SC 631: Games and Information **Autumn 2014**

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10.1 Non Zero Sum Games

In this chapter, we are going to learn about games where the payoff of all the players need not sum to zero for all strategies. More specifically,

- There are only **two** players.
- Both players have **finite** number of **pure** strategies.
- All the players try to **minimize** according to their payoff matrices.

Convention: *In the case of Non-Zero Sum (NZS) matrix games all players* ***minimize*** *their pay-off.*

10.1.1 2 Player NZS Matrix Game

A 2 player NZS matrix games, the objective function of each player is represented using a pair of matrices, $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{m \times n}$ for the Row and Column player respectively. Here, row player has m no. of pure strategies and column player has n no. of pure strategies.

P1: Row player choose to minimise it's payoff with respect to the matrix A.

P2: Column player choose to minimise it's payoff with respect to the matrix B.

10.1.2 Nash Equilibrium (NE) in NZS Matrix Games (for pure strategies)

A pair of strategies i^* and j^* (for the Row and Column players respectively) is a Nash Equilibrium of the given game if

$$a_{i^*j^*} \le a_{ij^*} \qquad \forall i$$
$$b_{i^*j^*} \le b_{i^*j} \qquad \forall j$$

10.1.3 Example 1

Consider the following game,

$$A = \begin{array}{c|c|c|} & z_1 & z_2 \\ \hline y_1 & \boxed{1} & 0 \\ \hline y_2 & 2 & \boxed{-1} \\ \hline \end{array} \qquad B = \begin{array}{c|c|c|} & z_1 & z_2 \\ \hline y_1 & \boxed{2} & 3 \\ \hline y_2 & 1 & \boxed{0} \\ \hline \end{array}$$

The strategy pairs (y_1, z_1) and (y_2, z_2) are the Nash Equilibria of this game. The above example shows that there exists 2-Nash equilibrium in this NZS matrix game, and pay-off is **different** for the two players unlike in Zero Sum games.

Unlike Zero Sum games it can be seen that in NZS games there can exist strategies which are *win-win* for both the players. A player may possibly gain by letting the other player to win and not necessary that their interests are at loggerheads. In this example, both players would *prefer* the (y_2, z_2) NE over the other NE since both them will be *better* off.

10.1.4 Battle of the Sexes (BoS)

$$A\text{ (Husband)}= \begin{array}{c|c|c|} & \text{cricket} & \text{movies} \\ \hline \text{cricket} & \boxed{-2} & 0 \\ \hline \text{movies} & 0 & \boxed{-1} \\ \hline \end{array} \qquad B\text{ (Wife)}= \begin{array}{c|c|c|} & \text{cricket} & \text{movies} \\ \hline \text{cricket} & \boxed{-1} & 0 \\ \hline \text{movies} & 0 & \boxed{-2} \\ \hline \end{array}$$

The strategy pairs $(cricket, cricket)$ and $(movies, movies)$ are the Nash Equilibria of this game. Each player might *prefer* a different NE than the other in this particular example compared to the previous example.

10.1.5 Other Examples

- Manufacturers agreeing to adopt common standards . Manufacturing a product to one's own specification may save some cost. But having products that interoperate with the other's product may boost total sales. e.g. A CD manufacturer and a CD drive manufacturer deciding to adopt a common standard so that CD and the drive are compatible. In this example there is no conflict of interest since the products are complementary and the situation becomes win-win.

- A cooperative game *with* conflict is *bargaining*. The seller wants to sell at a higher price while the buyer prefers a lower price. But both of them cooperate with each other to reach at a final decision which may be social welfare maximization.

- A cooperative game *without* conflict is a group of workers forming a union.

10.1.6 Multiple Nash Equilibria

As it can be seen in the examples, there can be multiple *Nash Equilibria* for the same game each with a different "flavour" of *cooperation/non-cooperation*. Selection of a more likely outcome from this set of multiple *Nash Equilibria* is a topic still under research. However we will see how some strategies can be more likely as outcome than others.

10.1.7 Definition : Better Strategy

A pair of strategies (i_1, j_1) is said to be **better** than another strategy pair (i_2, j_2) if

$a_{i_1 j_1} \leq a_{i_2 j_2}$ & $b_{i_1 j_1} \leq b_{i_2 j_2}$ & at least one, otherwise strict.

10.1.8 Definition : Admissible Nash Equilibrium

A Nash Equilibrium is said to be **admissible** if there is **no better** NE.

10.1.8.1 Examples

- In the first example given above the NE strategy (y_2, z_2) is *better* than the other NE strategy and hence the only admissible NE for that game.
- In the second example given above (BoS), both NE are admissible as neither of them is *better* than the other.

10.1.8.2 Aumann theorem on Admissibility

In case of NZS mixed game, if the rational players discusses among each other, and reaches a pre-play agreement to play and admissible NE which gives better payoff to all players, it still does not mean that the NE strategy will be played.

10.1.9 MinMax Strategy for NZS Games

If each player assumes that the other player tries to maximise the damage to the first player, then each player can opt to play a *security strategy* i^* or j^* such that

Row player, i^* $\quad \max_j a_{i^* j} \leq \max_j a_{ij} \qquad \forall i$

Column player, j^* $\quad \max_i b_{ij^*} \leq \max_i b_{ij} \qquad \forall j$

10.1.9.1 MinMax Strategies and NE for NZS Games

- In the first example, the player1 does not choose strategy y_2 even though it contains the minimum value since the same strategy also contains 2 as a value in the first column. This is because of the assumption that the player2 is trying to do damage to player1.
- In the game (similar to Prisoner's dilemma) $A = \begin{pmatrix} \boxed{8} & 0 \\ 30 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} \boxed{8} & 30 \\ 0 & 2 \end{pmatrix}$, neither player chooses their second strategy (which is better for both) under the assumption that the other player may play the first strategy.

It can be seen from above examples that, in general, there is no relationship between the MinMax strategy and the NEs of a NZS game.

10.1.10 Mixed Strategies in NZS Games

In mixed strategies, as already introduced in the section on Zero Sum Games, the player chooses to play one of the available strategies randomly according to some probability distribution so as to minimise their pay-off **Expectation**.

The Row player has a set of m *pure* strategies and an associated probability distribution $\mathbf{y} \in \mathbb{R}^m$ on the strategy set such that $\mathbf{1}^T\mathbf{y} = 1$ and $\mathbf{y} \geq 0$.

The Column player has a set of n *pure* strategies and an associated probability distribution $\mathbf{z} \in \mathbb{R}^n$ on the strategy set such that $\mathbf{1}^T\mathbf{z} = 1$ and $\mathbf{z} \geq 0$.

10.1.10.1 NE in Mixed Strategies

The Nash Equilibrium strategy $\mathbf{y}^*$ and $\mathbf{z}^*$ such that

$$\begin{aligned} {\mathbf{y}^*}^T A\mathbf{z}^* &\leq \mathbf{y}^T A\mathbf{z}^* && \forall \mathbf{y} \in Y \\ {\mathbf{y}^*}^T B\mathbf{z}^* &\leq {\mathbf{y}^*}^T A\mathbf{z} && \forall \mathbf{z} \in Z \end{aligned}$$

10.1.11 Finding Nash Equilibrium:

Consider the 2 player, 2 strategy, NZS matrix game given below

$$A = \begin{array}{c|c|c|} & z_1 & z_2 \\ \hline y_1 & 1 & 0 \\ \hline y_2 & 2 & -1 \\ \hline \end{array} \qquad B = \begin{array}{c|c|c|} & z_1 & z_2 \\ \hline y_1 & 2 & 3 \\ \hline y_2 & 1 & 0 \\ \hline \end{array}$$

Since there are only two strategies for each player, the second strategy y_2 and z_2 can be eliminated using the constraint that sum of all probabilities is one. i.e., $y_2 = 1 - y_1$ and $z_2 = 1 - z_1$.

$$\begin{aligned} y_1^* + 3z_1^* - 2y_1^*z_1^* - 1 &\leq y_1 + 3z_1^* - 2y_1z_1^* - 1 && \forall y_1 \in [0,1] \\ y_1^* - z_1^* + 2y_1^*z_1^* + 1 &\leq y_1 - z_1 + 2y_1^*z_1 + 1 && \forall z_1 \in [0,1] \end{aligned}$$

The NE strategies are found by minimising the RHS of both equations. To minimise the equations, the points to be checked are the boundary points and the values where the first derivative becomes zero. All the players chooses its strategies so that, its pay-off is independent of other players strategy. And like, ZS mixed strategy matrix games, in NZS games there exists atleast one saddle point.

For the first player, $\frac{\partial(y_1 - 2y_1z_1^*)}{\partial y_1} = 0 \Rightarrow z_1^* = \frac{1}{2}$.

For the second player, $\frac{\partial(-z_1 + 2y_1^*z_1)}{\partial z_1} = 0 \Rightarrow y_1^* = \frac{1}{2}$.

We can see that the minima with respect to a player is decided by the equilibrium strategy of the **other** player. The contour plot of the payoff function in plotted in Figure 10.1. The NE found by the above is not the only NE since some of the *pure* strategies (corresponding to boundary points) may also form a Nash Equlibrium.

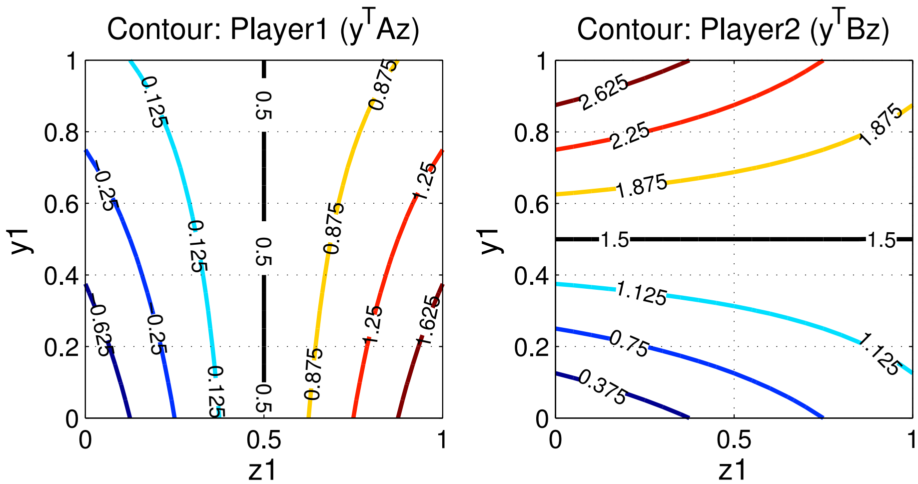

Figure 10.1: Contour of Payoffs plotted against y1 and z1

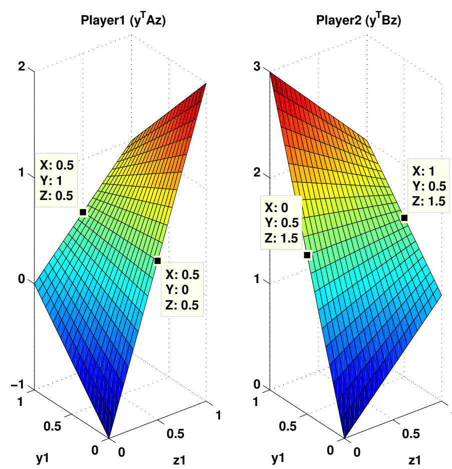

Figure 10.2: Plot of Payoffs

Theorem 10.1 *Let, $\mathring{Y}$ and $\mathring{Z}$ denote the **Interiors** of Y and Z.*

$\mathring{Y} = \{\mathbf{y}|\mathbf{y} > 0, \mathbf{1}^T\mathbf{y} = 1\}$ *and* $\mathring{Z} = \{\mathbf{z}|\mathbf{z} > 0, \mathbf{1}^T\mathbf{z} = 1\}$

Then, if (A, B) admits a mixed strategy NE in $\mathring{Y}$ and $\mathring{Z}$, then this mixed strategy NE is also a mixed strategy NE for $(-A, -B)$.

10.2 Next topic