Marketing resource allocation in duopolies over social networks

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March 7, 2019

Traditional marketing

1. Broadcast via TV/Radio etc,
2. Exploit influence of celebrities
3. Not customized for viewers

Modern social media marketing

1. SM platforms collect user data
2. Firms compete online to provide ads based on user data
3. Exploits individual preferences and opinions.

SoA and novelty

State of the art

- Marketing games classical literature [L. Friedman et al. 1958], [Butters et al. 1977],
- Targeted ads (second price auction game) [Edelman et al. 2007],
- Opinion dynamics aware targeted ads, but with all to all graph [Masucci et al. 2014].

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1. Study the strategies of competing firms marketing over social networks,
2. firms are aware of the social network graph and opinions.

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- Action of firm $i \in \{1, 2\}$ is $a_i \in \mathcal{A}_i$ and

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- Agents interact over a directed and weighted graph $(\mathcal{V}, \mathcal{E}, \Omega)$, with $\Omega_{m,n}$ describing the influence of agent m on n.

The Laplacian of the graph is given by

$$L_{m,n} = \begin{cases} \sum_{n=1}^{N} \Omega_{m,n} \text{ if } m = n \\ -\Omega_{m,n} \text{ if } m \neq n \end{cases} . \quad (2)$$

Dynamics model

$$\begin{cases} \dot{x}(t) & = & -\mathbf{L}x(t) & \forall t \in \mathbb{R} \setminus \{0\} \\ x_n(t_k^+) & = & \phi(x_n(t_k), a_{1,n}, a_{2,n}) & \forall n \in \mathcal{V}, t_k \in \mathcal{T} \end{cases} . \quad (3)$$

$$\phi(x_{0,n}, a_{1,n}, a_{2,n}) = \frac{x_{0,n} + a_{1,n}}{1 + a_{1,n} + a_{2,n}}, \ \forall n \in \{1, ..., N\}. \quad (4)$$

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Motivation and interpretation for this rule

1. If $x_n(t)$ is seen as the probability of agent n picking the product of Firm 1, $\phi(\cdot)$ corresponds to a Bayessian update rule on the opinion,
2. $a_{i,n}$ is the increase in the odds of agent n choosing Firm i,
3. Nice properties like symmetry, asymptotic limits etc.

Agent influential power

The AIP of Agent n is given by $\rho_n > 0$. When the profits are an integral of $x(t)$, the calculation of ρ is given in [Varma et al, CDC 2017].

Revenue per firm: $\mathcal{T} = \{0\}$

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The revenue functions are taken as

$$u_1(x_0, a_1, a_2) := \rho\Phi(x_0, a_1, a_2) - \lambda_1 1_N^\top a_1, \quad (5)$$

$$u_2(x_0, a_1, a_2) := \rho\Phi(1 - x_{0,n}, a_{2,n}, a_{1,n}) - \lambda_2 1_N^\top a_2. \quad (6)$$

where $\lambda_i \geq 0$ is the advertising efficiency or pricing factor for Firm i.

Static game formulation: $\mathcal{T} = \{0\}$

The strategic form of the *static game* of interest therefore writes as:

$$\mathcal{G} = (\{1, 2\}, \{\mathcal{A}_1, \mathcal{A}_2\}, \{u_1, u_2\}) , \tag{7}$$

where:

- $\{1, 2\}$ is the set of players (i.e., Firms 1 and 2);
- $\mathcal{A}_i$ defined in (1) is the set of pure actions for Player i;
- u_i as defined per (5) (6) is the utility function for Firm i.

Introduction

System model

One-shot game analysis

Repeated campaigns

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Definition (Pure NE)

A strategy profile $(a_1^\star, a_2^\star) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a pure NE for $\mathcal{G}$ for a given x_0 if $\forall i \in \{1,2\}$,

$$\forall a_i \in \mathcal{A}_i,\ u_i(x_0, a_i^\star, a_{-i}^\star) \geq u_i(x_0, a_i, a_{-i}^\star). \tag{8}$$

Theorem

The game $\mathcal{G}$ has a pure and unique NE.

Characterization of the NE of $\mathcal{G}_n$

Theorem

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Main steps of the proof:

1. Action space $\mathcal{A}_i$ is a convex and compact set.
2. We show the utility function is concave w.r.t actions, enabling us to use the existence theorem in [Rosenthal et al 1965].
3. We show that the property of diagonally strict concavity in [Rosenthal et al 1965] is satisfied for uniqueness.

Let $\beta_i(x_{0;i}, a_{-i})$ be the BR. Then, at the NE

$$a_1^\star \in \beta_1(x_{0;1}, a_2^\star)\,,\; a_2^\star \in \beta_2(x_{0;1}, a_1^\star) \tag{9}$$

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Proposition

The best-response functions are given by

$$\beta_{i,n}(x_{0;i}, a_{-i}) = \min\{b_i, \max\{0, \alpha_{i,n}(x_{0;i}, a_{-i})\}\} \tag{10}$$

$$\alpha_{i,n}(x_{0;i}, a_{-i}) = \sqrt{\frac{\rho_n(x_{0,n;-i} + a_{-i,n})}{\mu_{0;i} + \lambda_i}} - 1 - a_{-i,n} \tag{11}$$

for all $n \in \mathcal{V}$, *and* $\mu_0 \in \mathbb{R}_{\geq 0}$ *is such that*

$$\sum_{n=1}^{N} \beta_{i,n}(x_{0;i}, a_{-i}) \leq B_i, \mu_{0;i}\Big(\sum_n \beta_{i,n}(x_{0;i}, a_{-i}) - B_i\Big) = 0 \tag{12}$$

Characterization of the NE

Proposition

For each $n \in \mathcal{V}$, *the NE* $(a^\star_{1,n}, a^\star_{2,n})$ *is given by*

- $(y, 0)$ *(or* $(0, y)$*) if* $\exists y \in [0, b_1]$ *(or* $[0, b_2]$ *respectively) such that* (9) *is satisfied by one of these pairs,*

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- *or* $(a^\star_{1,n}, a^\star_{2,n}) \in (0, b_1) \times (0, b_2)$ *and is given by*

$$a^\star_{i,n} = \left(\frac{k_i}{k_i + k_{-i}}\right)^2 k_{-i}\rho_n - x_{0,n;i}, \qquad (13)$$

where $k_i = \frac{1}{\lambda_i + \mu_{0;i}}$ *and* $\mu_{0;i}$ *is a common constant for all* $n \in \mathcal{V}$ *given by* (12).

Uniform broadcasting allocation (UBA)

When firm i uses UBA strategy, then

$$a_i^{\mathrm{UBA}} := \min \left\{ b_i, \frac{B_i}{N} \right\}. \tag{14}$$

Numerical examples: Comparison

Uniform broadcasting allocation (UBA)

When firm i uses UBA strategy, then

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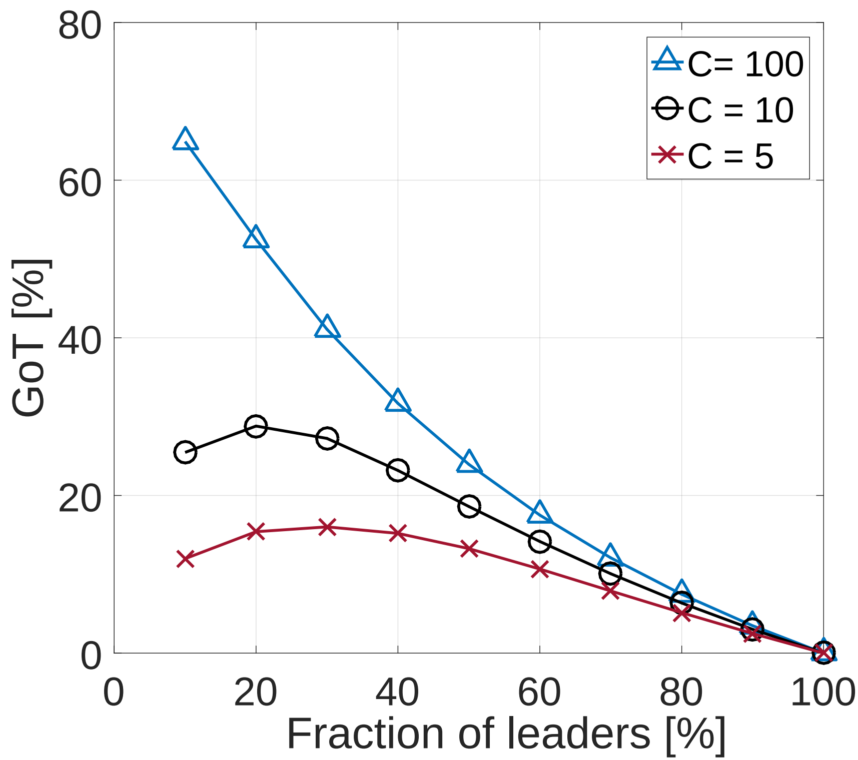
The resulting difference in utility between the two strategies is referred to as the *gain of targeting* (GoT), and is measured as

$$\mathrm{GoT} := \frac{u_1(x_0, \beta_1(a_2^{\mathrm{UBA}})), a_2^{\mathrm{UBA}}) - u_1(x_0, a_1^{\mathrm{UBA}}, a_2^{\mathrm{UBA}})}{u_1(x_0, a_1^{\mathrm{UBA}}, a_2^{\mathrm{UBA}})}. \tag{15}$$

For this simulation, we consider $N = 100$ with $\rho_n \in \{1, C\}$. C being the AIP of leaders.



NE for a given graph and initial opinions

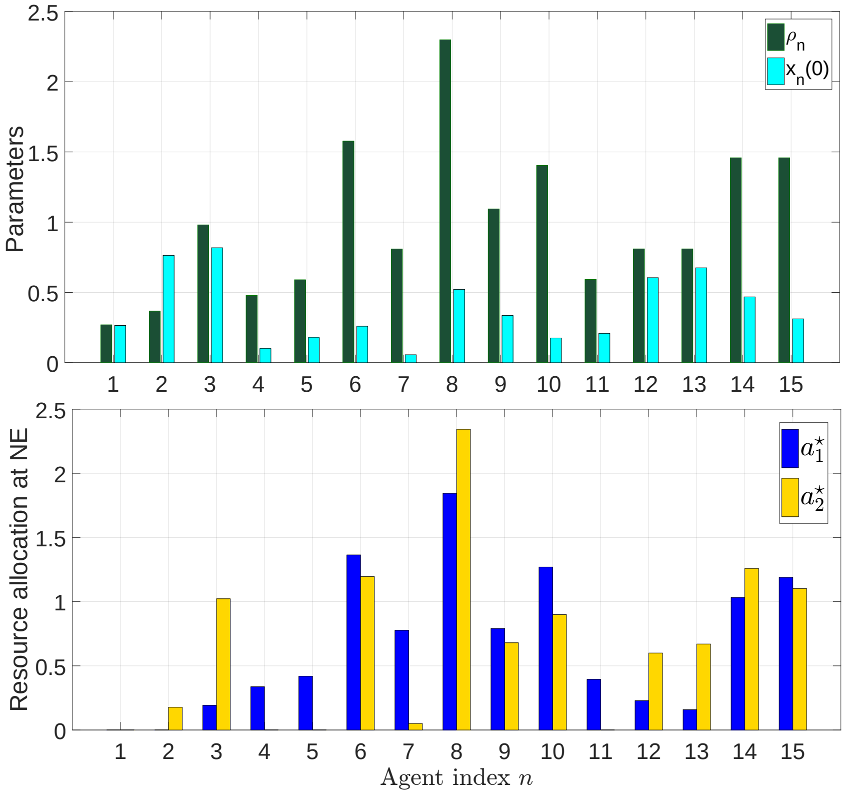
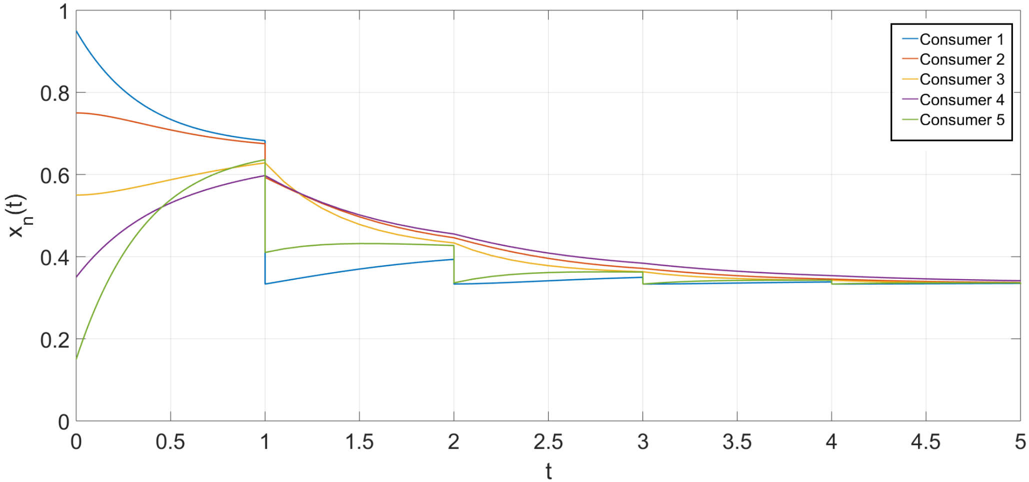


Figure 1: The sub-figure on top shows the AIP ρ_n and initial opinion $x_n(0)$, the sub-figure on the bottom shows the a_i^*.

Repeated campaigns: applying the OS-NE 16/22



Here, we take $N = 5$, $\lambda_1 = 2$, $\lambda_2 = 1$ and

$$\rho = (6.8910, 1.9202, 1.30631.1036, 3.7789)^T.$$

Repeated campaigns: key definitions

Campaigns instances given by t_k, $k \in \mathcal{K} \subset \mathbb{Z}$.
Net utilities given by

$$U_i = \sum_{k \in \mathcal{K}} u_i(x(t_k), a_1(k), a_2(k)) \tag{16}$$

Let

$$\overline{X}_n := \left\{ y \in \mathbb{R} : y > 1 - \frac{\lambda_1}{\rho_n}, y < \frac{\lambda_2}{\rho_n} \right\} \tag{17}$$

and

$$\eta := \frac{\lambda_2}{\lambda_1 + \lambda_2}. \tag{18}$$

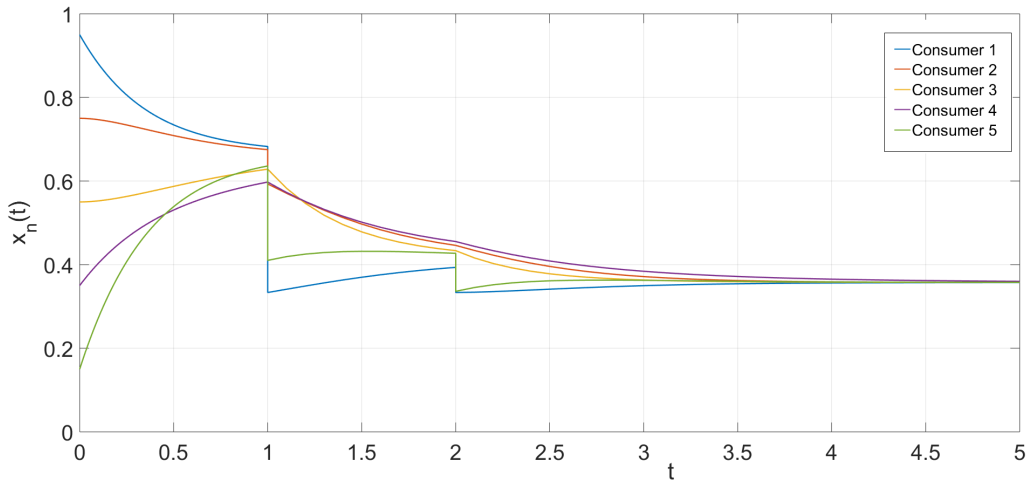
Theorem

Let $\rho_{\max} := \min_{k\in\mathcal{K}} \max_{n\in\mathcal{N}} \rho_n(k)$. Assume Marketer i, $i \in \{1,2\}$, implements the marketing strategy $\sigma_i^\star$. Assume the graph associated with the matrix $\mathbf{L}$ to be strongly connected. Then the dynamical system (3) has at least one (system) equilibrium which verifies the following:

- *If $\frac{\rho_{\max}}{\lambda_1+\lambda_2} > 1$, then $x_n = \eta$, $n \in \mathcal{N}$, is the unique equilibrium.*
- *If $\frac{\rho_{\max}}{\lambda_1+\lambda_2} \leq 1$, then any $x_n \in \overline{X}_{\max}$, $n \in \mathcal{N}$ is an equilibrium, where $\overline{X}_{\max}$ is defined by replacing ρ_n with $\rho_{\max}$ in (17).*

- Apply the OS NE strategy for K_1 stages
- For all $k > K_1$, both players agree to do $a_1(k) = a_2(k) = 0$.

Coopetition plan is sustainable if it pareto-dominates the OS NE strategy.
Trivially, if $x(t_k) = \eta \mathbf{1}_N$ at some k, applying 0 will pareto-dominate the NE.



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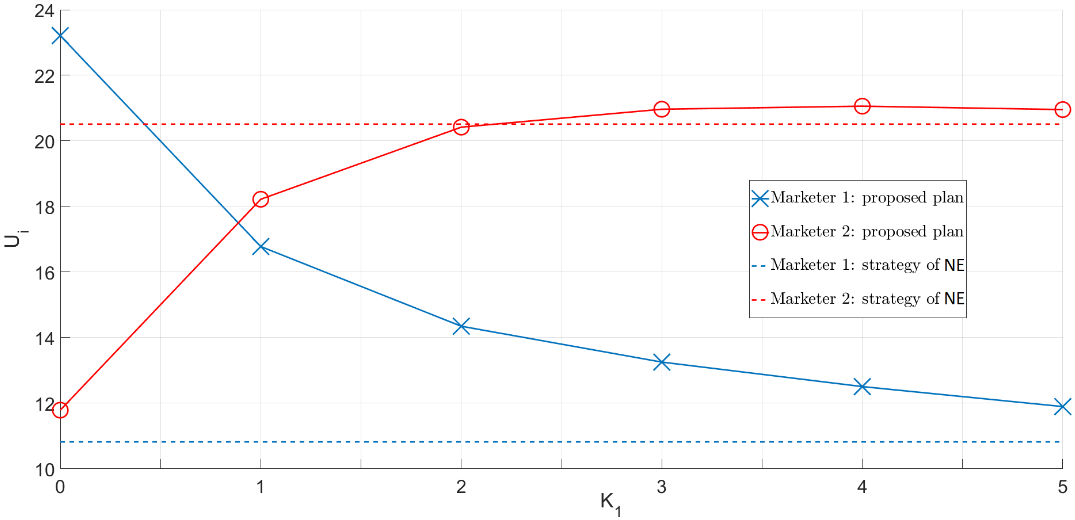

Here, we take $N = 5$, $\lambda_1 = 3$, $\lambda_2 = 1.5$ and

$$\rho = (6.8910, 1.9202, 1.30631.1036, 3.7789)^T.$$

Final remarks

Future directions

- imperfect/noisy information on ρ (or L) and $x(0)$.
- continuous control

Thanks for your attention