Systems & Control Engg.	PhD Entrance Exam.
IIT Bombay, Powai	Date
Mumbai 400076	***Name:***
Maharashtra, India	Time: 180 min

Instructions:

- This question paper contains **8** questions; attempt **all** of them.
- Please write your name clearly at the head of the question paper.
- Please staple your question paper to your answer script before turning in the latter.

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1. (20 points) Let $x_1, x_2, x_3, x_4 \in \mathbb{R}$. On the set $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, find the smallest and largest values of the function $J : \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by

$$J(x_1, x_2, x_3, x_4) \coloneqq x_1^2 - 2x_1x_2 + x_2^2 + 10x_3^2 - 10x_4^2.$$

2. A proper transfer function $G(s)$ is said to be *strictly positive real* if all its poles have a negative real part and the real part of $G(j\omega)$ is positive for all $\omega \in \mathbb{R}$.

   (a) (4 points) Is the following transfer function strictly positive real?

$$G(s) = \frac{1}{s+1}.$$

   Draw the Nyquist plot of $G(s)$.

   (b) (8 points) Give an example of a second-order transfer function $H(s)$ which is strictly positive real. (The transfer function $H(s)$ must also be proper, i.e., the degree of its denominator must be 2 and its numerator must be less than or equal to 2.) Draw the Nyquist plot of $H(s)$.

   (c) (8 points) Prove that if $G_1(s)$ and $G_2(s)$ are strictly positive real transfer functions, then all the poles of $(I + G_1(s)G_2(s))^{-1}$ have a negative real part.
   [*Hint*: Nyquist plot of $G_1(s)G_2(s)$, Nyquist stability criterion.]

3. Let F be an integer-valued function defined on the set of all integers $\{0, \pm 1, \pm 2, \ldots\}$ and having the following properties:

   - $F(0) = 1$.
   - $F(F(n)) = n$ for all integers n;
   - $F(F(n + 2) + 2) = n$ for all integers n.

   (a) (8 points) Find $F(5)$.

   (b) (2 points) Suggest a formula for $F(n)$ based on your calculations.

4. Let $U \subset \mathbb{R}^n$ be a nonempty, open, and convex set, and let $V : U \longrightarrow \mathbb{R}$ be a smooth function. Recall that the gradient, $\operatorname{grad} V$, of V is the $\mathbb{R}^n$-valued function defined by

$$\operatorname{grad} V(x) \coloneqq \begin{pmatrix} \frac{\partial V}{\partial x_1}(x) \\ \vdots \\ \frac{\partial V}{\partial x_n}(x) \end{pmatrix} \quad \text{for } x \in U.$$

   Recall also that a point $y \in U$ is said to be a *regular point* of V if $\operatorname{grad} V(y) \neq 0$, and for each $c \in \mathbb{R}$, the *c-level set* of V is defined by

$$V^{-1}(c) \coloneqq \{x \in \mathbb{R}^n \mid V(x) = c\}.$$

Consider the dynamical system, called the *gradient system*,

$$\dot{x} = -\operatorname{grad} V(x), \quad x \in U.$$

(a) (10 points) Suppose that V is strictly convex. If $p \in U$ is a minimizer of V, then show that p is an asymptotically stable equilibrium of the gradient system.

(b) (10 points) If $q \in U$ is a regular point of V, justify that the vector $-\operatorname{grad} V(q)$ is perpendicular to the $V(q)$-level set of V.

5. In the two parts below, we assume $A \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$, $c \in \mathbb{R}^{1\times 2}$.

   (a) (8 points) If (A, b) is given and this pair is *not controllable*, then is it always possible to choose a c so that (c, A) is observable? If yes, justify; if not, provide a counterexample.

   (b) If (A, b) is given and this pair is *controllable*, then

      i. (6 points) is it always possible to choose a c so that (c, A) is observable, and

      ii. (6 points) is it always possible to pick a c such that $(c, A^\top)$ is observable?

   If yes, justify; if not, provide a counterexample.

6. (20 points) A right triangle ABC on the plane has a right angle at C, with $|AC| = 1$ and $\angle BAC = \theta$. The point D is chosen on AB so that $|AC| = |AD| = 1$, and the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$.

   [*Notation*: In a triangle PQR on the plane, the length of the side PQ is denoted by $|PQ|$, and the angle between PQ and QR is denoted by $\angle PQR$.]

7. Let n be a positive integer and let the vector space $\mathbb{R}^{n\times n}$ be equipped with the norm $\|A\| := \sqrt{\operatorname{trace}(AA^\top)}$. Recall that a map $f : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ is

   - *directionally differentiable at x along $y \in \mathbb{R}^{n\times n}$* if the limit $\lim\limits_{t\to 0} \frac{1}{t}(f(x+ty) - f(x))$ exists, in which case this limit is the *directional derivative* of f at x along y;
   - *differentiable* at $x \in \mathbb{R}^{n\times n}$ if the limit $\lim\limits_{y\to 0} \frac{1}{\|y\|}(f(x+y) - f(x))$ exists, in which case this limit is the *derivative* of f at x.

   Consider the map $\Phi : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ defined by $\Phi(X) := X^\top AX + BX \in \mathbb{R}^{n\times n}$.

   (a) (10 points) Show that Φ is directionally differentiable everywhere on $\mathbb{R}^{n\times n}$ and compute its directional derivative at $Z \in \mathbb{R}^{n\times n}$ along $Y \in \mathbb{R}^{n\times n}$.

   (b) (10 points) Show that Φ is differentiable everywhere on $\mathbb{R}^{n\times n}$ and compute its derivative at $Z \in \mathbb{R}^{n\times n}$.

8. Let D be an unbiased four-sided die such that three of its four faces are coloured *Red* (R), *Green* (G), and *Blue* (B), respectively, and the fourth face is coloured in equal proportions (of the area of the face) with these three colours. Suppose that D is thrown once, and consider the events

$$A_C := \{\text{the colour } C \text{ is visible on top}\} \quad \text{where } C \text{ may be } R, G, B.$$

   (a) (5 points) Find $\mathsf{P}(A_R)$.

   (b) (5 points) Find the probabilities of the events $A_G \cap A_B$, $A_B \cap A_R$, and $A_R \cap A_G$.

   (c) (10 points) Are the events A_R, A_G, and A_B independent? Justify.