SC 631: Games and Information **Autumn 2014**

Lecture 16: September 30

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16.1 Dynamic Game In Extensive Form

An extensive form of dynamic game is a tree with

1. A specific vertex indicating the starting point,
2. A payoff for each player at each terminal node,
3. A partition of the nodes of the tree into N player sets, and
4. A subpartition of each player set into information sets η^i_j such that the same number of branches emanates from each node belonging to the same information set and no node follows another node in same information set.

where η^i_j is j^{th} information set for player i. At each node there can only be one player. A game is called a **single act game** when each path from root node intersects to terminal node of the player set of each player at-most once.

16.2 Nash Equilibrium In Zero Sum Single Act Dynamic Game

There are different methods to compute NE in a Zero sum dynamic game. One method is by converting to normal form and finding the NE.

Example 1 Consider a zero sum dynamic game, the payoffs are given in figure 1. What is/are Nash equilibrium/equilibria of this Game?

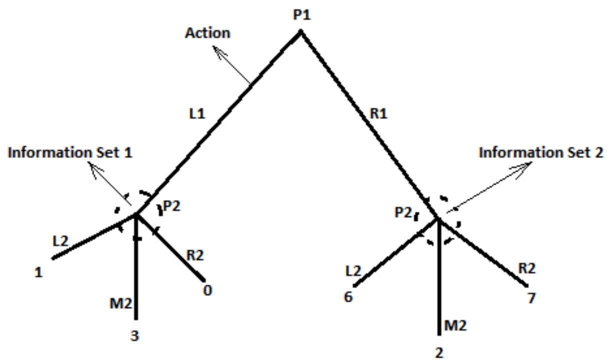

Fig:1

Solution: We will solve this game by converting it into normal form. Let $\gamma_1^1 = L_1, \gamma_2^1 = R_1$ be strategies of player one. Second player P_2 has strategies as a function of his information set. Here P_2 has two information set namely η_1^2 and η_2^2(incircled left and right respectively in figure)and there are three actions available for each information set. So the number of strategies of P_2 is product of action(s) available for P_2 on each information set. Therefore P_2 has (3×3) nine strategies. Which are given below

$$\gamma_1^2 = \begin{cases} \text{L}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{L}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_2^2 = \begin{cases} \text{L}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{M}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_3^2 = \begin{cases} \text{L}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{R}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_4^2 = \begin{cases} \text{M}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{L}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_5^2 = \begin{cases} \text{M}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{M}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_6^2 = \begin{cases} \text{M}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{R}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_7^2 = \begin{cases} \text{R}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{L}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_8^2 = \begin{cases} \text{R}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{M}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_9^2 = \begin{cases} \text{R}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{R}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

Now normal form of game is represented in terms of payoff matrix, which is given in the table. Now the

	γ_1^2	γ_2^2	γ_3^2	γ_4^2	γ_5^2	γ_6^2	γ_7^2	γ_8^2	γ_9^2
γ_1^1	1	1	1	3^*	3	3^*	0	0	0
γ_2^1	6	2	7	6	2	7	6	2	7

P_1 (rows), P_2 (columns)

saddle points or Nash equilibria of this zero sum dynamic game are $\{\gamma_1^1, \gamma_4^2\}$ and $\{\gamma_1^1, \gamma_6^2\}$. □

A game G is **informationally inferior** to game G' if for all i and $\eta^i_{G'}$ there exist η^i_G such that $\eta^i_{G'} \subseteq \eta^i_G$ with atleast one inclusive is strict. In next example we consider an informationally inferior game of the above game.

Example 2 (Modified game) In the above game, what happens when P_2 has only one information set? What are the Nash equilibria?

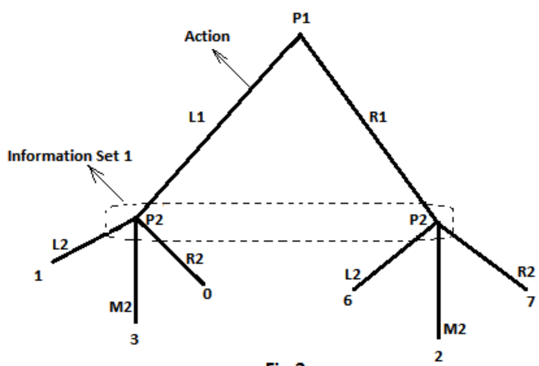

Fig:2

Solution: In this reference, this modified game become a static game or simultaneous move game. Strategies for P_1 are L_1 and R_1 and P_2 has only one information set and there are three possible actions at this information set. There will only be three strategies for P_2 namely L_2, M_2 and R_2. If we convert this game to its normal form then payoff matrix is given below. In this matrix of the game, there is no saddle point.

	L_2	M_2	R_2
L_1	1	3	0
R_1	6	2	7

P_1 (rows), P_2 (columns)

Hence the modified game has no saddle point. □

Note The modified game which is informationally inferior to original game may not have a saddle point event if the original game has one.

Example 3 Consider a two player $Z - S$ game, the payoffs of the are given in the figure 3. Find all Nash equilibria?

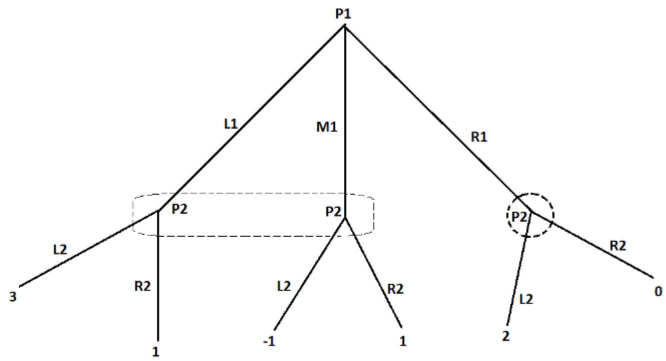

Fig:3

Solution: Here P_1 has three strategies say $\gamma_1^1 = L_1$, $\gamma_2^1 = M_1$ and $\gamma_3^1 = R_1$ and P_2 has strategies as a function of his information set. Further P_2 has two information sets η_1^2 and η_2^2(inscribed left and right in figure resp.)and there are two action in each information set. So P_2 has 2×2 strategies. which are listed below

$$\gamma_1^2 = \begin{cases} \text{L}_2 & if \text{ P}_1 \text{ plays } L_1 \text{ or } M_1 \\ \text{L}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_2^2 = \begin{cases} \text{L}_2 & if \text{ P}_1 \text{ plays } L_1 \text{ or } M_1 \\ \text{R}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_3^2 = \begin{cases} \text{R}_2 & if \text{ P}_1 \text{ plays } L_1 \text{ or } M_1 \\ \text{L}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

$$\gamma_4^2 = \begin{cases} \text{R}_2 & if \text{ P}_1 \text{ plays } L_1 \text{ or or } M_1 \\ \text{R}_2 & if \text{ P}_1 \text{ plays } R_1 \end{cases}$$

And the payoffs corresponding to these strategies are given below in the table. Nash equilibrium or saddle

	γ_1^2	γ_2^2	γ_3^2	γ_4^2
γ_1^1	3	3	1	1
γ_2^1	-1	-1	1*	1
γ_3^1	2	0	2	0

P_1

P_2

point of this matrix game is $\{\gamma_2^1, \gamma_3^2\}$. That is P_1 plays M_1 and P_2 plays R_2 in Nash equilibrium.

Instead of converting extensive form to normal form of the game to find NE(s), one can do it in a faster and easier way, which is known as **Backward Induction Method**.

Algorithm of Backward Induction Method

1. In a zero sum game there can be atmost two players. Consider the last acting player in the game and divide the actions for that player based on his information structure.

2. Find the best strategies corresponding to each information set and note down the values and move backward.

3. Now using these strategies solve the game for first acting player.

Illustration of backward induction method
Look at one portion of the game, which corresponds to η_1^2. This part of game can be thought of static game. At node 2 and node 3, since P_2 is maximizer, so optimal action for her is to play L_2 and R_2 respectively. At initial node (node 1), since P_1 is minimzer, so optimal action for her is to play M_1. So, here we end up with optimal action of player one and player two, which is $\{M_1, R_2\}$. Now in the second half of game the optimal action for P_2 is to play L_2, which, anyway, P_1 not going to play R_1. So the strategy in Nash equilibrium is $\{M_1, R_2\}$.

16.3 Nash Equilibrium In Non Zero Sum Single Act Dynamic Game

In the previous section we learn the techniques to find Nash equilibria in $Z - S$ single act game. Here we will study of behavior of Nash equilibria in non zero sum game($N - Z - S$). To illustrate the procedure let us look at some examples. Note that here each player plays for minimizing the cost.

Example 4 Consider a non zero sum single act dynamic game as shown in figure 4. Find all Nash equilibria of this game?

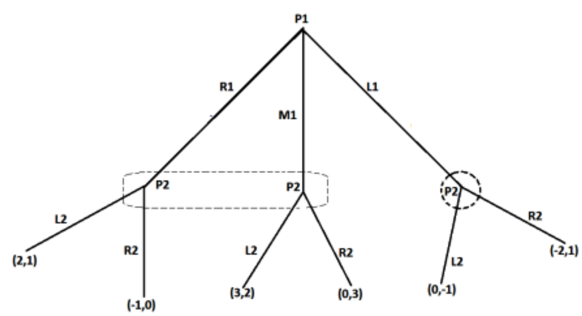

Fig: 4

Solution: Player one has three strategies namely $\gamma_1^1 = L_1$, $\gamma_2^1 = M_1$ and $\gamma_3^1 = R_1$ and player has two information sets η_1^2 and η_2^2. Let L_2 and R_2 are actions in each information set of P_2, so P_2 has a 2×2 strategies and these are listed below

$$\gamma_1^2 = \begin{cases} \text{L}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{L}_2 & if \text{ P}_1 \text{ plays } R_1 \text{ or } M_1 \end{cases}$$

$$\gamma_2^2 = \begin{cases} \text{L}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{R}_2 & if \text{ P}_1 \text{ plays } R_1 \text{ or } M_1 \end{cases}$$

$$\gamma_3^2 = \begin{cases} \text{R}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{L}_2 & if \text{ P}_1 \text{ plays } R_1 \text{ or } M_1 \end{cases}$$

$$\gamma_4^2 = \begin{cases} \text{R}_2 & if \text{ P}_1 \text{ plays } L_1 \\ \text{R}_2 & if \text{ P}_1 \text{ plays } R_1 \text{ or } M_1 \end{cases}$$

The payoffs table for players is as shown below So Nash equilibria of this matrix game are $\{\gamma_1^1, \gamma_1^2\}$ and

	γ_1^2	γ_2^2	γ_3^2	γ_4^2	
γ_1^1	(0,-1)*	(0,-1)	(-2,1)	(-2,1)	
γ_2^1	(3,2)	(0,3)	(3,2)	(0,3)	P_1
γ_3^1	(2,1)	(-1,0)*	(2,1)	(-1,0)	

P_2

$\{\gamma_3^1, \gamma_2^2\}$.

Example 5 (modified game) Find all Nash equilibria of the game in the above question, if the information set of player two (P_2) is singleton (i.e.Nash equilibria of the informationally inferior game)?

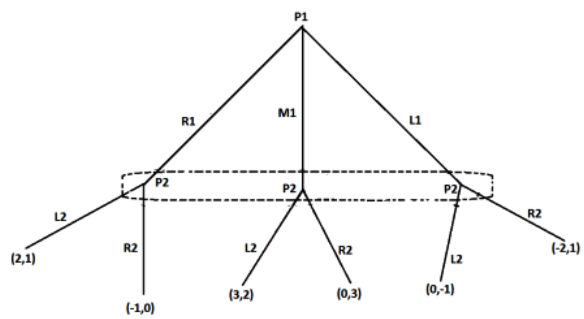

Fig: 5

Solution: Here P_1 strategies will remain same $\gamma_1^1 = L_1$, $\gamma_2^1 = M_1$ and $\gamma_3^1 = R_1$. While P_2 has only one information set and there are two actions in information set. So P_2 also has two strategies say $\gamma_1^2 = L_2$ and $\gamma_2^2 = R_2$. The payoffs matrix for the players is Clearly the Nash equilibrium of the game is $\{L_1, L_2\}$. Which

	γ_1^2	γ_2^2	
γ_1^1	(0,-1)	(-2,1)	
γ_2^1	(3,2)	(0,3)	P_1
γ_3^1	(2,1)	(-1,0)	

P_2

was also one of Nash equilibria of its informationally superior game!
Note: Nash equilibria of informationally inferior game is also Nash equilibria of its informationally superior game but converse is not true.

16.4 Finite Games of Perfect Information

A game is called a game of **perfect information** if each information set is singleton.

Theorem Every finite game of perfect information has a Nash equilibrium.

Let us illustrate it with an example.

Example 6 Consider a non zero sum single act dynamic game of perfect information with three players. The payoffs are shown in figure 6. Find all Nash equilibria of this game?(Note: Here player are minimizing their cost)

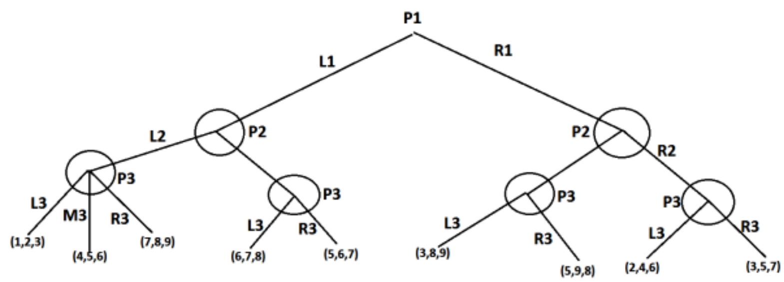

Fig:6

Solution: We will do it by the method of backward induction. Here one can appreciate the ease of this method.
Let P_1, P_2 and P_3 be three players. There are two strategies for player one namely $\gamma_1^1 = L_1$ and $\gamma_2^1 = R_2$ and for player two, there are two information set and each information set there are two actions. So P_2 has 4 strategies. Similarly P_3 has 24 strategies.

Let nodes are labeled from left to right(in Figure 6). Starting from the terminal node, P_3 has three choices at node 4, so optimal action for her is to pick L_3, at node 5 P_3 will choose R_3. Coming to node 6 and 7, P_3 will pick R_3 and L_3 respectively. Schematic diagram is shown in figure 7.

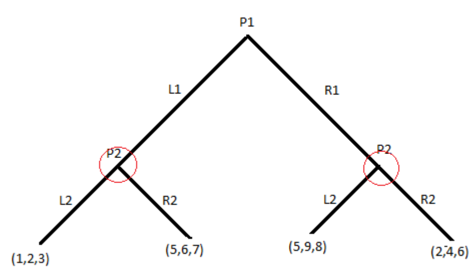

Figure 7

Now at node 2, P_2 will pick L_2. At node 3, P_2 will pick R_2. Schematic diagram is shown in figure

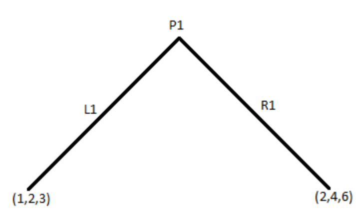

Figure 8

At root node P_1 will choose L_1. Thus we are left with a strategy $\{L_1, L_2, L_3\}$, corresponding to which the payoff is (1,2,3) for P_1, P_2 and P_3, is in Nash equilibrium.