Systems & Control Engg.	Ph.D. Entrance Exam.	
IIT Bombay, Powai	4 Dec 2018	
Mumbai 400076	***Name:***	
Maharashtra, India	Time: 180 min	

Instructions:

- This question paper contains **9** questions; attempt **all** of them.
- Please write your name clearly at the head of the question paper.
- Please staple your question paper to your answer script before turning in the latter.

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1. (a) (10 points) For $x_1^2 + x_2^2 = 2$, find the smallest and largest value of

$$J(x_1, x_2) = x_1^2 - 2x_1x_2 + 4x_2^2.$$

(b) (5 points) Find the rank of the following matrix A. Also find the dimension of its range space and null space.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) (5 points) Find a square matrix $A \in \mathbb{R}^{3\times 3}$ such that $A^2(I - A) = 0$ but $A^2 \neq A$.

2. Let L be the set of all lines on a plane. Consider the following distance function for $l_1, l_2 \in L$.

$$d(l_1, l_2) = \begin{cases} \frac{s}{1+s} & \text{if } l_1 \text{ and } l_2 \text{ are parallel and } s \text{ is the distance betweeen them,} \\ \theta + 1 & \text{otherwise, where } \theta \in (0°, 90°] \text{ is the smallest angle of intersection between } l_1, l_2 \end{cases}$$

We denote the x-axis as l_x and y-axis as l_y in L.

(a) (5 points) Let $N(r) = \{l \in L \,|\, d(l_x, l) < r\}$. Draw $N(0.5)$ and $N(2)$.

(b) (15 points) Given three lines l_1, l_2 and l_3 prove that $d(l_1, l_3) \leq d(l_1, l_2) + d(l_2, l_3)$. This is known as the *triangle inequality.*

3. (a) (5 points) Consider a real-valued continuous function f defined on the interval $[-1, 1]$. Suppose that $f(\frac{1}{n}) = 5$ for all positive integers $n = 1, 2, 3 \ldots$. Find $f(0)$.

(b) Let g be a continuous real-valued function defined on the interval $[-3, 3]$ such that

$$g(0) = 1 \quad \text{and} \quad g(x) = g(x^2) \quad \text{for every } x \text{ belonging to the interval } [-3, 3].$$

i. (5 points) Prove that $g(0.5) = 1$.
ii. (5 points) Prove that $g(1) = 1$.
iii. (5 points) Prove that $g(2) = 1$.

Your proof should work for any g and not for some fixed g that you select.

4. (20 points) Consider a completely observable system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

Define the observability matrix as N:

$$N = \begin{bmatrix} C^T & A^T C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix}$$

Show that

$$N^T A (N^T)^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n$ are the coefficients of the characteristic polynomial

$$\det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n.$$

5. (a) (10 points) Show that if $a_1, \ldots, a_n, w_1, \ldots, w_n > 0$ and $\sum_{k=1}^n w_k = 1$ then,

$$\left(\sum_{k=1}^n a_k w_k \right)^2 \leq \sum_{k=1}^n a_k^2 w_k.$$

(b) (10 points) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x, y, z) = 2x^2 + y^2 + z^2/2$$

and the functions $g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}$ where

$$g_1(x, y, z) = x + y + z \qquad g_2(x, y, z) = x - y.$$

Find the minimum value of f under the condition that $g_1(x, y, z) = 10$ and $g_2(x, y, z) = 5$.

6. (20 points) Let X be a standard normal random variable, i.e., the probability density function f_X of X is given by $f_X(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}$. Let Y be a random variable independent of X and taking values in $\{-1, 1\}$ with $\mathsf{P}(Y = 1) = 1/2$. Define a new random variable $Z := XY$. What is the probability density function of Z?

7. Positive limit points play a significant role in the theory and applications of dynamical systems. Consider a dynamical system

$$\dot{x} = f(x); \quad x(t) \in \mathbb{R}^n$$

where $f(\cdot)$ is smooth and $x(t) = \Phi(t, x_0), t \geq 0$ denotes the solution (or trajectory) of the system with the initial condition $x(0) = x_0$. A point p is said to be a *positive limit point* of the trajectory $\Phi(t, x_0)$ if there exists an increasing sequence of times $\{t_k\}$ ($t_1 < t_2 < \ldots < t_k < \ldots$), with $t_k \to \infty$ (as $k \to \infty$), satisfying

$$\lim_{t_k \to \infty} \Phi(t_k, x_0) = p$$

The set of all positive limit points of a trajectory $\Phi(t, x_0)$ forms the positive limit set of *the trajectory* $\Phi(t, x_0)$.

(a) (5 points) Consider the linear oscillator

$$\ddot{z} + 2\psi\omega_0 \dot{z} + \omega_0^2 z = 0$$

where $\omega_0 > 0$ and $0 \leq \psi \leq 1$. Choose the states as $(x_1, x_2) = (z, \dot{z})$ and cast the system as $\dot{x} = f(x)$.

(b) (15 points) Obtain the positive limit set of the trajectory with initial condition $(3, 0)$ for the following two cases: (a) $\psi = 0$ (b) $\psi \neq 0$.

8. (20 points) The open loop transfer function of a feedback control system is given by

$$G(s) = \frac{K(s+1)}{s(1+Ts)(1+2s)}, \qquad K > 0.$$

Suppose the closed loop transfer function is given by $\frac{G(s)}{1+G(s)}$. Using the Routh-Hurwitz criterion, determine the region of the $K-T$ plane in which the closed-loop system is stable.

9. Let $T \in \mathbb{R}$. Consider the linear system $\dot{x} = A(t)x$, where $A(t+T) = A(t)$ for all t. Recall that the state-transition matrix $\Phi(t, t_0)$ is a matrix whose product with the state vector $x(t_0)$ at a time t_0 gives the state $x(t)$ a time t. $\Phi(.,.)$ satisfies the following properties

$$\Phi(t, t_0) = \Phi(t, \tau)\Phi(\tau, t_0)$$
$$\Phi(t, t_0)^{-1} = \Phi(t_0, t)$$

Define a constant matrix B via the equation $\exp(BT) = \Phi(T, 0)$ and let $P(t) = \exp(Bt)\Phi(0, t)$. Show that

(a) (15 points) $P(t+T) = P(t)$.

(b) (5 points) $\Phi(t, \tau) = P(t)^{-1} \exp[(t-\tau)B]P(\tau)$