**SC 631: Games and Information** **Autumn 2014**

# Lecture 14: September 23

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## 14.1 Dynamic Games

The assumption in games studied till now was that all players play simultaneously. We now allow players to play possibly sequentially (some may play sequentially, some simultaneously). Players may play after one another or players may even play multiple times. All of this will be allowed.

**Example**: Two players have coins and they have to put the coin in either heads or tails facing upwards.

| | H | T |
|---|---|---|
| H | (+1, -1) | (-1, +1) |
| T | (-1, +1) | (+1, -1) |

If the game is simultaneous, none of the players has the option of seeing what the other player has played and then decide his own strategy. Only if we allow a player certain information before he picks his strategy, only then will the game be fundamentally different from a simultaneous game.

**Example**: *Cournot Game* - Two firms $P_1$ and $P_2$, producing $q_1$ and $q_2$ quantities of an item respectively. For some constant $c$, we have:

$$U_i(q_1, q_2) = [1 - (q_1 + q_2)]q_i - cq_i$$

Where $U_i$ denotes the payoff of $P_i$.
Suppose $P_2$ observes the action of $P_1$. Let $Q_1$ = space of $q_1$, $Q_2$ = space of $q_2$.
A strategy for $P_2$ is a function $\gamma^2 : Q_1 \to Q_2$; whereas, a strategy for $P_1$ is just a quantity $q_1$, or equivalently $Q_1$ valued functions that are constants.

Can we predict what gets played? It makes sense to think that given $P_1$'s response $q_1$, $P_2$'s response will be the best response possible (the one which gives the maximum payoff to $P_1$ given that $q_1$ was played). Let $R_i(\cdot)$ denote $P_i$'s best response strategy, then assuming that $R(\cdot)$ is a single value map:

$$\gamma_2^*(q_1) = R_2(q_1) = \underset{q_2'}{\text{argmax}}\ U_2(q_1, q_2')$$

$$q_1^* = \underset{q_1}{\text{argmax}}\ U_1(q_1, R_2(q_1))$$

Thus, the game can be thought of as a static game in the space of functions.

## 14.2 Nash Equilibrium of a Dynamic Game

**Definition 14.1** *Nash equilibrium: Functions $\gamma_1^*, \gamma_2^*$ such that*

$$U_1(\gamma_1^*, \gamma_2^*) \geq U_1(\gamma_1, \gamma_2^*) \quad \forall \textit{ constant } \gamma_1 \tag{14.1}$$
$$U_2(\gamma_1^*, \gamma_2^*) \geq U_2(\gamma_1^*, \gamma_2) \quad \forall \textit{ functions } \gamma_2 \textit{ of } q_1 \tag{14.2}$$

We now show that $\gamma_1^* = q_1^*$, $\gamma_2^* = R_2$ is a Nash equilibrium of the dynamic game:

$$\begin{aligned} U_1(q_1^*, R_2(q_1^*)) &= \max_{\gamma_1} U_1(\gamma_1, \gamma_2^*(\gamma_1)) \\ &= \max_{q_1} U_1(q_1, \gamma_2^*(q_1)) \\ &= \max_{q_1} U_1(q_1, R_2(q_1)) \end{aligned}$$

And,

$$\begin{aligned} U_2(\gamma_1^*, \gamma_2^*) &= \max_{\gamma_2} U_2(q_1^*, \gamma_2(q_1^*)) \\ &= U_2(q_1^*, R_2(q_1^*)) \end{aligned}$$

### 14.2.1 Another Nash Equilibrium

If $(q_1^{**}, q_2^{**})$ is a static Nash Equilibrium of the game, then the equivalent dynamic strategies are $(\gamma_1^{**}, \gamma_2^{**}) \equiv (q_1^{**}, q_2^{**})$. It is worthwhile to note that in this case, $P_2$'s strategy is a constant (independent of what $P_1$ plays). $P_2$ though blessed with the information of what $P_1$ plays, decides to ignore that.

We now try to show that $(q_1^{**}, q_2^{**})$ is also the Nash equilibrium of the dynamic game. Consider:

$$\begin{aligned} U_1(\gamma_1^{**}, \gamma_2^{**}) &= U_1(q_1^{**}, q_2^{**}) \\ &= \max_{q_1} U_1(q_1, q_2^{**}) \end{aligned}$$

where the second equality holds due to the definition of a static Nash equilibrium strategy.

$$= \max_{\gamma_1} U_1(\gamma_1, \gamma_2^{**})$$
$$\therefore \quad U_1(\gamma_1^{**}, \gamma_2^{**}) \geq U_1(\gamma_1, \gamma_2^{**}) \quad \forall \textit{ constant } \gamma_1 \tag{14.3}$$

which is a reminiscent of Eq. (14.1)

Now consider:

$$\begin{aligned} U_2(\gamma_1^{**}, \gamma_2^{**}) &= U_2(q_1^{**}, q_2^{**}) \\ &= \max_{q_2} U_2(q_1^{**}, q_2) \\ &= \max_{\gamma_2} U_2(q_1^{**}, \gamma_2(q_1^{**})) \end{aligned} \tag{14.4}$$

Any function $\gamma_2$ that satisfies $\gamma_2(q_1^{**}) = q_2^{**}$ $(= R_2(q_1^{**}))$ will maximize $U_2(q_1^{**}, \gamma_2(q_1^{**}))$. Thus, the function $\gamma_2 \equiv q_2^{**}$ is also a maximizer. This proves the truth of the last equality – which is equivalent to Eq. (14.2).

Hence, by Eq. (14.3), (14.4), we have shown that $(q_1^{**}, q_2^{**})$ is a Nash equilibrium of the dynamic game. Note that, however every function which satisfies $\gamma_2(q_1^{**}) = q_2^{**}$ will not necessarily satisfy Eq. (14.1) and isn't necessarily a Nash equilibrium strategy (the *constant* function $\gamma_2 \equiv q_2^{**}$ does).

We can also show that the payoff for $P_1$ from the dynamic equilibrium is at least as good as that from the static equilibrium.

$$\begin{aligned} U_1(q_1^*, R_2(q_1^*)) &= \max_{q_1} U_1(q_1, R_2(q_1)) \\ &\geq U_1(q_1^{**}, R_2(q_1^{**})) \\ &= U_1(q_1^{**}, q_2^{**}) \end{aligned}$$

### 14.2.2 Interpretation

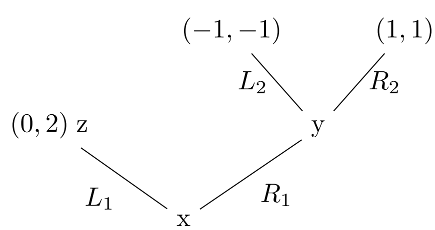
We can interpret the static equilibrium as if $P_2$ is disregarding what $P_1$ is doing and is fixated on playing $q_2^{**}$. This is as if $P_2$ is issuing $P_1$ a threat – a threat to play irrationally at a certain node. It is irrational in the sense if $P_1$ had played anything other than $q_1^{**}$ then playing $q_2^{**}$ would not be optimal any more for $P_2$. But this induces $P_1$ to play $q_1^{**}$ in which case, $q_2^{**}$ is the best response. So, $q_2^{**}$ is irrational only at certain 'meta situations' which actually do not arise!

In essence, there are two Nash equilibria: 1) one which exploits the information structure of the game – dynamic equilibrium; and the 2) other equilibrium which is of the static game.

It is also interesting to observe that $P_1$ will gain in going from a static equilibrium to a dynamic one – $P_1$ has an advantage in playing first. But there's also a possibility of a trick when $P_2$ plays a seemingly irrational strategy.

## 14.3 Another example

Players $P_1$ and $P_2$ play the following game with $P_1$ making the first move starting at node x. If $P_1$ plays $L_1$, the game ends. Else if $P_1$ plays $R_1$, $P_2$ has to move next at node y.




In the following description, $\gamma_j^i$ denotes the $j^{\text{th}}$ strategy of $i^{\text{th}}$ player.
Let us enumerate available strategies for $P_1$:

$$\begin{aligned} \gamma_1^1 &\equiv L_1 \\ \gamma_2^1 &\equiv R_1 \end{aligned}$$

Let $DN$ denotes the the action of doing nothing when $P_2$ is at node z, then $\gamma_2 : \{y, z\} \rightarrow \{L_2, R_2, DN\}$ will be $P_2$'s strategy satisfying:

$$\gamma_2(y) \in \{L_2, R_2\}$$

$$\gamma_2(z) = DN$$

Thus,

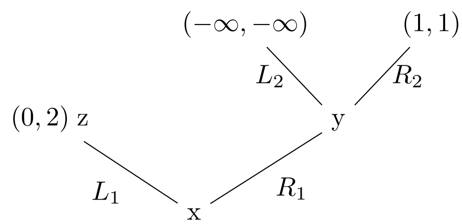
$$\gamma_1^2 = \begin{cases} L_2 & \text{at y} \\ DN & \text{at z} \end{cases}$$

$$\gamma_2^2 = \begin{cases} R_2 & \text{at y} \\ DN & \text{at z} \end{cases}$$

are the two strategies for $P_2$. It is left as an exercise for the motivated reader to check that $(\gamma_1^1, \gamma_1^2)$ and $(\gamma_2^1, \gamma_2^2)$ are the Nash equilibria for the above dynamic game.

Here we obtain $(\gamma_2^1, \gamma_2^2)$ by the dynamic analysis and the other equilibrium $(\gamma_1^1, \gamma_1^2)$ which can be interpreted similar to the above discussion – $P_2$ issues a threat to $P_1$ of playing irrationally.

An interesting variant of the above game is the following:




Here if $P_2$ issues a threat of playing $L_2$, ie. irrationally, then $P_1$ must adjust to the equilibrium $(0, 2)$. Suppose $P_1$ and $P_2$ were nuclear armed countries, then the use of nuclear weapons by $P_2$ is destructive for both $P_1$ and $P_2$. Thus, the threat of $P_2$ using nuclear warfare forces $P_1$ to play $L_1$, thus reaching the equilibrium $(0, 2)$. This is the essence of the doctrine of Mutually Assured Destruction (MAD).