**SC 631: Games and Information** **Autumn 2014**

# Lecture 12: September 16

*Instructor: Ankur A. Kulkarni* *Scribes: Siddhartha, Shivam, Aditya, Siddhant*

**Note**: *LaTeX template courtesy of UC Berkeley EECS dept.*



## 12.1 Introduction

In the last lecture, we showed that there always exists a Nash Equilibrium in a finite mixed strategy N-player game using Katukani's fixed point theorem. However, we have seen a number of pure strategy game examples where the Nash Equilibrium doesn't exist. If players had infinitely many strategies(with some additional conditions), such a problem wouldn't arise.

In this lecture, we shall dwell upon the existence of Nash Equilibrium in continuous kernel games.

## 12.2 Continuous Kernel Games

**Definition 1** *Games in which each player has a continuum of pure strategies are known as Continuous Kernel Games.*

Examples:

1. The Box Pulling problem we saw in Lecture 1.
2. Setting the price of a service/good.
3. Player's trying to decide on when to meet.

**Definition 2** *Consider an N-player Game. We define:*

- $U^i$ *= Set of pure strategies of* $P_i$
- $J^i(u^{-i}, u^i)$ *= Payoff of* $P_i$ *when* $P_i$ *plays* $u^i$ *and other players play* $u^{-i}$
- *Best Response* $R_i(u^{-i}) = \underset{u^i \in U^i}{\text{ArgMin}}\ J^i(u^i, u^{-i})$
- $R(u) = \prod_{i=1}^{N} R_i(u^{-i})$

## 12.3 Graphical Analysis of 2-Player Games

**Definition 3** *A contour line of a function of two variables is a curve along which the function has a constant value. A contour map is a collection of such curves where each curve corresponds to a particular value of the function. These curves need not be closed.*

In the following figure, Player 1 can chose a strategy from $U^1$ and Player 2 can choose his strategy from $U^2$. The pair of strategies lying on the same contour line have equal payoffs. There are two contour maps in the figure $J^1$ and $J^2$ showing the payoffs of Player 1 and Player 2 respectively.

Suppose, we were to find the best response for Player 2 i.e. $R_2(u_1)$ with respect to some strategy $u' \in U^1$ chosen by Player 1. In this case, our domain would be $u^1$ and range would be $u^2$. Thus, the plot can be drawn on a 2-dimensional $u^1 - u^2$ graph.

Here, it is assumed that the value denoted by each contour decreases as we move in i.e. $k_2 < k_1$ and so on.

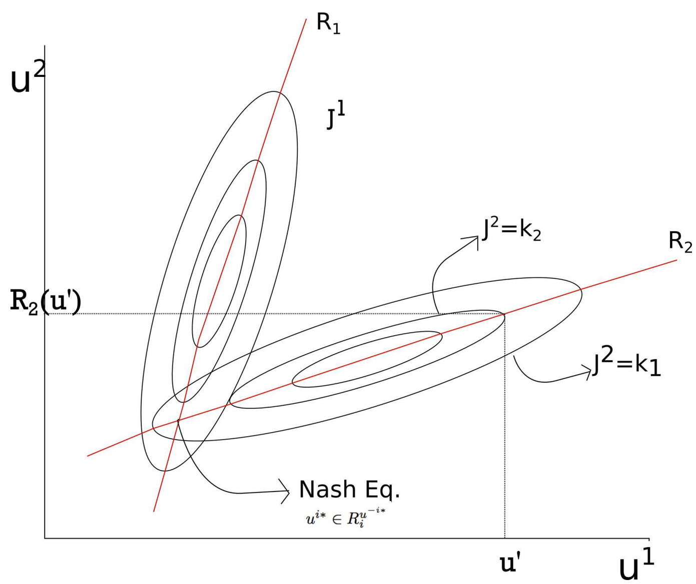


Figure 12.1: Obtaining Best Response curves via contour plots

For some $u' \in U^1$ chosen by Player 1 and Player 2's response $R_2(u') \in U^2$ to it, $(u', R_2(u'))$ lies on the intersection of $x = u'$ and the contour lines in $J^2$. Since, the players in the case we are considering wish to minimise their respective payoffs, it would be in Player 2's best interest that he choses his response in such a way that $(u', R_2(u'))$ lies on the contour line with minimum payoff. This would happen to be the inner most contour line that $x = u'$ intersects. In this case, $(u', R_2(u'))$ would be the point of the tangency of the the line $x = u'$ and the inner most contour curve that the line is tangent to.

$R_2$ denotes the locus of $u' \in U^1$ played by Player 1 and its corresponding best response by Player 2. Via a similar reasoning $R_1$ can also be obtained.

Intersection of two curves $R_1$ and $R_2$ gives the Nash Equilibrium since no player has an incentive to deviate from this point as it corresponds to the best response of that player to the other player's move.

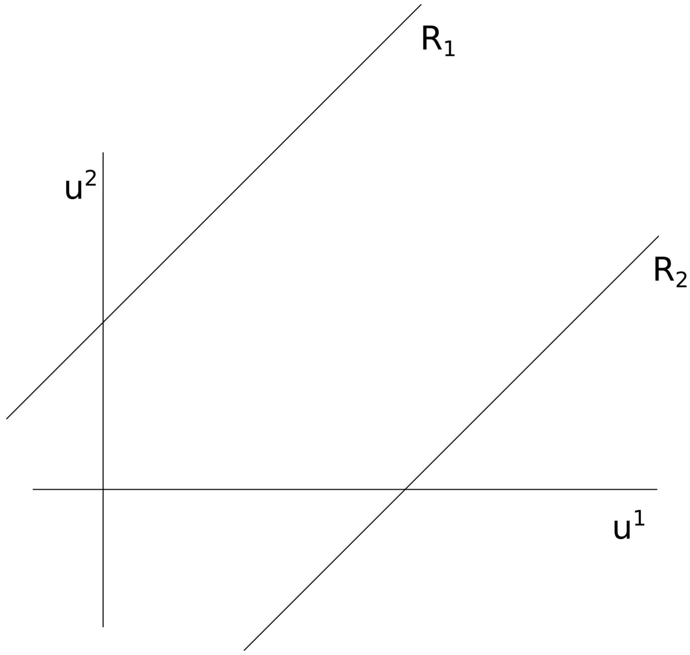


Figure 12.2: Parallel Response curves: No Nash Equilibrium

Note that Nash Equilibrium may not exist, for example, in cases where the response curves lie parallel to each other.

## 12.4 Stability

Sometimes, small perturbations in payoffs eliminate the existence of Nash Equilibrium while sometimes response curves are immune to such perturbations. Examples of such cases can be seen in Fig. 12.3. In Fig. 12.3 (a), if $R_2$ shifts to $R_2'$ we go from having infinite Nash Equilibria to no Nash Equilibria while in Fig. 12.3 (b), perturbations in payoffs does not threaten the finitely many Nash Equilibria that we have.

Situations as in Fig. 12.3 (a) where a Nash Equilibria is not guaranteed for all perturbations are known to be **unstable**.

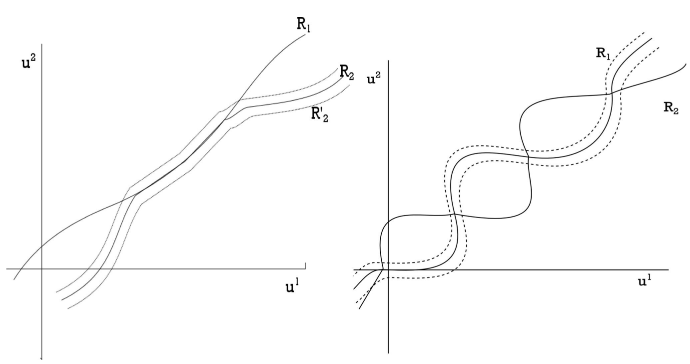


Figure 12.3: Parallel Response curves: (a) Unstable to perturbations (b) Stable Nash Equilibrium

Stability can be seen through the following three different view points.

1. The kind of stability we see in Nash Equilibrium wherein no rational player has an incentive to deviate from playing the strategy that leads to Nash Equilibrium.
2. Stability in the sense that small changes in payoffs does not endanger the existence of Nash Equilibrium. If making small changes to payoffs of a player vanishes the Nash Equilibrium (Refer to Figure 12.3), such a game is unstable.
3. If a player deviates from a strategy that leads to a Nash Equilibrium, the sequence of changes in strategies of other players again leads to the Nash Equilibrium.

The third view point of stability is illustrated in Figure 12.4. In such a case if a player deviates, the sequence of changes in the strategies of different players responding sequentially to this deviation will lead them to Nash Equilibrium.

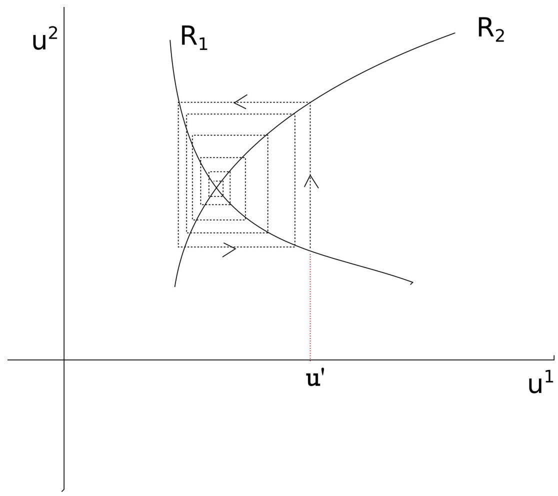


Figure 12.4: Stable Nash Equilibrium

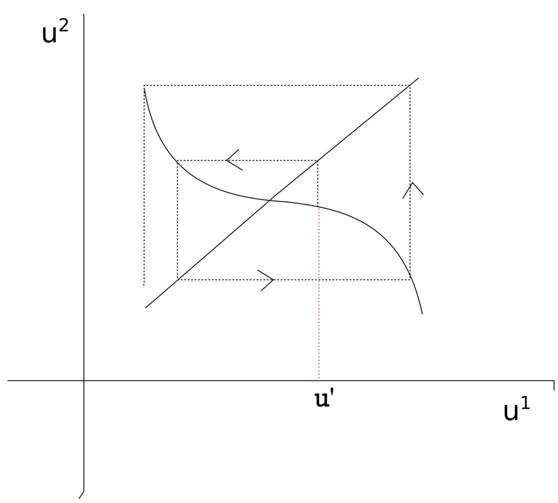


Figure 12.5: Unstable Nash Equilibrium

**Definition 4** *Let $S$ be an adjustment scheme. A nash equilibrium $u^*$ is said to be globally(i.e. for any starting point $u$) stable with respect to $S$ if it is the limit of the iteration*

$$u^{*i} = \lim_{k \to \infty} u^i(k) \tag{12.1}$$

$\forall i \in N$ *where*

$$u^i(k+1) = ArgMin_{u^i \in U^i} \; J^i(u^{-i}_{(s_k)}, u^i) \tag{12.2}$$

For Example:

- $u^{-i}_{s_k} = u^{-i}_k$ means all players simultaneously determine their $k+1^{th}$ strategy from the knowledge of strategies of other players at $k^{th}$ stage
- $u^{-i}_{s_k} = (u^1_{k+1}..., u^{i-1}_{k+1}, u^{i+1}_k..., u^N_k)$ means players determine their strategy at each iteration in a sequential and round robin manner

Note that these epsilon adjustments, tangents, taking turn ideas are only possible if a continuum of strategies is available. They are not possible for the discrete case.

## 12.5 Existence of Nash Equilibrium

Now we will state some theorems,lemmas (without proof) and definitions which will help us prove the existence of Nash Equilibrium in continuous kernel games (with some constraints).

**Theorem 12.1** *If $C \subseteq \mathbb{R}$ is convex and compact, $f : C \to C$ is a continuous function, then $f$ admits a fixed point. This theorem is known as **Brouwer's fixed-point theorem**.*

**Theorem 12.2** *Every bounded sequence has a convergent subsequence. This is a fundamental theorem known as **BolzanoWeierstrass theorem**.*

**Definition 5** *A function $f$ is convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{12.3}$$

$\forall x, y$ *and* $\forall \lambda \in [0, 1]$

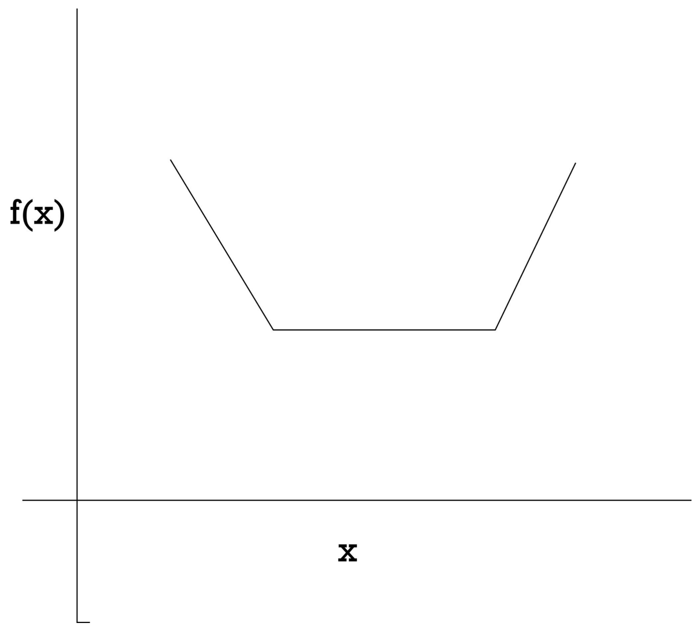


Figure 12.6: Convex Function (Not Strictly Convex)

**Definition 6** *A function $f$ is strictly convex if*

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y) \tag{12.4}$$

$\forall x \neq y$ *and* $\forall \lambda \in (0,1)$

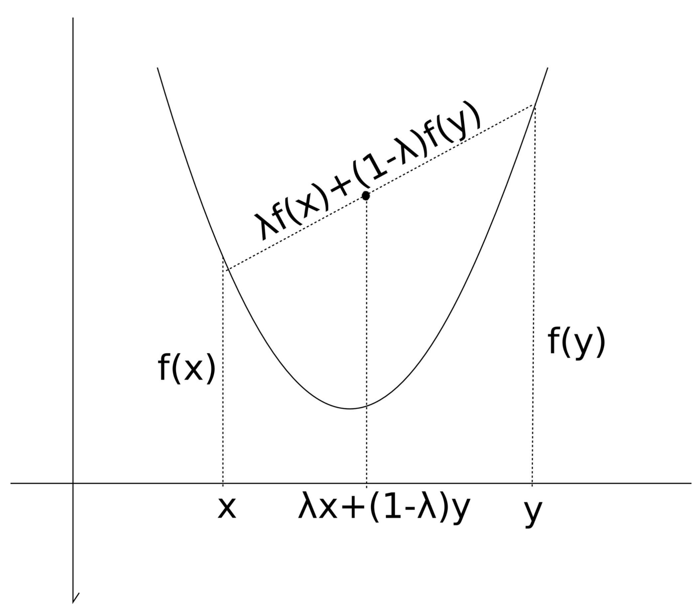


Figure 12.7: Strictly Convex Function

**Lemma 12.3** *A strictly convex function over a convex set will always have a unique minimum (if exists).*

Now comes the main theorem which proves the existence of Nash equilibrium in continuous kernel games.

**Theorem 12.4** *For each $i \in \mathbb{N}$ let $u^i$ be a compact and convex set in $\mathbb{R}^{m_i}$. Suppose $J^i : \prod_{k=1}^{N} U^i \to \mathbb{R}$ is the payoff of $P_i$ and $J^i(u^i, u^{-i})$ is strictly convex and continuos in $u^i$ $\forall u^{-i}$, then the associated N-person game has a Nash Equilibrium.*

**Proof.** Consider the best response function $R_i(u^{-i}) = ArgMin_{u^i \in U^i} \; J^i(u^i, u^{-i})$. Since $U^i$ is compact and $J^i$ is a continuous function in $u^i$, therefore $min_{u^i \in U^i} \; J^i(u^i, u^{-i})$ exists for every $u^{-i}$ (By Weierstrass theorem). $J^i(u^i, u^{-i})$ is strictly convex in $u^i$ and the domain $U^i$ is also convex. Therefore $R_i(u^{-i})$ is single valued for each $i$ (By Lemma 12.2). Hence, the function $R(u) = \prod_{i=1}^{N} R_i(u^{-i})$ is also single vauled.

Now we just need to show that $R$ is continuous. It is sufficient to show that $R_i$ is continuous for each $i$. Suppose $\{u_{(k)}^{-i}\}$ is a sequence in $U^{-i}$ such that

$$\underset{k \to \infty}{Lim} \; u_{(k)}^{-i} = \bar{u}^{-i} \tag{12.5}$$

We need to show that

$$\underset{k \to \infty}{Lim} \; R_i(u_{(k)}^{-i}) = R_i(\bar{u}^{-i}) \tag{12.6}$$

This equation will not hold in 2 cases:

- LHS does not exist
- LHS is not equal to RHS

We will show that both these things can't happen. $\{R_i(u_{(k)}^{-i})\}$ is a sequence in $U^i$. Since $U^i$ is compact, therefore this sequence has a convergent subsequence (By Bolzano Weierstrass theorem). Let $\{R_i(u_{(k_n)}^{-i})\}$ be a convergent subsequence of $\{R_i(u_{(k)}^{-i})\}$. We know that

$$J^i(R_i(u_{(k_n)}^{-i}), u_{(k_n)}^{-i}) \;\leq\; J^i(u^i, u_{(k_n)}^{-i}) \quad \forall u^i \in U^i \tag{12.7}$$

Taking limit on both sides.

$$\underset{n \to \infty}{Lim} \; J^i(R_i(u_{(k_n)}^{-i}), u_{(k_n)}^{-i}) \;\leq\; \underset{n \to \infty}{Lim} \; J^i(u^i, u_{(k_n)}^{-i}) \quad \forall u^i \in U^i \tag{12.8}$$

$J^i$ is continuous (polynomial), so we can take limits inside.

$$J^i(\underset{n \to \infty}{Lim} \; R_i(u_{(k_n)}^{-i}), \bar{u}^{-i}) \;\leq\; \underset{n \to \infty}{Lim} \; J^i(u^i, \bar{u}^{-i}) \quad \forall u^i \in U^i \tag{12.9}$$

This implies

$$\underset{n \to \infty}{Lim} \; R_i(u_{(k_n)}^{-i}) = R_i(\bar{u}^{-i}) \tag{12.10}$$

This proves that every converging subsequence of $\{R_i(u_{(k)}^{-i})\}$ converges to $R_i(\bar{u}^{-i})$. Now, we just need to show the existence of limit for $\{R_i(u_{(k)}^{-i})\}$. Suppose $\{R_i(u_{(k)}^{-i})\}$ doesn't converge to $R_i(\bar{u}^{-i})$. Then there exist a small ball around $R_i(\bar{u}^{-i})$ and a subsequence of $\{R_i(u_{(k)}^{-i})\}$ such that this subsequence never enters this small ball. But this subsequence will have a convergent subsequence that converges to $R_i(\bar{u}^{-i})$. We just proved that every convergent subsequence must converge to $R_i(\bar{u}^{-i})$. This is a contradiction! The limit indeed exists. Hence proved!