Bilevel Programming and MPCC : Connections and Counterexamples

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Bilevel Programming Problem

$$\min_{x} F(x, y), \quad \text{subject to} \quad x \in X, y \in S(x),$$

where for each $x \in X$ the set $S(x)$ is given as

$$S(x) = argmin_y\{f(x, y) : y \in K(x)\},$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $K(x)$ is a closed convex set in $\mathbb{R}^m$ depending on $x \in X$.
In our presentation we shall restrict ourselves to the case where $X = \mathbb{R}^n$ im most situations. The set $K(x)$ will often be given as

$$K(x) = \{y \in \mathbb{R}^m : g_i(x, y) \leq 0, i = 1, \ldots, k\},$$

here $y \mapsto g_i(x, y)$ is convex for each i.

Optimistic and Pessimistic Formulation

The optimistic formulation is given as follows : Consider that $S(x) \neq \emptyset$ for each x and define the function

$$\varphi_0(x) = \min_{y \in S(x)} F(x, y).$$

Then the *optimistic problem* is to minimize φ_0 over x. We shall refer to the optimistic problems as (BP_o).
The pessimistic formulation is given as follows : Let us define the function

$$\varphi_p(x) = \max_{y \in S(x)} F(x, y).$$

Thus the pessimistic bilevel problem consist of minimizing φ_p over $\mathbb{R}^n$. Note that the pessimistic formulation of a bilevel problem is viewed as one where the follower does not cooperate with the leader.

Optimistic Bilevel Programming

This optimistic bilevel programming problem which is denoted as (OBP) is given as

$$\min_{x,y} F(x,y), \quad \text{subject to} \quad y \in S(x).$$

Most researchers speak of this formulation as the bilevel programming. How is this problem related to the original optimistic formulation. How is (BP_o) is related to (OBP).

Relation between optimistic formulation and OBP

Result 1 :
Let $\bar{x}$ be the local solution of the optimistic formulation (BP_o). Then for any $\bar{y} \in S(\bar{x})$, the vector $(\bar{x}, \bar{y})$ is a local minimum of (OBP) if $\bar{y}$ be such that $\varphi_o(\bar{x}) = F(\bar{x}, \bar{y})$.

Result 2 :
Let $(\bar{x}, \bar{y})$ be the global minimizer of (OBP). Then $\bar{x}$ is a global minimizer of the problem (BP_o).

Result 3 : Let $(\bar{x}, \bar{y})$ be the global minimizer of (OBP). Then $\bar{x}$ is a global minimizer of the problem (BP_o).

KKT reformulation of OBP

Let the set $K(x)$ be defined by convex inequalities. The KKT reformulation of (OBP) is given below and is called (OBP-KKT)

$$\min_{x,y} F(x,y), \quad \text{subject to} \quad x \in X, \nabla L(x,y,u) = 0, \quad u \geq 0, u^T g(x,y) = 0,$$

where $L(x,y,u)$ is the Lagrangian function associated with the lower-level problem. The set of Lagrangian multipliers of the lower-level problem is given as

$$\Lambda(x,y) = \{u : \nabla L(x,y,u) = 0, \quad u \geq 0, u^T g(x,y) = 0\}$$

The set set X is often $\mathbb{R}^n$.

The Global Case

Result 4 : Let $(\bar{x}, \bar{y})$ be a global minimizer of (OBP) and the Slater CQ holds for the lower-level problem at $x = \bar{x}$. Then for any $\bar{u} \in \Lambda(\bar{x}, \bar{y})$, we have that $(\bar{x}, \bar{y}, \bar{u})$ is a solution of (OBP-KKT).

Result 5 : Let $(\bar{x}, \bar{y}, \bar{u})$ be the global minimizer of (OBP-KKT) . Let us assume that the Slater constraint qualification holds true for the lower-level problem for each $x \in X$. Then $(\bar{x}, \bar{y})$ solves (OBP).

Example : Global Case

Consider the following (OBP) in $\mathbb{R}^2$.

$$\min_{x,y} (x-1)^2 + y^2, \quad x \in \mathbb{R}, y \in S(x),$$

$$S(x) = argmin_y \{x^2 y : y^2 \leq 0\}$$

Solution of (SBP) : $(\bar{x}, \bar{y}) = (0, 0)$.
Associated MPCC problem

$$\min_{x,y,\lambda} (x-1)^2 + y^2; \quad \text{subject to} \quad x^2 + 2\lambda y = 0, \quad \lambda \geq 0, \quad y^2 \leq 0, \lambda y^2 = 0.$$

All feasible points of the MPCC is of the form $(0, 0, \lambda)$ thus by solving the MPCC we cannot solve the bilevel problem in the context of global minimizers.

Example : Local case

Consider the following (OBP) in the $\mathbb{R}^2$.

$$\min_{x,y}(x-1)^2+(y-1)^2, \quad \text{subject to} \quad x \in \mathbb{R}, y \in S(x),$$

where

$$S(x) = argmin_y\{-y : x+y \leq 1, -x+y \leq 1\}.$$

The problem (OBP) has a unique global minimizer $(\bar{x}, \bar{y}) = (0.5, 0.5)$ and there are no local minimizers.

Example : Contd

The corresponding MPCC is given as

$$
\begin{aligned}
\min_{x,y,\lambda} (x-1)^2 + (y-1)^2, \\
\text{subject to} \\
-1 + u_1 + u_2 = 0, u_1 \geq 0, u_2 \geq 0 \\
u_1(x+y-1) = 0 \\
u_2(x+y-1) = 0 \\
x + y - \leq 0 \\
-x + y - 1 \leq 0.
\end{aligned}
$$

For example $(x^*, y^*, u_1^*, u_2^*) = (0, 1, 0, 1)$ is a local solution of MPCC but $(0, 1)$ is not a local solution of (OBP).

Main Result : Local Case

Let $\bar{x}$ be a point where Slater condition holds for the lower-level problem.
Let $\bar{y}$ be a solution of the lower-level problem corresponding to $\bar{x}$.
For each $\bar{u} \in \Lambda(\bar{x}, \bar{y})$ the point $(\bar{x}, \bar{y}, \bar{u})$
is a local minimizer of (OBP).
Then $(\bar{x}, \bar{y})$ is a local minimizer of (OBP).

Simple Bilevel Programming Problem

Let us consider the following Simple Bilevel Programming (SBP) problem

$$\begin{aligned} & \text{minimize} \quad f(x) \\ & \text{subject to} \\ & x \in \text{argmin}\{h(x) : g(x) \leq 0\}. \end{aligned} \tag{1}$$

KKT conditions

Now the question is if the lower level problem i.e.

$$\begin{aligned} &\text{minimize} \;\; h(x) \\ &\text{subject to} \\ &\qquad g(x) \leq 0. \end{aligned} \tag{2}$$

can be replaced by its Karush Kuhn Tucker conditions?

The answer is yes if the Slater's CQ condition holds for the lower level problem.

Simple MPCC problem

If the lower level problem of the SBP (1) is replaced by the KKT conditions then we get the following simple MPCC problem

$$\begin{aligned} &\text{minimize} \;\; f(x) \\ &\text{subject to} \\ &\qquad \nabla h(x) + \lambda^t \nabla g(x) = 0 \\ &\qquad g(x) \leq 0 \\ &\qquad \lambda \geq 0 \\ &\qquad \lambda^t g(x) = 0. \end{aligned} \tag{3}$$

For any $x \in \mathbb{R}^n$ such that $g(x) \leq 0$, Let us define

$$\Lambda(x) := \{\lambda \geq 0 : \nabla h(x) + \lambda^t \nabla g(x) = 0, \lambda^t g(x) = 0\}$$

Then (x, λ) is a feasible point of the problem (2).

Theorem

Let $\bar{x}$ is a global optimal solution of the simple bilevel programming problem and assume that the lower level problem satisfies the Slater's CQ condition i.e. $\exists x \in \mathbb{R}^n$ such that $g(x) < 0$. Then for any $\lambda \in \Lambda(\bar{x})$, the point $(\bar{x}, \lambda)$ is a global optimal solution of the corresponding MPCC Problem.

Theorem

Let the Slater's condition holds for the lower level problem (2) of the SBP . Then $(\bar{x}, \bar{\lambda})$ *is a local solution of the MPCC problem implies that* $\bar{x}$ *is a global solution of the SBP problem.*

Corollary

Let the Slater's condition holds for the lower level problem (2) of the SBP . Then $(\bar{x}, \bar{\lambda})$ is a local solution of the corresponding MPCC problem implies that $(\bar{x}, \bar{\lambda})$ is a global solution of the same.

Example

Slater's condition holds and the solution of SBP and MPCC are same.
Let

$$
\begin{aligned}
f(x) &= (x - \frac{1}{2})^2 \\
h(x) &= \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} \\
g_1(x) &= -x \\
g_2(x) &= x - 3
\end{aligned}
$$

Then Slater's condition holds as $g_1(2) < 0$ and $g_2(2) < 0$.
Here the feasible set for the MPCC problem is

$$\{(x, \lambda_1, \lambda_2) : 0 \leq x \leq 1, \lambda_1 = 0, \lambda_2 = 0\}$$

Hence the global optimal solution for the MPCC problem is $x = \frac{1}{2}$ with optimal value $f(\frac{1}{2}) = 0$.
The feasible set of the SBP problem is

$$\text{argmin}\{h(x) : 0 \leq x \leq 3\} = \{x : 0 \leq x \leq 1\}$$

Therefore the global solution of the SBP problem is same as the MPCC i.e. $x = \frac{1}{2}$.

The SBP and MPCC problems are different in general if the Slater's condition is not satisfied. Next we present some examples to show how they are different.

Example

SBP has unique solution but corresponding MPCC is not feasible (Slater's condition is not satisfied).
Let

$$\begin{aligned} f(x_1, x_2) &= x_1 + x_2 \\ h(x_1, x_2) &= x_1 \\ g_1(x_1, x_2) &= x_1^2 - x_2 \\ g_2(x_1, x_2) &= x_2 \end{aligned}$$

Clearly, $g_1(x_1) \leq 0$ and $g_2(x_1, x_2) \leq 0$ together imply that $x_1 = 0 = x_2$. Which implies that Slater's condition fails for the lower level problem of the SBP.
Now, the feasible set for the SBP problem is

$$\text{argmin}\{h(x_1, x_2) : x_1 = 0 = x_2\} = \{(0, 0)\}$$

Therefore, $(0, 0)$ is the solution of the SBP problem.
But for $x_1 = 0 = x_2$, there does not exists $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that

$$\nabla h(x_1, x_2) + \lambda_1 \nabla g_1(x_1, x_2) + \lambda_2 \nabla g_2(x_1, x_2) = 0$$

Therefore the MPCC problem is not feasible even when the SBP has unique solution in case of the failure of Slater's condition.

Example

SBP and the corresponding MPCC both are feasible but have different solution sets (Slater's condition is not satisfied).
Let

$$f(x,y) = (x-1)^2 + y^2$$
$$h(x,y) = x^2 y$$
$$g_1(x,y) = y^2$$
$$g_2(x,y) = -x$$

Now, $g_1(x,y) \leq 0$ and $g_2(x,y) \leq 0$ together implies that

$$x \geq 0 \quad \text{and} \quad y = 0.$$

Therefore,

$$\operatorname{argmin}\{h(x,y) : x \geq 0, y = 0\} = \{(x,0) : x \geq 0\}$$

Hence, $(1,0)$ is the solution of the SBP problem with optimal value $f(1,0) = 0$.
Now for the MPCC problem and $x \geq 0, y = 0$

$$\nabla h(x,y) + \lambda_1 \nabla g_1(x,y) + \lambda_2 \nabla g_2(x,y) = 0$$

holds true if $x = 0, y = 0, \lambda_1 \geq 0, \lambda_2 = 0$.
Therefore, $x = 0, y = 0$ is the optimal solution for the MPCC problem with optimal value $f(0,0) = 1$ which is different from the SBP problem.

References

Dempe, S.; Dutta, J. Is bilevel programming a special case of a mathematical program with complementarity constraints? Math. Program. 131 (2012), no. 1-2, Ser. A, 3748.

THANK YOU