SC 631: Games and Information **September 2014**

Lecture 11: September 5

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This lecture's notes follows Nash's proof for existence of a nash equillibrium for N person game with finite strategies

11.1 Mixed strategies for N player games with finite strategies

1. let $\aleph = \{1, 2, ..., N\}$ set of players
2. M_i is the finite set of pure strategies for each $i \in \aleph$
3. $a^i_{x_1,x_2,...,x_N}$ = payoff for player i, denoted as P_i, when the strategies chosen by $P_1, P_2, \cdots, P_n$ are $x_1, x_2, \cdots, x_N$ respectively.
4. Y_i : set of probability distributions on M_i for P_i.

We first consider the following:

$Y_i \subseteq \Re^{m_i}$, where $m_i = |M_i|$ = number of pure strategies available to P_i.

$y^i \in Y_i \subseteq \Re^{m_i}$ is a vector such that

$$1^T y^i = 1$$
$$y^i \geq 0$$

where $y^i = (y^i_1, y^i_2, ..., y^i_{m_i})$ and $1 \in \Re^{m_i}$ is a vector where each component is 1.

The function that player i minimises is his expected payoff which is defined as:

$$J^i(y^1, y^2, ..., y^N) = \sum_{x_1 \in M_1, x_2 \in M_2, ..., x_N \in M_N} a^i_{x_1,...,x_N} y^1_{x_1} ... y^N_{x_N} \tag{11.1}$$

Recalling the definition of Nash Equilibrium from earlier, we say that $(y^{1*}, y^{2*}, \cdots, y^{N*}) \in Y_1 \times Y_2 \times \cdots Y_N$ is the Nash Equilibrium (as per our set-up as above), if,

$$J^i(y^{1*}, y^{2*}, \cdots, y^{i*}, \cdots, y^{N*}) = J^i(y^{i*}, y^{-i*}) \leq J^i(y^i, y^{-i*}) \ \forall \ y^i \in Y_i \ \forall \ i \in \aleph \tag{11.2}$$

11.2 Existence of Nash Equilibrium

In 1950, John F. Nash, proved that for any $N-$person game with finite strategies, there exists at least one Nash Equilibrium in mixed strategies. In what follows, we will prove this famous existence result, but we will not be using the principles which Nash used to prove the result in his paper.

We now delve right into the proof. In order to understand the the proof, it is imminent that we understand the meanings and significance of some crucial terminology and concepts. They are discussed in the following sub-section:

11.2.1 Preliminaries

1. **Best Response:**

   The best response for the i^{th} player, given that other players play the strategy profile y^{-i} is defined as

   $$R_i(y^{-i}) = \{y^i \in Y_i \mid J^i(y^i, y^{-i}) \leq J^i(y^{i'}, y^{-i}) \ \forall \ y^{i'} \in Y_i\} = argmin[J^i(y^i, y^{-i})], y^i \in Y_i \tag{11.3}$$

   2^{Y_i} is used to denote the power set of Y_i and $Y^{-i} = \prod_{i \neq j} Y_j$. Therefore, as is clear from the above definition, R_i is a set valued map. $R_i : Y^{-i} \to 2^{Y_i}$. Using the best response function for each of the i^{th} player, we can naturally define another set valued map $R : Y \to 2^Y, Y = \prod_{i \in N} Y_i$, such that,

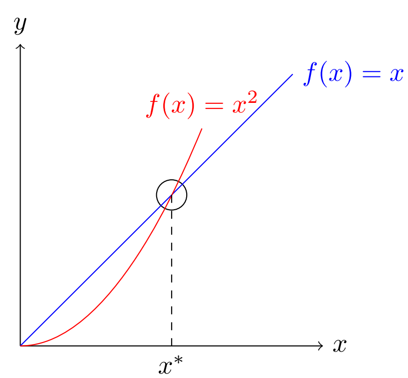
   $$R(y) = R_1(y^{-1}) \times R_2(y^{-2}) \ldots \times R_N(y^{-N}) \tag{11.4}$$

   Now, if y^* is a Nash Equilibrium, then $y^{*1} \in R_1(y^{-1*}), y^{*2} \in R_2(y^{-2*}) \ldots, y^{*N} \in R_N(y^{-N*})$, and hence we can say that $y^* \in R(y^*)$. The converse is also true, by definition. Hence we obtain the following lemma:

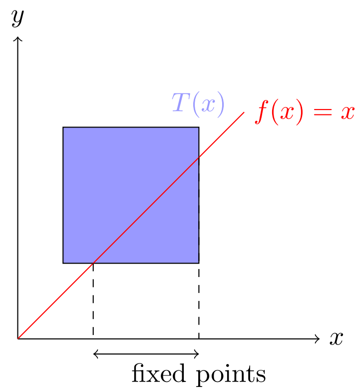
   Lemma 11.1 *y^* is a Nash Equilibrium $\Leftrightarrow y^* \in R(y^*)$*

2. **Kakutani's Fixed Point Theorem:**

   For a function $f : \mathbb{R} \to \mathbb{R}$, x^* is a fixed point if $x^* = f(x^*)$



   Similarly, for a set valued function $T : X \to 2^X$, x^* is a fixed point if $x^* \in T(x^*)$. To illustrate this, consider the set valued function $T(x)$ in the following figure. The graph of T is $\{(x, y) \mid y \in T(x), x \in \mathbb{R}\}$. In this case, the graph is a rectangle. The fixed points for this function T are all those $x \in \mathbb{R}$ for which $y = x$.



When does a set-valued map as described, admits a fixed point? This question has been considered extensively, but the one that suits our purpose of proving the existence of the Nash Equilibrium is due to Kakutani.

Theorem 11.2 ***(Kakutani's Fixed Point Theorem)*** *Let $S \subseteq \mathbb{R}^n$ be convex and compact, and $T : S \to 2^S$ be a set valued map which*

(a) is convex (and non-empty) valued

(b) has a closed graph

Then, T admits a fixed point.

11.2.2 Proving the Existence of Nash Equilibrium

We apply Kakutani's Fixed Point Theorem to the set valued map $R : Y \to 2^Y$, which has been defined in Equation (11.4). First, we need to check whether R suffices all of the conditions required by Kakutani's Fixed Point Theorem.

1. We need to prove that the domain of R i.e. Y is compact. Since $Y = Y_1 \times Y_2 \times \cdots \times Y_N$ and each of Y_i is compact (as per previous lecture), we therefore have, Y **is compact.**

2. Now, we have to show that the domain of R i.e. Y is convex. We first show that for every $i \in \aleph$, Y_i is convex.

   Let $y^i, z^i \in Y_i$. Therefore, from the definition of the vectors y^i and z^i, we have, $\mathbf{1}^T.y_i = 1 = \mathbf{1}^T.z^i$ and $y^i, z^i \geq 0$. Now, for any $\alpha \in [0, 1]$,

$$\begin{aligned} &\mathbf{1}^T(\alpha y^i + (1-\alpha)z^i) = \alpha + 1 - \alpha = 1 \\ &\implies (\alpha y^i + (1-\alpha)z^i) \in Y_i \\ &\implies Y_i \; \textit{is convex} \; \forall \; i \end{aligned}$$

   Now, we use the fact that if two sets A and B are convex, then $A \times B$ is also convex. Since each of Y_i is convex (as we have just shown), $Y = \prod_{i \in \aleph} Y_i$ **is also convex**.

3. We need to prove now, that the set valued map R is convex valued. We start off by revisiting its definition:

$$R(y) = R_1(y^{-1}) \times \cdots \times R_N(y^{-N})$$

For some fixed y^{-i}, $R_i(y^{-i}) = \{y^i \in Y_i | J^i(y^i, y^{-i}) \leq J^i(y^i, y^{-i}) \quad \forall y^i \in Y_i\}$. Let $y^i \in R_i(y^{-i})$, $z^i \in R_i(y^{-i})$ and $\alpha = [0,1]$.

Here, if we say the y_i is in the set then it means that,

$$J^i(y^i, y^{-i}) \leq J^i(y^{i'}, y^{-i}) \quad \forall y^{i'} \in Y_i \tag{11.5}$$

Similarly we can say for $z^i \in Y_i$ that,

$$J^i(z^i, y^{-i}) \leq J^i(y^{i'}, y^{-i}) \quad \forall y^{i'} \in Y_i \tag{11.6}$$

Furthermore, we know that

$$J^i(y) = \sum a^i_{x_1, x_2, \ldots\ldots x_N} \; y^1_{x_1} \ldots\ldots y^N_{x_N}$$

Therefore, we have shown that $J^i(y^i)$ is linear for any fixed y^{-i}. Using linearity, we can write that

$$\alpha J^i(y^i, y^{-i}) + (1-\alpha) J^i(z^i; y^{-i}) = J^i(\alpha y^i + (1-\alpha) z^i; y^i)$$

Using equations-(11.5) and 11.6), we have, for each $y^i \in Y_i$,

$$J^i(\alpha y^i + (1-\alpha) z^i; y^{-1}) \leq J^i(y^i, y^{-i})$$

Since Y_i is convex, we also have, $\alpha y^i + (1-\alpha) z^i \in R_i(y^{-i})$. Therefore, $R_i(y^{-i})$ is convex valued $\Longrightarrow$ **R is convex valued**, since it is the Cartesian product of R_i's.

4. Finally, we need to prove that the R has a closed graph. If we denote the graph of R by $GraphR$ then,

$$GraphR = \{(y, z) | y \in R(y), y \in Y\}$$

and we need to prove that $GraphR$ is closed. Let, $\{y^{(k)}, z^{(k)}\}_{k \in N}) \in GraphR$ be a sequence of points such that $(\bar{y}, \bar{z}) = \lim_{k \to \infty} (y^{(k)}, z^{(k)})$. Therefore, we now need to show that $(\bar{y}, \bar{z}) \in GraphR$. We have,

$$\begin{aligned} & z^{(k)} \in R(y^{(k)}) \ \forall k = 1, 2, \ldots \\ \Longrightarrow\ & z^{(k)} \in Y (= Y_1 \times Y_2 \times \ldots . \times Y_N) \\ \Longrightarrow\ & z^{i(k)} \in R_i(y^{-i(k)}) \end{aligned}$$

Furthermore, $\forall k$ *and* $\forall i \in N$, we also have,

$$J^i(z^{i(k);y^{-i(k)}} \leq J^i(Y^i(y^{i'}, y^{-i(k)}) \ \forall \ y^i \in Y_i \ and \ z^{i(k)} \in Y_i$$

Now if we fix $i, y^{i'} \in Y_i$, then,

$$\lim_{k \to \infty} J^i(z^{i(k)}, y^{-i(k)}) \leq \lim_{k \to \infty} J^i(y^i, y^{-i(k)})$$

Since J^i is continuous polynomial (of degree $\sum_{i=1}^N m_i$) we can take the limit inside and thus have the following implication:

$$\begin{aligned} & J^i(\lim_{k \to \infty} (z^{i(k)}, y^{-i(k)})) \leq \lim_{k \to \infty} J^i(y^i, y^{-i(k)}) \\ \Longrightarrow\ & J^i(\bar{z}^i, \bar{y}^{-i}) \leq J^i(y^i, y^{-i(k)}) \ \forall \ i \in N, \ \forall \ y^i \in Y_i \\ \Longrightarrow\ & \bar{z} \in R(y) \ \& \ (\bar{y}, \bar{z}) \in GraphR \\ \Longrightarrow\ & \bar{z}^i \in Y_i \\ \Longrightarrow\ & Y_i \ is \ compact \\ \Longrightarrow\ & GraphR \ is \ closed \end{aligned}$$

Since we have showed that all the four assumptions of Kakutani's fixed point theorem is satisfied by the set valued map R, using Theorem (11.2), we can say that R must have a fixed point. Let us denote this fixed point by y^*. Therefore, we have, $y^* \in R(y^*)$. But, that is exactly the definition of Nash Equilibrium. Hence, we have proved that, for an $N-$person non-cooperative game there always exists a Nash Equilibrium.

11.3 Some Comments

Let us suppose that we have $y^* \in Y$, such that $y^* \in R(y^*)$. Let us say that, $y_1^* \in Y_{-1*} = \{a, b\}$. Then, we ask the following question: if (a, y_1^*) is Nash Equilibrium, then is (b, y_1^*) also a Nash Equilibrium?

Answer: No! We have to check whether on account of a change from a to b by Player 1, the y_1^* values are still optimal for all the other players. It is not easy to compute Nash Equilibrium in the case of mixed strategies as the computational complexity of this problem is of PPAD (Polynomial Parity Arguments on Directed graphs) complexity.

Next, we look at $\widehat{Y}_i = \{y \ | \ 1^T y = 1 \ ; \ y > 0\} \ = \ $ Interior of Y_i. We state an interesting observation in the following theorem, regarding $\widehat{Y}$.

Theorem 11.3 *Consider a N person joint strategy sum game. A point $y^* \in Y$ is NE iff*

$$\sum_{x_2 \in M_2, x_3 \in M_3 \ldots x_N \in M_N} y_{x_2}^{2*} \ldots y_{x_N}^{N*} (a^1_{x_1,\ldots,x_N} - a^1_{1,x_2,\ldots,x_N}) = 0 \quad \forall\ x_1 \in M_1 \tag{11.7}$$

$$\sum_{x_2 \in M_2, x_3 \in M_3 \ldots x_N \in M_N} y_{x_1}^{1*} y_{x_3}^{3*} \ldots y_{x_N}^{N*} (a^2_{x_1,\ldots,x_N} - a^2_{x_1,1,x_3\ldots,x_N}) = 0 \quad \forall\ x_2 \in M_2 \tag{11.8}$$

$$\sum_{x_2 \in M_2, x_3 \in M_3 \ldots x_N \in M_N} y_{x_1}^{1*} y_{x_2}^{2*} \ldots y_{x_{N-1}}^{N-1*} (a^N_{x_1,\ldots,x_N} - a^2_{x_1,x_2,\ldots,x_{N-1},1}) = 0 \quad \forall\ x_N \in M_N \tag{11.9}$$

Hence, $J(x_i; y^{-1*}) = J(i, y^{-1*})$ $\forall\ x_i$ i.e. every player is indifferent among his strategies, when we have $y^* \in \widehat{Y}$.

11.4 An Example

Suppose we have, $N = 3$. Instead of a 3-dimensional matrix, we draw two matrices.

	L	R
L	$(1, -1, 0)$	$(0, 1, 0)$
R	$(2, 0, 0)$	$(0, 0, 1)$

$x_3 = \mathrm{L}$

	L	R
L	$(1, 0, 1)$	$(0, 0, 0)$
R	$(0, 3, 0)$	$(-1, 2, 0)$

$x_3 = \mathrm{R}$

$y^1 = (y_L^1, y_R^1) \quad ; \quad y^2 = (y_L^2, y_R^2) \quad ; \quad y^3 = (y_L^3, y_R^3)$

Using Theorem (11.3), we therefore obtain 3 equations which are:

$$1 - y_R^{2*} - 2y_R^{3*} + y_R^{2*} y_R^{3*} = 0$$
$$2 - y_R^{1*} - 2y_R^{3*} + y_R^{1*} y_R^{3*} = 0$$
$$y_R^{1*} + y_R^{2*} - 1 = 0$$

such that $0 < y_R^{1*}, y_R^{2*}, y_R^{3*} < 1$. On solving, they admit the solution: $y_R^{1*} = \sqrt{3} - 1$, $y_R^{2*} = 2 - \sqrt{3}$ and $y_R^{3*} = 1 - \frac{1}{3}\sqrt{3}$