SC 631: Games and Information	**Autumn 2014**

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1.1 What is Game Theory?

When two or more rational decision makers (agents) interact, one would like to have a mathematical model for what goes on, under certain assumptions. It seems natural to consider these as "games" where both agents adopt a plan of action after analyzing the other person's options. Game theory is the study of such interactions involving strategic decision making.

1.2 Motivation

1.2.1 Prisoner's Dilemma

Consider two prisoners A and B, each confined in a solitary room, who are given a choice to either testify or maintain their silence with the following consequences-

- If A and B both testify, each of them serves 2 years in prison
- If one of them opts to remain silent but the other testifies, then the prisoner who testified will be set free and the one who opted to remain silent will serve 3 years in prison
- If both of them remain silent, then each of them will serve 1 year each in prison

	silent	testify
silent	(1,1)	(3,0)
testify	(0,3)	(2,2)

The value in each cell of the above table represents the number of years of punishment the prisoners need to serve corresponding to the choice of a particular strategy. The rows correspond to A and the columns to B. In a particular pair, the first number represents the punishment of A and the second that of B. Lesser the punishment, greater the payoff.

Let us summarize and understand the implications of the above information with an example, picking silent from the row and testify from the column i.e. A remains silent and B testifies. The concerned cell has (3,0) as the payoff- A has to serve three years and B zero. Both the prisoners are trying to minimize their punishments.

It is crucial here to realise *what* question we are trying to answer.

Question - i What will the prisoner do ?
Question - ii What should the prisoner do ?

1.2.2 Hunter's Dilemma

The game consists of two hunters who can choose to hunt either deer or rabbits with following rules:

- If A and B both choose to hunt deer, each of them get 2 deer.
- If one of them opts to hunt deer but the other choses to hunt rabbits, then the hunter who decides to hunt rabbit will get one, while the one who went for deer will get nothing!
- If both of them choose rabbits, then each of them will receive half a rabbit each

	deer	rabbit
rabbit	(2,2)	(0,1)
rabbit	(1,0)	(0.5,0.5)

The table is similar to the one in the prisoner's dilemma but with some changes. Unlike the previous case the cells of the above table represents the benefits (say the amount of meat) of the hunters corresponding to the choice of a particular animal they hunt. The rows and number correspondence remain the same for A and B. But in this case they need to maximize their benefit or payoff. What should each hunter do?

Game theory tries to analyze these situations, which we term as games. To begin, we must formally define certain concepts.

1.3 Definitions

Game A game comprises of

- A set of Players - 1,2,...N
- For each player, a set of strategies S_i for i = 1,2,...N
- A payoff/utility function π_i: $S \longrightarrow R$ where $S = \prod_{i=1}^{N} \; S_i$
- $x \in S$,that is, $x = (x_1, x_2, ...x_N)$ where $x_i \in S_i$

Furthermore, games can be classified into 2 categories:

- Cooperative Game
  - Any amount of communication allowed between players involved in the game
  - Players can have binding agreements between them
- Noncooperative Game
  - No communication between the players involved
  - No binding agreement between the players

The Prisoner's dilemma we considered earlier falls in the category of non cooperative games if no communication is allowed between the prisoners.

Game theory is, in a way, *physics for the social sciences.* It is important to note that we are only **observers** in the games we study, and must not approach them as players. In other words, it is essential to keep track of all assumptions and not let biases based on intuition and personal thought processes creep into our analysis.

We now need a principle or concept to work with that will let us define an optimum to aim for.

Solution Concept A solution concept of a game from a family of games is a function that maps each game to a set of profile of strategies in that game

- A solution concept should be consistent with the assumptions of the game (discussed in further lectures).
- It must be consistent with itself.
- (Theoretician's bias) It should be regardable as an "outcome".

1.4 Nash Equilibrium

The Nash equilibrium represents the best a player can do given the set of other players' strategies in a non co-operative game. Formally, A profile of strategies $x = (x_1, x_2, ..., x_N)$ where $x_i \in S_i$ is said to be a Nash Equilibrium of game if,

$$\Pi_i(x) \geq \Pi_i(\bar{x}_i, x^{-i}) \qquad \forall \bar{x}_i \in S_i, \forall i = 1, 2, ..., N$$

where $\bar{x}_i, x^{-i}$ denotes $(x_1, x_2, \cdot, \bar{x}_i, x_{i+1} \cdot, x_N)$ which is a profile of strategies (i.e.) if only player i shifts from x_i to $\bar{x}_i$, then his payoff reduces. The player i judges unilaterally(ie, without any communication or knowledge about strategies picked by other players), picks the Nash equilibrium and has no incentive to change as the payoff is the highest as per his unilateral judgement. In order to find the Nash equilibrium, we can iterate through all the possibilities, if they are finite in number.
It is essential to consider unilateral deviations in non co-operative games. This is because in the absence of communication and binding agreements, it is reasonable to assume that one is likely to profit by deviating from one's current strategy only if the payoff increases when other players do not change their strategy.
The Nash equilibrium is *stable* by definition, meaning that a particular player cannot profitably deviate from his current strategy if all other players hold their strategy constant. This makes it an equilibrium state in a non co-operative game.

Note:

- There could be multiple Nash equilibria.
- Also, it is possible that the game has no Nash equlibrium.
- We do not care how we arrived there. We cannot yet derive this for general games.
- This is an solution concept *only* for non cooperative games

1.4.1 Examples

Let us try to find the Nash equilibrium in the previously discussed example of Prisoner's Dilemma.

	silent	testify
silent	(1,1)	(3,0)
testify	(0,3)	(2,2)

If you carefully observe, in this case the Nash equilibrium is the (testify,testify) state. Remember that here we are trying to minimize the pay off hence instead of the maximum point as we found in the definition we try to find the minimum one here.So,greater the punishment lesser the payoff. As the number of possibilities is finite, one can just parse through all the possibilities and verify the equilibrium condition.
Explanation:
As the number of possibilities of combination of strategies is finite (i.e) total of four in this case, one can

just parse through all the possibilities and verify the equilibrium condition. There are four possible set of actions as follows:
(silent,silent) : The prisoner A could always shift to testify to increase/decrease the payoff/punishment to 0 years while prisoner B is still silent. Hence this is surely not a Nash equilibrium.
(silent,testify) or (testify,silent): At (silent,testify) the prisoner A can increment/decrement his payoff/punishment from 3 to 2 years by switching to testify, while prisoner B still testifies implying they are not Nash equilibrium. Similarly for (testify,silent) prisoner B can shift to testify and decrement his punishment, thus again defying Nash Equilibrium.
(testify,testify): Prisoners A and B can shift to begin silent but that would only decrease/increase their payoffs/punishments to 3 years. For Example, if prisoner A shifts to silent , then his payoff increases to 3 years. Similar increment happens for B.Since all the conditions of of Nash equilibrium are satisfied.

Similarly let us look at the Nash equilibium for the Hunter's Dilemma.

	deer	rabbit
deer	(2,2)	(0,1)
rabbit	(1,0)	(0.5,0.5)

Explanation: There are two Nash equilibria in this case which are, (deer, deer) and (rabbit, rabbit). This is also an example of finite possibilities. Here, we try to maximize the payoff.The case of (deer,rabbit) is not an equilibrium because hunter A can switch from deer to the rabbit to improve his payoff from 0 to 0.5 while the hunter 2 is still at rabbit, hence defying the rules of Nash equilibrium. Similarly one could have explained why (rabbit,deer) won't be a Nash equilibrium.(deer, deer) and rabbit, rabbit) satisfy all the properties and hence form the Nash equilibria.

1.5 Box Pulling Problem

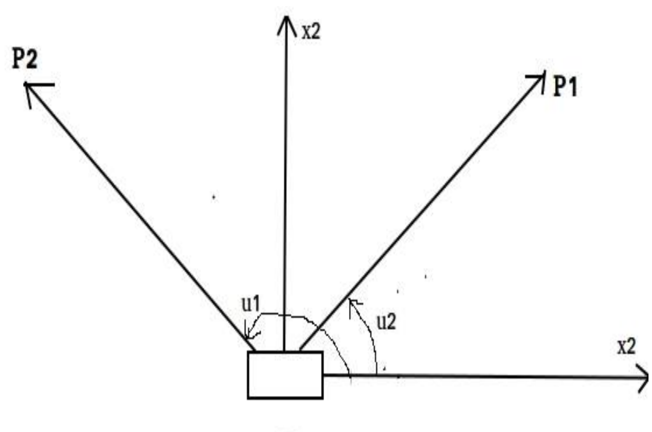

Figure 1.1: Figure

Problem Description There are two players who are exerting unit force each, on a box kept on a two dimensional plane. Let the coordinate axes be x_1 and x_2, as shown in the figure. Person 1 exerts a force P_1 at an angle u_1 from x_1. Person 2 exerts a force P_2 at an angle u_2 from x_2. Let the coordinates of the box at time t be denoted by (x_1(t), x_2(t)).

We are given the following: $x_1(0) = 0$; $x_2(0) = 0$ $x'_1(0) = 0$; $x'_2(0) = 0$
The acceleration at t=0: $x''_1(0) = \cos u_1 + \cos u_2$ $x''_2(0) = \sin u_1 + \sin u_2$
Assuming that the acceleration is constant for a small time interval of 1 second, coordinates at t=1 are: $x_1(1) = \frac{1}{2}(\cos u_1 + \cos u_2)^2$ $x_2(1) = \frac{1}{2}(\sin u_1 + \sin u_2)^2$

Aim of the game: Person 1 wants to minimise $x_1(1)$; Person 2 wants to minimise $x_2(1)$.

Observation

- Nash Equilibrium in this example is achieved when $u_1 = \pi$ and $u_2 = \frac{-\pi}{2}$
- The strategy for Nash Equilibrium is not the most optimum strategy i.e. it does not give the highest payoff. This is achieved when $u_1 = \frac{5\pi}{4}$ and $u_2 = \frac{5\pi}{4}$
- Hence it is not necessary that the most optimum strategy will be a Nash Equilibrium, if it exists.

We can draw an analogy between laws of motion and laws of game theory. Just as the laws of motion act as axioms to let us calculate the behaviour of entire systems of bodies, a similar role is played by the laws of game theory with games, *under the condition that the assumptions hold.*

1.6 Fields of Game Theory

- **Descriptive Game Theory:** It is an empirical science, based on what actually happened in the past. It mainly uses experimental data to predict future results, and also involves humanities and psychology.
- **Normative Game Theory:** This speaks more about what players *should* do, assuming what we already know. It could perhaps be useful in consultancy.
- **Theoretical Game Theory:** It deals with things which we can say logically about the game. We will be studying *Theoretical game theory* in this course.