

- Ensures the convergence of the state trajectories $x\,(t)$ to the largest invariant subset M of the manifold where the time derivative of the Lyapunov function takes no value.
- $M \subset \mathbf{R}^n$ is an invariant set of (14) if, for all $x_0 \in M$, the trajectories initialized at x_0 at some time t_0 remain in M for all $t > t_0$.
- Proven for autonomous continuous dynamic systems
- In general, not valid for non-autonomous systems
- Non-extendible to general differential inclusions and, particularly, to discontinuous dynamic systems, possibly, due to their ambiguous behavior.

The invariance principle remains in force for autonomous VSS

$$\dot{x} = \varphi(x),$$

whose solutions are uniquely continuable to the right.

Sufficient right uniqueness conditions for solutions of the above system and continuous dependence of the solutions on their initial data have been carried out by Filippov.

Application to Frictional Oscillator

- Mathematical model

$$m\ddot{y} + P(\dot{y}) + ky = 0$$

Application to Frictional Oscillator

- Mathematical model

$$m\ddot{y} + P\left(\dot{y}\right) + ky = 0$$

- Coulomb friction

$$P\left(\dot{y}\right) = \begin{cases} +P_0 & \text{if } \dot{y} > 0 \\ -P_0 & \text{if } \dot{y} < 0 \end{cases}$$

Application to Frictional Oscillator

- Mathematical model

$$m\ddot{y} + P(\dot{y}) + ky = 0$$

- Coulomb friction

$$P(\dot{y}) = \begin{cases} +P_0 & \text{if } \dot{y} > 0 \\ -P_0 & \text{if } \dot{y} < 0 \end{cases}$$

- $P_0 > 0$ is the Coulomb friction level

Application to Frictional Oscillator

- Mathematical model

$$m\ddot{y} + P(\dot{y}) + ky = 0$$

- Coulomb friction

$$P(\dot{y}) = \begin{cases} +P_0 & \text{if } \dot{y} > 0 \\ -P_0 & \text{if } \dot{y} < 0 \end{cases}$$

- $P_0 > 0$ is the Coulomb friction level
- Dimensionless model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - sign\ x_2 \end{pmatrix} \tag{19}$$

Application to Frictional Oscillator

- Mathematical model

$$m\ddot{y} + P(\dot{y}) + ky = 0$$

- Coulomb friction

$$P(\dot{y}) = \begin{cases} +P_0 & \text{if } \dot{y} > 0 \\ -P_0 & \text{if } \dot{y} < 0 \end{cases}$$

- $P_0 > 0$ is the Coulomb friction level
- Dimensionless model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - sign\ x_2 \end{pmatrix} \tag{19}$$

- Lyapunov function $V(x) = \left(x_1^2 + x_2^2\right)/2 \ \Rightarrow \ \dot{V}(x) = -|x_2|$

- The discontinuity manifold

$$S = \{x \in \mathbf{R}^2 : x_2 = 0\}$$

- The discontinuity manifold

$$S = \left\{x \in \mathbf{R}^2 : x_2 = 0\right\}$$

- The phase plane $\mathbf{R}^2$ is partitioned into two regions

$$G^+ = \left\{x \in \mathbf{R}^2 : x_2 > 0\right\} \text{ and } G^- = \left\{x \in \mathbf{R}^2 : x_2 < 0\right\}.$$

- The discontinuity manifold

$$S = \left\{ x \in \mathbf{R}^2 : x_2 = 0 \right\}$$

- The phase plane $\mathbf{R}^2$ is partitioned into two regions

$$G^+ = \left\{ x \in \mathbf{R}^2 : x_2 > 0 \right\} \text{ and } G^- = \left\{ x \in \mathbf{R}^2 : x_2 < 0 \right\}.$$

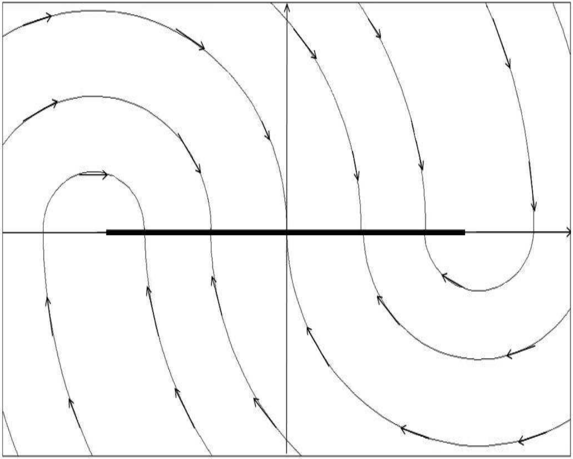
- In the discontinuity manifold S the vector fields

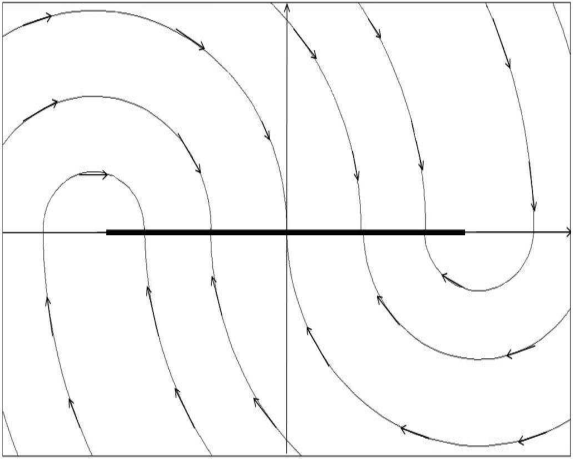
$$\varphi^+ (x_s) = \lim_{x \to x_s,\ x \in G^+} \varphi (x) = \begin{pmatrix} 0 \\ -x_1 - 1 \end{pmatrix},$$

$$\varphi^- (x_s) = \lim_{x \to x_s,\ x \in G^-} \varphi (x) = \begin{pmatrix} 0 \\ -x_1 + 1 \end{pmatrix} \tag{20}$$

are directed to opposite directions inside the segment $|x_1| \leq 1$ and they point toward the same region (G^+ for $x < -1$ and G^- for $x > 1$) outside the segment.

Phase Portrait





- By invariance principle, any trajectory of the unforced system (19) converges to the segment $I = \{x : |x_1| \leq 1,\ x_2 = 0\}$ rather than the whole x_1-axis.

The controlled oscillator

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - sign\ x_2 + u \end{pmatrix} \tag{21}$$

is asymptotically stabilizable by the control law

$$u(x) = -sign\ x_1. \tag{22}$$

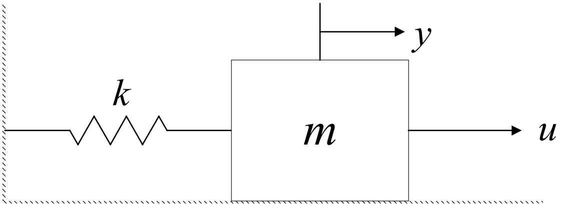


Figure: One-degree-of-freedom mechanical oscillator.

- The closed-loop vector field

$$\varphi(x) = \begin{pmatrix} x_2 \\ -x_1 - sign\ x_1 - sign\ x_2 \end{pmatrix},\ x = (x_1, x_2)^T \in \mathbf{R}^2, \qquad (23)$$

- The closed-loop vector field

$$\varphi(x) = \begin{pmatrix} x_2 \\ -x_1 - sign\ x_1 - sign\ x_2 \end{pmatrix}, \ x = (x_1, x_2)^T \in \mathbf{R}^2, \qquad (23)$$

- The Discontinuity Manifolds

$$S_1 = \left\{x \in \mathbf{R}^2 : x_1 = 0\right\}, \ S_2 = \left\{x \in \mathbf{R}^2 : x_2 = 0\right\}$$

- The closed-loop vector field

$$\varphi(x) = \begin{pmatrix} x_2 \\ -x_1 - sign\ x_1 - sign\ x_2 \end{pmatrix}, \ x = (x_1, x_2)^T \in \mathbf{R}^2, \qquad (23)$$

- The Discontinuity Manifolds

$$S_1 = \left\{ x \in \mathbf{R}^2 : x_1 = 0 \right\}, \ S_2 = \left\{ x \in \mathbf{R}^2 : x_2 = 0 \right\}$$

- The phase plane $\mathbf{R}^2$ is partitioned into four regions

$$\begin{aligned} G_1 &= \{x : x_1 > 0, x_2 > 0\}, \ G_2 = \{x : x_1 > 0, x_2 < 0\}, \\ G_3 &= \{x : x_1 < 0, x_2 > 0\}, \ G_4 = \{x : x_1 < 0, x_2 < 0\}, \end{aligned}$$

- The closed-loop vector field

$$\varphi(x) = \begin{pmatrix} x_2 \\ -x_1 - sign\ x_1 - sign\ x_2 \end{pmatrix},\ x = (x_1, x_2)^T \in \mathbf{R}^2, \tag{23}$$

- The Discontinuity Manifolds

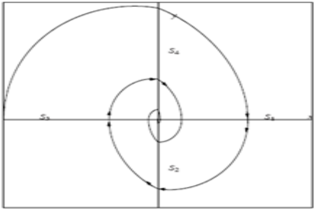
$$S_1 = \left\{x \in \mathbf{R}^2 : x_1 = 0\right\},\ S_2 = \left\{x \in \mathbf{R}^2 : x_2 = 0\right\}$$

- The phase plane $\mathbf{R}^2$ is partitioned into four regions

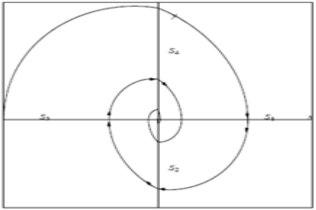
$$\begin{aligned} G_1 &= \{x : x_1 > 0, x_2 > 0\},\ G_2 = \{x : x_1 > 0, x_2 < 0\}, \\ G_3 &= \{x : x_1 < 0, x_2 > 0\},\ G_4 = \{x : x_1 < 0, x_2 < 0\}, \end{aligned}$$

- The velocity vectors in these regions are such that the trajectories of the closed-loop system cross the discontinuity manifolds S_1 and S_2 everywhere except the origin $x = 0$, which is the only equilibrium point of the system.

Asymptotic Stability of the Controlled Oscillator



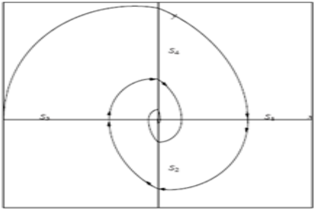
- The closed-loop system meets the right uniqueness property and the invariance principle is applicable



- The closed-loop system meets the right uniqueness property and the invariance principle is applicable
- The nonsmooth Lyapunov function

$$V\,(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + |x_1| \Rightarrow \dot{V} = -|x_2|$$

Asymptotic Stability of the Controlled Oscillator



- The closed-loop system meets the right uniqueness property and the invariance principle is applicable
- The nonsmooth Lyapunov function

$$V\left(x_1, x_2\right) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + |x_1| \Rightarrow \dot{V} = -|x_2|$$

- By invariance principle, the **closed loop system is asymptotically stable** because the largest invariant manifold is now reduced to the origin.

Finite-time Stability of Uncertain Homogeneous and Quasihomogeneous Systems

Perturbed VSS'

$$\dot{x} = \varphi(x,t) + \psi(x,t), \;\; x \in \mathbf{R^n} \tag{24}$$

Perturbed VSS'

$$\dot{x} = \varphi(x,t) + \psi(x,t), \ x \in \mathbf{R^n} \tag{24}$$

- Disturbance $\psi(x,t) = (\psi_1(x,t), \ldots, \psi_n(x,t))^T$ is piece-wise continuous

Perturbed VSS'

$$\dot{x} = \varphi(x,t) + \psi(x,t),\ x \in \mathbf{R^n} \tag{24}$$

- Disturbance $\psi(x,t) = (\psi_1(x,t), \ldots, \psi_n(x,t))^T$ is piece-wise continuous
- Uniform Boundedness of Admissible Disturbances

$$|\psi_i(x,t)| \leq M_i,\ i = 1, \ldots, n \tag{25}$$

for almost all $(x,t) \in B_\delta \times \mathbf{R}$ and some constants $M_i \geq 0$, fixed *a priori*.

Perturbed VSS'

$$\dot{x} = \varphi(x,t) + \psi(x,t),\ x \in \mathbf{R^n} \tag{24}$$

- Disturbance $\psi(x,t) = (\psi_1(x,t), \ldots, \psi_n(x,t))^T$ is piece-wise continuous
- Uniform Boundedness of Admissible Disturbances

$$|\psi_i(x,t)| \leq M_i,\ i = 1, \ldots, n \tag{25}$$

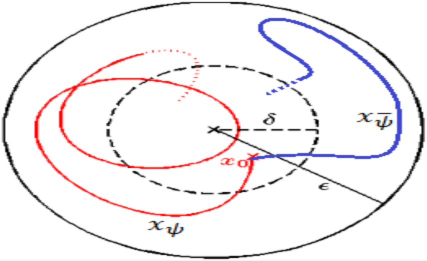
  for almost all $(x,t) \in B_\delta \times \mathbf{R}$ and some constants $M_i \geq 0$, fixed *a priori.*
- The above equation (24) is viewed as an uncertain differential equation with rectangular uncertainties, whose Filippov solutions x_ψ are associated with an admissible disturbance ψ

Uncertain Systems and Equiuniform Stability

Definition

The equilibrium point $x = 0$ of the uncertain system (24), (25) is *equiuniformly stable* iff for each $t_0 \in \mathbf{R}$, $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, dependent on ε and independent of t_0 and ψ, such that each solution $x_\psi(t, t_0, x^0)$ of (24), (25) with the initial data $x^0 \in B_\delta$ exists for all $t \geq t_0$ and satisfies the inequality

$$\|x_\psi(t, t_0, x^0)\| < \varepsilon, \; t_0 \leq t < \infty.$$



Uncertain Systems and Equiuniform Stability

Definition

The equilibrium point $x = 0$ of the uncertain system (24), (25) is said to be *equiuniformly asymptotically stable* if it is equiuniformly stable and the convergence

$$lim_{t\to\infty}\|x_\psi(t, t_0, x^0)\| = 0 \tag{26}$$

holds for all solutions of (24), (25) initialized within some B_δ, uniformly in the initial data t_0 and x^0, and all the solutions $x_\psi(\cdot, t_0, x^0)$. If this convergence remains in force for each $\delta > 0$ the equilibrium point is said to be *globally equiuniformly asymptotically stable.*

Definition

The equilibrium point $x = 0$ of the uncertain system (24), (25) is said to be *globally equiuniformly finite-time stable* if, in addition to the global equiuniform asymptotical stability, the limiting relation

$$x_\psi(t, t_0, x^0) = 0 \tag{27}$$

holds for each solution $x_\psi(\cdot, t_0, x^0)$ and all $t \geq t_0 + T(t_0, x^0)$ where the *settling time function*

$$T(t_0, x^0) = \sup_{x_\psi(\cdot, t_0, x^0)} \inf\{T \geq 0 : x_\psi(t, t_0, x^0) = 0 \ for \ all \ t \geq t_0 + T\} \tag{28}$$

is such that

$$\mathcal{T}(B_\delta) = sup_{t_0 \in \mathbf{R}, \ x^0 \in B_\delta} T(t_0, x^0) < \infty \ for \ each \ \delta > 0.$$

.

Homogeneous Functions

Definition

A piece-wise continuous function $\varphi(x,t)$ is called *locally homogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1, \ldots, r_n)$ where $r_i > 0,\ i = 1, \ldots, n$ if there exist a constant $c_0 > 0$ and a ball $B_\delta \subset \mathbf{R}^n$ such that

$$\varphi_i(c^{r_1}x_1, \ldots, c^{r_n}x_n, c^{-q}t) = c^{q+r_i}\varphi_i(x_1, \ldots, x_n, t) \tag{29}$$

for all $c \geq c_0$ and almost all $(x,t) \in B_\delta \times \mathbf{R}$.

Homogeneous Functions

Definition

A piece-wise continuous function $\varphi(x,t)$ is called *locally homogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1,\ldots,r_n)$ where $r_i > 0,\ i = 1,\ldots,n$ if there exist a constant $c_0 > 0$ and a ball $B_\delta \subset \mathbf{R}^n$ such that

$$\varphi_i(c^{r_1}x_1,\ldots,c^{r_n}x_n,c^{-q}t) = c^{q+r_i}\varphi_i(x_1,\ldots,x_n,t) \tag{29}$$

for all $c \geq c_0$ and almost all $(x,t) \in B_\delta \times \mathbf{R}$.

- Constructive definition admits analytical verification!

Homogeneous Functions

Definition

A piece-wise continuous function $\varphi(x,t)$ is called *locally homogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1,\ldots,r_n)$ where $r_i > 0,\ i = 1,\ldots,n$ if there exist a constant $c_0 > 0$ and a ball $B_\delta \subset \mathbf{R}^n$ such that

$$\varphi_i(c^{r_1}x_1,\ldots,c^{r_n}x_n,c^{-q}t) = c^{q+r_i}\varphi_i(x_1,\ldots,x_n,t) \tag{29}$$

for all $c \geq c_0$ and almost all $(x,t) \in B_\delta \times \mathbf{R}$.

- Constructive definition admits analytical verification!
- The twisting and supertwisting vector functions

$$\varphi_{tw} = \begin{pmatrix} x_2 \\ -\alpha sign\ x_1 - \beta sign\ x_2 \end{pmatrix},\ \varphi_{stw} = \begin{pmatrix} x_2 - \mu\sqrt{x_1} sign\ x_1 \\ -\nu sign\ x_1 \end{pmatrix} \tag{30}$$

with constant $\alpha,\beta,\mu,\nu \in \mathbf{R}$ are homogeneous of degree $q = -1$ with respect to dilation $r = (2,1)$.

Quasihomogeneous Uncertain Systems

Definition

The uncertain system

$$\dot{x} = \varphi(x,t) + \psi(x,t)$$

with rectangular uncertainties

$$|\psi_i(x,t)| \leq M_i, \ i = 1, \ldots, n$$

is called *locally quasihomogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1, \ldots, r_n)$ where $r_i > 0, \ i = 1, \ldots, n$ if there exist a constant $c_0 > 0$, called a lower estimate of the homogeneity parameter, and a ball $B_\delta \subset \mathbf{R}^n$, called a homogeneity ball, such that any solution $x_\psi(t)$ of the uncertain system, evolving within the ball B_δ, generates a parameterized set of solutions $x^c(t)$ of the same system (but affected by another admissible disturbance $\psi_c(t)$!) with parameter $c \geq c_0$ and components

$$x_i^c(t) = c^{r_i} x_i(c^q t). \tag{31}$$

Homogeneous Functions Generate Quasihomogeneous Uncertain Systems

Lemma (Orlov, CDC'2003)

Let a piece-wise continuous function $\varphi(x,t)$ *be locally homogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1, \dots, r_n)$. *Then the uncertain system*

$$\dot{x} = \varphi(x,t) + \psi(x,t)$$

with rectangular uncertainties

$$|\psi_i(x,t)| \leq M_i, \; i = 1, \dots, n$$

is locally quasihomogeneous of the same degree $q \in \mathbf{R}$ *with respect to the same dilation* $(r_1, \dots, r_n)$.

Homogeneous Functions Generate Quasihomogeneous Uncertain Systems

Lemma (Orlov, CDC'2003)

Let a piece-wise continuous function $\varphi(x,t)$ *be locally homogeneous of degree* $q \in \mathbf{R}$ *with respect to dilation* $(r_1, \ldots, r_n)$. *Then the uncertain system*

$$\dot{x} = \varphi(x,t) + \psi(x,t)$$

with rectangular uncertainties

$$|\psi_i(x,t)| \leq M_i, \ \ i = 1, \ldots, n$$

is locally quasihomogeneous of the same degree $q \in \mathbf{R}$ *with respect to the same dilation* $(r_1, \ldots, r_n)$.

- **Proof** is based on embedding a quasihomogeneous uncertain system into an appropriate framework of homogeneous differential inclusion.

Finite-time Stability of Quasihomogeneous Systems

While being globally asymptotically stable, a locally homogeneous vector field $\varphi(x,t)$ of degree $q<0$ generates a globally finite-time stable uncertain VSS

Theorem (Quasihomogeneity Principle; Orlov, CDC'2003; SIAM'2005)

1. *the right-hand side of an uncertain differential equation*

$$\dot{x} = \varphi(x,t) + \psi(x,t) \tag{32}$$

consists of a locally homogeneous piece-wise continuous function φ of degree $q<0$ with respect to dilation $(r_1,\ldots,r_n)$ and a piece-wise continuous function ψ whose components $\psi_i,\ i=1,\ldots,n$ are locally uniformly bounded by constants $M_i \geq 0$ within a homogeneity ball;

Finite-time Stability of Quasihomogeneous Systems

While being globally asymptotically stable, a locally homogeneous vector field $\varphi(x,t)$ of degree $q<0$ generates a globally finite-time stable uncertain VSS

Theorem (Quasihomogeneity Principle; Orlov, CDC'2003; SIAM'2005)

1. *the right-hand side of an uncertain differential equation*

$$\dot{x} = \varphi(x,t) + \psi(x,t) \tag{32}$$

   consists of a locally homogeneous piece-wise continuous function φ of degree $q<0$ with respect to dilation $(r_1,\dots,r_n)$ and a piece-wise continuous function ψ whose components ψ_i, $i=1,\dots,n$ are locally uniformly bounded by constants $M_i \geq 0$ within a homogeneity ball;

2. $M_i = 0$ *whenever* $q + r_i > 0$;

Finite-time Stability of Quasihomogeneous Systems

While being globally asymptotically stable, a locally homogeneous vector field $\varphi(x,t)$ of degree $q<0$ generates a globally finite-time stable uncertain VSS

Theorem (Quasihomogeneity Principle; Orlov, CDC'2003; SIAM'2005)

1. *the right-hand side of an uncertain differential equation*

$$\dot{x} = \varphi(x,t) + \psi(x,t) \tag{32}$$

   consists of a locally homogeneous piece-wise continuous function φ of degree $q<0$ with respect to dilation $(r_1,\dots,r_n)$ and a piece-wise continuous function ψ whose components ψ_i, $i=1,\dots,n$ are locally uniformly bounded by constants $M_i \geq 0$ within a homogeneity ball;
2. $M_i = 0$ *whenever* $q + r_i > 0$;
3. *the uncertain system* (32) *is globally equiuniformly asymptotically stable around the origin.*

   Then VSS (32) *is globally equiuniformly finite-time stable.*

Settling Time Estimate

Orlov, SICON 2005

Going through this route yields:

Upper Estimate

$$T(t_0, x^0) \le \tau(x^0, E_R) + \frac{1}{1-2^q}(\delta R^{-1})^q s(\delta)$$

of the settling-time function

$$T(t_0, x^0) = \sup_{x(\cdot, t_0, x^0)} \inf\{T \ge 0 : x(t, t_0, x^0) = 0 \; for\ all\ t \ge t_0 + T\}$$

in terms of the reaching-time function

$$\tau(x^0, E_R) = \sup_{x(\cdot, t_0, x^0)} \inf\{T \ge 0 : x(t, t_0, x^0) \in E_R \; for\ all\ t_0 \in \mathbf{R},\ t \ge t_0 + T\}$$

of attaining the ellipsoid

$$E_R = \{x \in \mathbf{R}^n : \sqrt{\Sigma_{i=1}^n \left(\frac{x_i}{R^{r_i}}\right)^2} \le 1\},$$

and the semidistance-time function

$$s(\delta) = \sup_{x^0 \in E} \tau(x^0, E_{\frac{1}{2}\delta})$$

Arsenal of Finite-time Stability Analysis Tools

Trivial First-order Quasihomogeneous System

The quasihomogeneous first-order VSS

$$\dot{x} = -\alpha sign\ x + w(x,t) \tag{33}$$

of degree $q = -1$ with respect to dilation $r = 1$.

Trivial First-order Quasihomogeneous System

The quasihomogeneous first-order VSS

$$\dot{x} = -\alpha sign\ x + w(x,t) \tag{33}$$

of degree $q = -1$ with respect to dilation $r = 1$.

- Uniform Upper Bound on Disturbance magnitude

$$|w(x,t)| \leq N$$

Trivial First-order Quasihomogeneous System

The quasihomogeneous first-order VSS

$$\dot{x} = -\alpha sign\ x + w(x,t) \tag{33}$$

of degree $q = -1$ with respect to dilation $r = 1$.

- Uniform Upper Bound on Disturbance magnitude

$$|w(x,t)| \leq N$$

- The higher switching magnitude is chosen:

$$\alpha > N > 0$$

The quadratic Lyapunov function

$$V(x) = x^2$$

Time derivative along the solutions of $\dot{x} = -\alpha sign\ x + w(x,t)$:

$$\begin{aligned}\dot{V}(x(t)) = -2|x(t)|[\alpha - w(x(t),t) sign\ x(t)] \leq -2(\alpha - N)|x(t)| \\ = -2(\alpha - N)\sqrt{V(x(t))}.\end{aligned} \tag{34}$$

The quadratic Lyapunov function

$$V(x) = x^2$$

Time derivative along the solutions of $\dot{x} = -\alpha sign\ x + w(x,t)$:

$$\begin{aligned}\dot{V}(x(t)) = -2|x(t)|[\alpha - w(x(t),t) sign\ x(t)] \leq -2(\alpha - N)|x(t)| \\ = -2(\alpha - N)\sqrt{V(x(t))}.\end{aligned} \quad (34)$$

- The global equiuniform asymptotic stability is thus ensured.

The quadratic Lyapunov function

$$V(x) = x^2$$

Time derivative along the solutions of $\dot{x} = -\alpha sign\ x + w(x,t)$:

$$\begin{aligned} \dot{V}(x(t)) = -2|x(t)|[\alpha - w(x(t),t) sign\ x(t)] \leq -2(\alpha - N)|x(t)| \\ = -2(\alpha - N)\sqrt{V(x(t))}. \end{aligned} \tag{34}$$

- The global equiuniform asymptotic stability is thus ensured.
- By quasihomogeneity principle, the global equiuniform finite time stability is guaranteed.

The quadratic Lyapunov function

$$V(x) = x^2$$

Time derivative along the solutions of $\dot{x} = -\alpha sign\ x + w(x,t)$:

$$\begin{aligned}\dot{V}(x(t)) = -2|x(t)|[\alpha - w(x(t),t) sign\ x(t)] &\leq -2(\alpha - N)|x(t)| \\ &= -2(\alpha - N)\sqrt{V(x(t))}.\end{aligned} \tag{34}$$

- The global equiuniform asymptotic stability is thus ensured.
- By quasihomogeneity principle, the global equiuniform finite time stability is guaranteed.
- Remark, the decay rate (34) itself results in the same conclusion. Indeed

Finite-time Stability of Useful Differential Inequality

Lemma

Let an everywhere non-negative function $V(t)$ *meet the differential inequality*

$$\dot{V}(t) \leq -2\gamma V^{\beta}(t) \tag{35}$$

for all $t \geq 0$ *and for some constants* $\gamma > 0$ *and* $\beta \in (0, 1)$. *Then* $V(t) = 0$ *for all* $t \geq [2\gamma(1-\beta)]^{-1} V^{1-\beta}(0)$.

Finite-time Stability of Useful Differential Inequality

Lemma

Let an everywhere non-negative function $V(t)$ meet the differential inequality

$$\dot{V}(t) \leq -2\gamma V^{\beta}(t) \tag{35}$$

for all $t \geq 0$ and for some constants $\gamma > 0$ and $\beta \in (0,1)$. Then $V(t) = 0$ for all $t \geq [2\gamma(1-\beta)]^{-1} V^{1-\beta}(0)$.

- **Proof** is based on the comparison principle: an arbitrary non-negative solution $V(t)$ of inequality (35) is dominated $V(t) \leq V_0(t)$ by the solution

$$V_0(t) = \begin{cases} [V^{(1-\beta)}(0) - 2\gamma(1-\beta)t]^{\frac{1}{1-\beta}} & \text{if } t \in [0, \frac{V^{(1-\beta)}(0)}{2\gamma(1-\beta)}] \\ 0 & \text{if } t \geq \frac{V^{(1-\beta)}(0)}{2\gamma(1-\beta)} \end{cases} \tag{36}$$

  of the differential equation

$$\dot{V}_0(t) = -2\gamma V_0^{\beta}(t),$$

  specified with the same initial condition $V_0(0) = V(0)$.

Finite-time Stability of Supertwisting Algorithm

Orlov, Aoustin, Chevallereau, IEEE TAC 2011

The homogeneous second-order VSS

$$\begin{aligned} \dot{x} &= y - \mu\sqrt{|x|}sign\ x,\ \ \mu > 0 \\ \dot{y} &= -\nu sign\ x,\ \ \nu > 0. \end{aligned} \tag{37}$$

of degree $q = -1$ with respect to dilation $(1, 2)$.

Finite-time Stability of Supertwisting Algorithm

Orlov, Aoustin, Chevallereau, IEEE TAC 2011

The homogeneous second-order VSS

$$\dot{x} = y - \mu\sqrt{|x|}sign\ x,\ \ \mu > 0$$
$$\dot{y} = -\nu sign\ x,\ \ \nu > 0. \tag{37}$$

of degree $q = -1$ with respect to dilation $(1, 2)$.

- Lyapunov function and its derivative

$$V = \nu|x| + \frac{1}{2}y^2 \quad \Rightarrow \quad \dot{V} = -\mu\nu\sqrt{|x|}$$

Finite-time Stability of Supertwisting Algorithm

Orlov, Aoustin, Chevallereau, IEEE TAC 2011

The homogeneous second-order VSS

$$\begin{aligned} \dot{x} &= y - \mu\sqrt{|x|}sign\ x,\ \mu > 0 \\ \dot{y} &= -\nu sign\ x,\ \nu > 0. \end{aligned} \tag{37}$$

of degree $q = -1$ with respect to dilation $(1, 2)$.

- Lyapunov function and its derivative

$$V = \nu|x| + \frac{1}{2}y^2 \quad \Rightarrow \quad \dot{V} = -\mu\nu\sqrt{|x|}$$

- Invariance principle is applicable (no sliding modes on $x = 0$, verified by the invalidity of $y\dot{y} < 0$ as $x \to 0$)

$$\Downarrow$$

Finite-time Stability of Supertwisting Algorithm

Orlov, Aoustin, Chevallereau, IEEE TAC 2011

The homogeneous second-order VSS

$$\begin{aligned} \dot{x} &= y - \mu\sqrt{|x|}sign\ x,\ \ \mu > 0 \\ \dot{y} &= -\nu sign\ x,\ \ \nu > 0. \end{aligned} \tag{37}$$

of degree $q = -1$ with respect to dilation $(1, 2)$.

- Lyapunov function and its derivative

$$V = \nu|x| + \frac{1}{2}y^2 \quad \Rightarrow \quad \dot{V} = -\mu\nu\sqrt{|x|}$$

- Invariance principle is applicable (no sliding modes on $x = 0$, verified by the invalidity of $y\dot{y} < 0$ as $x \to 0$)

$$\Downarrow$$

- (37) is GAS $\Rightarrow$ (37) is FTS due to homogeneity

The perturbed second-order VSS

$$\dot{x} = y - \mu\sqrt{|x|}sign\ x,\ \ \mu > 0$$
$$\dot{y} = -\nu sign\ x + \omega(t),\ \ \nu > 0.$$

is no longer homogeneous.

The perturbed second-order VSS

$$\dot{x} = y - \mu\sqrt{|x|} sign\ x,\ \ \mu > 0$$
$$\dot{y} = -\nu sign\ x + \omega(t),\ \ \nu > 0.$$

is no longer homogeneous.

- External disturbances are uniformly bounded by some $M > 0$:

$$ess\ \sup_{t>0} |\omega(t)| \leq M < min\left\{\frac{\mu}{2}, \frac{\mu\nu}{1+\mu}\right\}$$

Robust Finite-time Stability of Supertwisting Algorithm

Moreno, Osorio, CDC'2008 & Orlov, Aoustin, Chevallereau , IEEE TAC 2011

The perturbed second-order VSS

$$\dot{x} = y - \mu\sqrt{|x|}sign\ x,\ \ \mu > 0$$
$$\dot{y} = -\nu sign\ x + \omega(t),\ \ \nu > 0.$$

is no longer homogeneous.

- External disturbances are uniformly bounded by some $M > 0$:

$$ess\ \sup_{t>0}|\omega(t)| \leq M < min\left\{\frac{\mu}{2}, \frac{\mu\nu}{1+\mu}\right\}$$

- Lyapunov function

$$V = \nu|x| + \frac{1}{2}y^2 + \frac{1}{2}(y - \mu\sqrt{|x|}sign\ x)^2$$

The Lyapunov function is shown to meet the useful differential inequality

$$\dot{V} \leq -\gamma\sqrt{V}$$

with

$$\gamma = \sqrt{2\nu} \cdot min\left\{\frac{2(\mu\nu - M - M\nu)}{4\nu + 3\mu^2}, \frac{\mu - 2M}{4}\right\}$$

Robust Finite-time Stability of Supertwisting Algorithm

Orlov, Aoustin, Chevallereau, IEEE TAC 2011

The Lyapunov function is shown to meet the useful differential inequality

$$\dot{V} \leq -\gamma\sqrt{V}$$

with

$$\gamma = \sqrt{2\nu} \cdot min\left\{\frac{2(\mu\nu - M - M\nu)}{4\nu + 3\mu^2}, \frac{\mu - 2M}{4}\right\}$$

-

$$\Downarrow$$

Robust (equiuniform) finite time stability is thus guaranteed with the settling time estimate

$$T(x_0, y_0) \leq 2\sqrt{V(x_0, y_0)}\gamma^{-1}$$

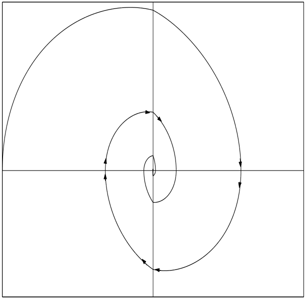
Finite-time Stability of Twisting Algorithm

Orlov, CDC'2003

The homogeneous second-order VSS

$$\dot{x} = y,\ \dot{y} = -asign\ x - bsign\ y, \quad a > b > 0. \tag{38}$$

of degree $q = -1$ with respect to dilation $(1, 2)$.



1. Beyond the origin, no sliding modes on axes $\Rightarrow$ solutions are uniquely determined to the right
2. Nonstrict Lyapunov function $V = a|x| + \frac{1}{2}y^2$
3. Time derivative $\dot{V} = -b|y|$
4. Invariance principle $\Rightarrow$ GAS
5. Quasihomogeneuity principle $\Rightarrow$ FTS

Robust Finite-time Stability of Twisting Algorithm

Orlov, CDC'2003

The perturbed *nonautonomous* second-order VSS

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a\,sign\ x - b\,sign\ y - hx - py + \omega(x,y,t), \quad h,p > 0 \ \&\ a > b > 0 \end{aligned}$$

is no longer homogeneous.

Robust Finite-time Stability of Twisting Algorithm

Orlov, CDC'2003

The perturbed *nonautonomous* second-order VSS

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -a\,sign\ x - b\,sign\ y - hx - py + \omega(x,y,t), \quad h,p > 0\ \&\ a > b > 0\end{aligned}$$

is no longer homogeneous.

- External disturbances are uniformly bounded by some $M > 0$ such that:

$$ess\ \sup_{t>0} |\omega(t)| \le M < b < a - M.$$

The perturbed *nonautonomous* second-order VSS

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a\,sign\ x - b\,sign\ y - hx - py + \omega(x,y,t), \quad h,p>0\ \&\ a>b>0 \end{aligned}$$

is no longer homogeneous.

- External disturbances are uniformly bounded by some $M > 0$ such that:

$$ess\ \sup_{t>0} |\omega(t)| \le M < b < a - M.$$

- Nonstrict Lyapunov function $V = a|x| + \frac{1}{2}(hx^2 + y^2)$ posseses non-positive definite time derivative $\dot{V} \le -(b-M)|y|$.

Robust Finite-time Stability of Twisting Algorithm

Orlov, CDC'2003

The perturbed *nonautonomous* second-order VSS

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -a\,sign\ x - b\,sign\ y - hx - py + \omega(x,y,t), \quad h,p>0\ \&\ a>b>0\end{aligned}$$

is no longer homogeneous.

- External disturbances are uniformly bounded by some $M > 0$ such that:

$$ess\ \sup_{t>0} |\omega(t)| \le M < b < a - M.$$

- Nonstrict Lyapunov function $V = a|x| + \frac{1}{2}(hx^2 + y^2)$ posseses non-positive definite time derivative $\dot{V} \le -(b-M)|y|$.
- Given a specific disturbance ω, the composite function $V(x(t), y(t))$ is non-strictly monotonically decreasing along the solutions and GAS is still ensured by the invariance principle.

The perturbed *nonautonomous* second-order VSS

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -a\,sign\ x - b\,sign\ y - hx - py + \omega(x,y,t), \quad h,p>0\ \&\ a>b>0\end{aligned}$$

might be viewed as an *autonomous homogeneous differential inclusion*

$$\ddot{x} \in F(x,\dot{x}) + \Omega$$

(of the same homogeneity degree and dilation as the nominal unperturbed system!) with Filippov convex hull $F(x,\dot{x})$ and a class Ω of admissible external disturbances subject to rectangular restrictions

$$ess\ \sup_{t>0} |\omega(t)| \le M < b < a - M. \tag{39}$$

The perturbed *nonautonomous* second-order VSS

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -asign\ x - bsign\ y - hx - py + \omega(x,y,t), \quad h,p > 0\ \&\ a > b > 0 \end{aligned}$$

might be viewed as an *autonomous homogeneous differential inclusion*

$$\ddot{x} \in F(x, \dot{x}) + \Omega$$

(of the same homogeneity degree and dilation as the nominal unperturbed system!) with Filippov convex hull $F(x, \dot{x})$ and a class Ω of admissible external disturbances subject to rectangular restrictions

$$ess \sup_{t>0} |\omega(t)| \leq M < b < a - M. \tag{39}$$

- Semiglobal Strict Lyapunov functions are subsequently involved to prove Equiuniform GAS.

R-parameterized family of Lyapunov functions

$$V_R(x,y) = a|x| + \frac{1}{2}(y^2 + hx^2) + \kappa_R xy, \quad \kappa_R > 0 \tag{40}$$

The weight parameter $\kappa_R > 0$ is chosen according to

$$\kappa_R < \min\{1, \frac{2a^2}{R}, \frac{a(b-M)}{a\sqrt{2R}+pR}\}. \tag{41}$$

to ensure that the Lyapunov function $V_R(x,y)$ is positive definite on the corresponding compact set

$$D_R = \{(x,y) \in \mathbf{R}^2 : a|x| + \frac{1}{2}(hx^2 + y^2) \le R\} \tag{42}$$

Differentiating V_R along the solutions yields

$$\dot{V}_R(x_\omega(t), y_\omega(t)) \leq -K_R V_R(x_\omega(t), y_\omega(t)) \tag{43}$$

where

$$\begin{aligned} K_R &= \frac{2ac_R}{\max\{2a^2 + hR, a\sqrt{2R} + 2\kappa_R R\}} > 0 \\ c_R &= \min\{b - M - \kappa_R(\sqrt{2R} + \frac{pR}{a}), \kappa_R(a - b - M)\} > 0 \end{aligned}$$

Differentiating V_R along the solutions yields

$$\dot{V}_R(x_\omega(t), y_\omega(t)) \le -K_R V_R(x_\omega(t), y_\omega(t)) \tag{43}$$

where

$$\begin{aligned} K_R &= \frac{2ac_R}{\max\{2a^2 + hR, a\sqrt{2R} + 2\kappa_R R\}} > 0 \\ c_R &= \min\{b - M - \kappa_R(\sqrt{2R} + \frac{pR}{a}), \kappa_R(a - b - M)\} > 0 \end{aligned}$$

- Semiglobal (not global!) equiuniform exponentail stability is established:

  $$V_R(x_\omega(t), y_\omega(t)) \le V_R(x_\omega(t_0), y_\omega(t_0))e^{-K_R(t-t_0)}$$

Differentiating V_R along the solutions yields

$$\dot{V}_R(x_\omega(t), y_\omega(t)) \leq -K_R V_R(x_\omega(t), y_\omega(t)) \tag{43}$$

where

$$\begin{aligned} K_R &= \frac{2ac_R}{\max\{2a^2 + hR, a\sqrt{2R} + 2\kappa_R R\}} > 0 \\ c_R &= \min\{b - M - \kappa_R(\sqrt{2R} + \frac{pR}{a}), \kappa_R(a - b - M)\} > 0 \end{aligned}$$

- Semiglobal (not global!) equiuniform exponentail stability is established:

$$V_R(x_\omega(t), y_\omega(t)) \leq V_R(x_\omega(t_0), y_\omega(t_0)) e^{-K_R(t-t_0)}$$

- and Equiuniform GAS follows

The perturbed nonautonomous VSS

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a sign\ x - b sign\ y - hx - py + \omega(x, y, t), \quad h, p > 0\ \&\ a > b > 0 \end{aligned}$$

being represented in the form of the *autonomous homogeneous differential inclusion*

$$\ddot{x} \in F(x, \dot{x}) + \Omega$$

of the homogeneity degree $q = -1$ and dilation $r = (2, 1)$ with Filippov convex hull $F(x, \dot{x})$ and a class Ω of admissible external disturbances subject to rectangular restrictions

$$ess \sup_{t>0} |\omega(t)| \leq M < b < a - M.$$

is shown to be equiuniformly GAS

The perturbed nonautonomous VSS

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a\,sign\ x - b\,sign\ y - hx - py + \omega(x,y,t), \quad h,p > 0\ \&\ a > b > 0 \end{aligned}$$

being represented in the form of the *autonomous homogeneous differential inclusion*

$$\ddot{x} \in F(x,\dot{x}) + \Omega$$

of the homogeneity degree $q = -1$ and dilation $r = (2,1)$ with Filippov convex hull $F(x,\dot{x})$ and a class Ω of admissible external disturbances subject to rectangular restrictions

$$ess\ \sup_{t>0} |\omega(t)| \le M < b < a - M.$$

is shown to be equiuniformly GAS

- By quasihomogeneity principle, equiuniform global FTS is concluded.

Supertwisting observer

Davila, Fridman, Levant, IEEE TAC 2005

The uncertain non-autonomous system

$$\dot{x} = y, \quad \dot{y} = u + \omega(t)$$

Finite-time Velocity Observer

$$\dot{\hat{x}} = \hat{y} + \mu\sqrt{|x - \hat{x}|}sign\ (x - \hat{x}), \quad \dot{\hat{y}} = u + \nu sign\ (x - \hat{x})$$

Supertwisting observer

Davila, Fridman, Levant, IEEE TAC 2005

The uncertain non-autonomous system

$$\dot{x} = y, \quad \dot{y} = u + \omega(t)$$

Finite-time Velocity Observer

$$\dot{\hat{x}} = \hat{y} + \mu\sqrt{|x - \hat{x}|}sign\ (x - \hat{x}), \quad \dot{\hat{y}} = u + \nu sign\ (x - \hat{x})$$

- The finite time error convergence $e_1 = x - \hat{x} \to 0, e_2 = y - \hat{y} \to 0$ as $t \to \infty$ is guaranteed under uniform disturbance magnitude constraint

$$ess \sup_{t>0} |\omega(t)| \leq M < min\left\{\frac{\mu}{2}, \frac{\mu\nu}{1+\mu}\right\} \text{ for some } M > 0$$

Supertwisting observer

Davila, Fridman, Levant, IEEE TAC 2005

The uncertain non-autonomous system

$$\dot{x} = y, \quad \dot{y} = u + \omega(t)$$

Finite-time Velocity Observer

$$\dot{\hat{x}} = \hat{y} + \mu\sqrt{|x - \hat{x}|}sign\ (x - \hat{x}), \quad \dot{\hat{y}} = u + \nu sign\ (x - \hat{x})$$

- The finite time error convergence $e_1 = x - \hat{x} \to 0, e_2 = y - \hat{y} \to 0$ as $t \to \infty$ is guaranteed under uniform disturbance magnitude constraint

$$ess \sup_{t>0} |\omega(t)| \leq M < min\left\{\frac{\mu}{2}, \frac{\mu\nu}{1+\mu}\right\} \text{ for some } M > 0$$

- **The idea behind the velocity observer:** the error dynamics $\dot{e}_1 = e_2 - \mu\sqrt{|e_1|}sign\ e_1, \quad \dot{e}_2 = -\nu sign\ e_1 + \omega$ are in the form of the FTS supertwisting algorithm.

Twisting State Feedback $u = -asign\ x - bsign\ y$ is coupled to the **Supertwisting-observer**

$$\dot{\hat{x}} = \hat{y} + \mu\sqrt{|x-\hat{x}|}sign\ (x-\hat{x}), \qquad \dot{\hat{y}} = u + \nu sign\ (x-\hat{x})$$

$$\Downarrow$$

Dynamic Position Feedback $u = -asign\ x - bsign\ \hat{y}$,
globally FT stabilizes the uncertain non-autonomous system

$$\dot{x} = y,\ \dot{y} = u + \omega(x,y,t)$$

Gain Tuning

$$\begin{aligned} ess\ \sup_{t>0}|\omega(t)| &\leq M < min\left\{\frac{\mu}{2}, \frac{\mu\nu}{1+\mu}\right\} \\ ess\ \sup_{t>0}|\omega(t)| &\leq M < b < a - M. \end{aligned}$$

Concluding Remarks

Orlov *Discontinuous Systems - Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions*, Springer-Verlag, London, 2009

Concluding Remarks

Orlov *Discontinuous Systems - Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions*, Springer-Verlag, London, 2009

- Analysis tools of discontinuous systems and sliding mode design methods of the first and second orders were presented.

Concluding Remarks

Orlov *Discontinuous Systems - Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions*, Springer-Verlag, London, 2009

- Analysis tools of discontinuous systems and sliding mode design methods of the first and second orders were presented.
- Capabilities of the methods and their robustness features were illustrated in mechanical applications