Lecture 17: October 7

Instructor: Ankur A. Kulkarni *Scribes: Anand, Ayush, Sushant, Anmol, Khushal*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

17.1 Recap: Extensive form

Extensive form of a game has the following properties

- It is a tree, with a root
- Payoff for each player is given at the leaf nodes
- Partition of nodes into player sets - nodes where a player was supposed to act
- Sub-partition of player sets into information sets, such that same number of branches emanate from each node in an information set and no node follows another node in the same information set. Clarifying further, if a node followed another node in the same information set, it would create a cycle.

Lemma 17.1. *Let $I^i \equiv$ set of all information sets of P_i, and $u^i_{\eta^i} \equiv$ set of actions at information set η^i for player P_i. Then a strategy for P_i is given by:*
$\gamma^i : I^i \to U^i$, *where* $U^i = \bigcup_{\eta^i \in I^i} U^i_{\eta^i}$ *such that* $\gamma^i(\eta^i) \in U^i_{\eta^i}$

This is a mapping of information sets to available actions at that information set.

17.2 Equilibria for inferior game

In this lecture we will show that equilibria of informationally inferior game are retained even in informationally richer game.

Definition 17.2.1. ***An inferior game***
Let (I) and (II) be extensive form games with the same tree and player sets and same payoffs. Game (I) is said to be informationally inferior to game (II), if $\forall i \in \mathcal{N}$ and $\forall \eta^i_{(II)} \in I^i_{(II)}$, $\exists\ \eta^i_{(I)}$ such that $\eta^i_{(II)} \subseteq \eta^i_{(I)}$ and at least one inclusion is strict.

It can be seen that Game (II) is richer than Game (I) and that each information set of the richer game is present in some information set of the weaker game. Note that

- Inferior game has bigger information sets than the richer game.
- Every information set of a richer game is included in an information set of the inferior game.

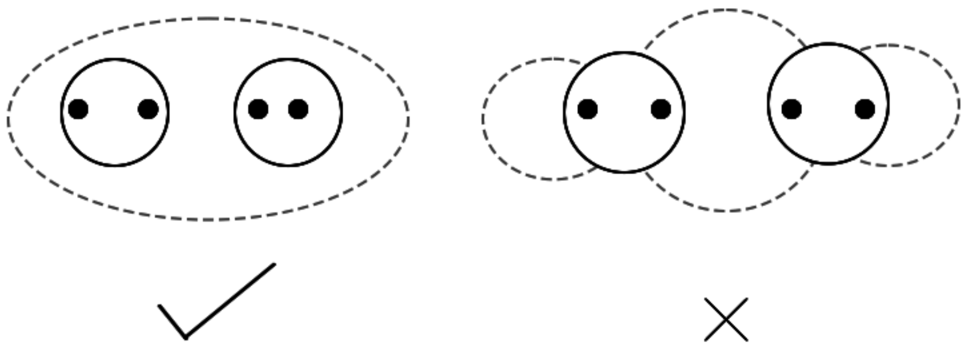


Figure 17.1:

- Every strategy in inferior game can be simulated using strategy in richer game. This can be done by ignoring the extra information in the richer game.

Figure 17.1 illustrates the basic property that every information set in richer game should be included in an information set of the inferior game. The solid circles represent the information sets of the richer game whereas the dotted circes represent those of the inferior game.
The 1^{st} figure is correct because each information set of the richer game is entirely contained in the information set of the inferior game. On the contrary, the 2^{nd} figure is incorrect because there exists at least one information set in the richer game which is not entirely contained in a single information set of the inferior game. In this case both the information sets of rich game are not contained entirely in the information sets of the weak game.

Theorem 17.2.1. *Let (I) and (II) be single act games, such that (I) is informationally inferior to (II). Then any Nash Equilibrium of (I) also constitutes a Nash Equilibrium of (II). That is, by simulating strategies in (I) to be strategies in (II).*
$\forall \eta^i_{(II)}$, $\exists \eta^i_{(I)}$ *(and there exists only one such* $\eta^i_{(I)}$ *because* $\eta^i_{(I)}$*'s form partitions) such that*

$$\eta^i_{(II)} \subseteq \eta^i_{(I)} \;\; and \;\; \gamma^i_{(II)}(\eta^i_{(II)}) = \gamma^i_{(I)}(\eta^i_{(I)}) \tag{17.1}$$

Proof. Suppose $(\gamma^{1*}, \gamma^{2*}, ...\gamma^{N*})$ is a Nash Equilibrium of (I).
Assume for sake of contradiction that $(\gamma^{1*}, \gamma^{2*}, ...\gamma^{N*})$ is not a Nash Equilibrium of (II).
$\Rightarrow \exists\ i(sayN)$ and a strategy $\tilde{\gamma}^N \in \Gamma^N_{(II)}$

$$J^N(\gamma^{1*}, \gamma^{2*}, ...\gamma^{N*}) > J^N(\gamma^{1*}, \gamma^{2*}, ...\tilde{\gamma}^N). \tag{17.2}$$

The strategies $(\gamma^{1*}, \gamma^{2*}, ...\tilde{\gamma}^N)$ determine a unique path in the tree.
Let $\tilde{\eta}^N_{(II)}$ be the information set of P_N that intersects this path.
Let the node where this has intersected be $\tilde{n}^N$.
Let $\tilde{\eta}^N_{(I)}$ be the information set of P_N containing $\tilde{n}^N$.
Consider $\bar{\gamma}^N$ such that

$$\bar{\gamma}^N(\tilde{\eta}^N_{(I)}) = \tilde{\gamma}^N(\tilde{\eta}^N_{(II)}) \tag{17.3}$$

$$J^N(\gamma^{1*}, \gamma^{2*}, ...\bar{\gamma}^{N*}) = J^N(\gamma^{1*}, \gamma^{2*}, ...\tilde{\gamma}^N) < J^N(\gamma^{1*}, \gamma^{2*}, ...\gamma^{N*}) \tag{17.4}$$

$\Rightarrow (\gamma^{1*}, \gamma^{2*}, ...\gamma^{N*})$ is not a Nash Equilibrium in (I). A contradiction □

The few points worth mentioning

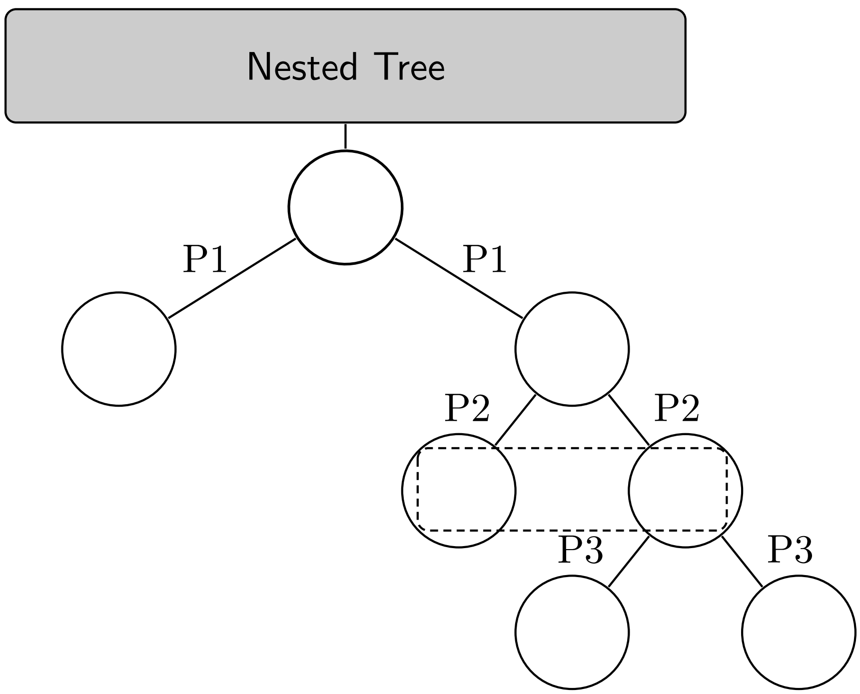
- Dynamic games in general have multiple equilibria.
- If the information structure permits, we can find same equilibrium by backward induction.
- Else most general way to find equilibrium is by writing it in normal form.

Definition 17.2.2. ***Nested Extensive Form***

An extensive form is nested (in single act game), if each player has access to the information acquired by all its precedents.

For any information set of a player, all player that have acted along the path leading to the information set are called its precedents.

Example :



The above figure represents a typical nested tree. Each of the branches labelled P1, P2 and P3 represent the possible choices available to the respective player. Based on this tree and by the definition of nested extensive form, the following observations hold valid :

- P2 has all information that P1 has
- P3 has more information than P2 as he knows what exactly P1 had played

We can see by the above example that in a nested extensive game, each player, indeed has access to information acquired by its precedents.