Systems & Control Engg.	PhD Entrance Exam.	
IIT Bombay, Powai	Date	
Mumbai 400076	***Name:***	
Maharashtra, India	Time: 180 min	

Instructions:

- This question paper contains **10** questions; attempt **all** of them.
- Please write your name clearly at the head of the question paper.
- Please staple your question paper to your answer script before turning in the latter.

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1. (a) (7 points) Consider the second-order transfer function

$$G(s) = \frac{as^2 + bs + c}{s^2 + s + d}.$$

Find $a, b, c, d \in \mathbb{R}$ with $d > 0$ such that $G(0) = -1$, $G(j) = 0$ and $\angle G(2j) = -90°$. Here $j = \sqrt{-1}$ and $\angle G(2j)$ denotes the phase of $G(2j)$ in degrees.

(b) Consider the transfer function

$$G(s) = \frac{1}{(s+1)^3}.$$

Denote the set of all complex numbers with non-negative real part by $\mathbb{C}^+$, i.e. $\mathbb{C}^+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s \geq 0\}$.

i. (4 points) Find the smallest number $p > 0$ such that $|G(s)| \leq p$ for all $s \in \mathbb{C}^+$. Here $|G(s)|$ denotes the magnitude of $G(s)$.

ii. (4 points) Using part i (or by any other means) find a number $m > 0$ such that for every k satisfying $-m < k < m$, the transfer function

$$H(s) = \frac{1}{1 + kG(s)}$$

has no poles in $\mathbb{C}^+$.

2. A cart of mass M has two inverted pendulums on it of lengths l_1 and l_2 both with bobs of mass m. For small $|\theta_1|$ and $|\theta_2|$, the equations of motion can be seen to be

$$M\dot{v}(t) = -mg\theta_1(t) - mg\theta_2(t) + u(t)$$
$$m(\dot{v}(t) + l_i\ddot{\theta}_i(t)) = mg\theta_i(t), \ i = 1, 2$$

where v is the velocity of the cart and u is an external force applied to the cart.

(a) (6 points) Is it always possible to control both pendulums, i.e, keep them both vertical such that $\theta_1 = \theta_2 = 0$, by using the input $u(\cdot)$? Justify.

(b) (4 points) Is the system observable with output $y = \theta_1$?

3. The population of two bacterial species A and B in a pond at any particular instant of time t is denoted by $x_A(t)$ and $x_B(t)$. Their growth rates are governed by the differential equations

$$\dot{x}_A(t) = k_1 x_A(t) - k_2 x_B(t),$$
$$\dot{x}_B(t) = k_3 x_A(t) - k_4 x_B(t),$$

where each of the k_is are constants. Assume that the initial populations are $x_A(0) = \alpha > 0$ and $x_B(0) = \beta > 0$.

(a) (3 points) Assume $k_1 = 0, k_2 = 1, k_3 = 1, k_4 = 0$.

   i. Qualitatively describe how the populations x_A and x_B change with time.

   ii. Plot the evolution of x_A (x-axis) with x_B (y-axis).

(b) (2 points) Suppose $k_1 = 1, k_2 = 1, k_3 = 0, k_4 = 1$.

   i. Qualitatively describe how the populations x_A and x_B change with time.

(c) (5 points) Let us assume that we are able to "control" this system and the equations are of the form

$$\dot{x}_A(t) = x_A(t) - x_B(t) + u(t),$$
$$\dot{x}_B(t) = -x_B(t),$$

where u is the control input.

   i. In a given time T, using an appropriate control action, can you achieve any arbitrary population of x_A and x_B? Justify your answer.

   ii. Suppose we wish to bring down the populations of both the species to zero after a very long time. Suggest a "control law" (or controller) that is just a function of x_A and x_B to achieve this objective.

4. Let A, M be two $n \times n$ matrices with real entries,

   (a) (5 points) Prove that, for all $t > 0$,

$$e^{(A+M)t} = e^{At} + \int_0^t e^{A(t-\tau)} M e^{(A+M)\tau} \mathrm{d}\tau.$$

   (b) (3 points) As a special case of the preceding part, prove that if $M = \gamma I$ for some scalar constant γ and I is the $n \times n$ identity matrix, then

$$e^{(A+\gamma I)t} = e^{\gamma t} e^{At} \quad \text{for all } t > 0.$$

5. Consider a function $f : [0, +\infty[\to \mathbb{R}$, and f is continuously differentiable on $]0, +\infty[$.

   (a) (5 points) Suppose that f satisfies $\lim_{t \to +\infty} \dot{f}(t) = 0$, where $\dot{f}$ denotes the derivative of f. Does this imply that $\lim_{t \to +\infty} f(t) = c$ for some finite constant c? If yes, justify. If not, provide a counterexample.

   (b) (5 points) Suppose that $\lim_{t \to +\infty} f(t) = c$ for some finite constant c. Does this imply that $\lim_{t \to +\infty} \dot{f}(t) = 0$? If yes, justify. If not, provide a counterexample.

6. Let $n \geqslant 2$ be a positive integer. Consider a *single-input*, *single-output*, linear time invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

   evolving on $\mathbb{R}^n$, wherein the matrix A is *diagonal* with *repeated eigenvalues*.

   (a) (4 points) Prove that the pair (A, B) cannot be controllable.

   (b) (3 points) Comment on (with justifications) whether the pair (A, C) can be observable or not for this setting.

7. (a) (2 points) Find the rank of the following matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) (3 points) Find the eigenvalues of the following matrix:

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1.5 & 1 & 0 & 0 \\ 2.5 & .5 & 7 & 0 \\ 11 & 2 & -4 & 10 \end{pmatrix}$$

(c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function.

i. (2 points) Let $g(x) := f(-x)$. Is g convex?

ii. (3 points) Let $A \in \mathbb{R}^{n \times n}$ be a matrix, and let $g : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. Let $h(x) := g(f(Ax))$. Is h convex? Justify.

8. (10 points) Let $X \in \mathbb{R}^n$ be a random vector and let $C := \mathbb{E}[XX^T]$.

(a) Show that C is positive semidefinite.

(b) Let A be a positive semidefinite matrix of size $n \times n$. Show that $\text{trace}(AC) \geq 0$.

9. Suppose that $A \in \mathbb{R}^{3\times 2}$ is a matrix with $\text{rank}(A) = 2$. Let a_1 and a_2 be the columns of A, and define $L := \text{span}\{a_1, a_2\}$.

(a) (1 point) Is L a subspace of $\mathbb{R}^3$? Justify.

(b) (1 point) If v is a vector in $\mathbb{R}^3$ and $v \mapsto P(v)$ is the map that produces the orthogonal projection of v to L, show that P is a linear map.

(c) (5 points) Compute the map P in terms of A.

(d) (5 points) What is the relationship between the trace of P and the dimension of L? Justify.

10. Let $[0, 1]$ be the unit interval, and let γ_0 be the curve $\gamma_0(t) = 0$ for $t \in [0, 1]$. The *length* of γ_0 is, as expected, 1. We iteratively define a sequence of saw-tooth curves $\{\gamma_n\}_{n=1}^{+\infty}$ on $[0, 1]$ starting as follows:

- The curve γ_1 is

$$\gamma_1(t) = \begin{cases} t & \text{if } t \in [0, 1/2[, \\ 1 - t & \text{if } t \in [1/2, 1]. \end{cases}$$

The *height* of γ_1 is $\frac{1}{2}$ and its *length*, defined to be the length of the inclined segments of the graph of γ_1 on the plane, is $2 \times \sqrt{2 \times \frac{1}{2^2}} = \sqrt{2}$.

- The curve γ_2 is

$$\gamma_2(t) = \begin{cases} t & \text{if } t \in [0, \frac{1}{2^2}[, \\ \frac{1}{2} - t & \text{if } t \in [\frac{1}{2^2}, \frac{2}{2^2}[, \\ -\frac{1}{2} + t & \text{if } t \in [\frac{2}{2^2}, \frac{3}{2^2}[, \\ 1 - t & \text{if } t \in [\frac{3}{2^2}, 1]. \end{cases}$$

The *height* of γ_2 is $\frac{1}{2^2}$ and its *length* is $4 \times \sqrt{2 \times \frac{1}{2^4}} = \sqrt{2}$.

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(a) (2 points) Sketch the first few curves γ_n defined above.

(b) (2 points) At iterate n, what is the *height* and *length* of γ_n?

(c) (2 points) Argue that $\gamma_n(t) \to 0$ as $n \to +\infty$ for each t.

(d) (2 points) Show that the area enclosed by γ_n shrinks to 0 as $n \to +\infty$ but the perimeter of the region enclosed between γ_n and the t-axis stays fixed.