SC 631: Games and Information **Autumn 2014**

Lecture 2: August 5

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2.1 Game

The following are needed in order to describe a game

1. A set of players
$$\mathcal{N} = \{1, ...N\}$$
2. A set of strategies for each player
$$S_i \text{ for each } i \in \mathcal{N}$$
3. payoff functions
$$\pi_i : S \to \mathbb{R} \text{ for each } i \in \mathcal{N} \text{ where } S = \prod_{i=1}^{N} S_i$$

2.2 Nash Equilibrium

We recall the definition of Nash Equilibrium: A profile of strategies $x = (x_1, x_2, ..., x_N)$, where $x_i \in S_i$, is said to be Nash equilibrium of the game, if

$$\pi_i(x) \geq \pi_i(\bar{x}_i, x^{-i}) \ \forall \bar{x}_i \in S_i \ , \forall i \in \mathcal{N} \tag{2.1}$$

where $(\bar{x}_i, x^{-i})$ denotes $(x_1, x_2, ., \bar{x}_i, x_{i+1}, ., x_N)$, which is profile of strategies when only player i shifts her strategy from x_i to $\bar{x}_i$.
Given the communication requirement, there will not be incentive for someone to deviate when the strategies being played is a Nash equilbrium. We only have a justification for the concept of Nash equilbrium. We do not know how to *derive* the Nash equilibrium. We also do not consider how the players have arrived to a Nash equilibrium. In a non cooperative game the player is not aware of which strategy the other player has played. Though, he is aware of the payoffs of other players at each strategy.

2.2.1 Justifications of Nash Equilibrium

If a game is non co-operative then the only solution concept is Nash Equilibrium.

1. **Stability**: A point that is not stable against unilateral deviation is cannot be an outcome.

2. **Self-fulfilling agreement**: if the players could communicate and decide to play Nash, the decision will hold. since none of the players will have an incentive to deviate.
3. **Normative Concept**: Nash Equilibrium gives the optimum payoff to all the players.
4. **Nature**: Nash equilibrium is seen in nature as evolution of life.

2.3 Security Dilema

Consider the game in which the USA and USSR have to decide on whether to have nuclear weapons or not. The payoff matrix is

Table 2.1: Payoff Matrix for the Security Dilemma Game

		USA	
		Yes	No
USSR	Yes	(2,2)	(3,1)
	No	(1,3)	(4,4)

Here, there are two Nash equilibrium strategies. Namely, (Yes, Yes) with rewards (2,2) and (No, No) with rewards (4,4). In terms of rewards, playing (No, No) is preferable for both players. But the risk involved in this strategy is higher since if the other player changes strategy, the balance of power is lost, with the other player getting reward of 3 while the one who sticks to "No" getting only 1. Hence, the strategy (No, No) is risk dominated and players play (Yes, Yes) though with lower rewards and lower risk.

2.4 Rationality

Definition Rationality implies that each player acts to optimize her payoff.

Definition A player is said to be rational if she optimizes her payoff.

Rationality is one of the most common assumptions in Game Theory, along with common knowledge of rationality. Common Knowledge is defined in section 2.6

2.5 Strictly Domianted Strategy

A strategy x_i for a player i is said to be strictly dominating ,if $\exists x_i^{'}$ such that

$$\pi_i(x_i^{'}, x^{-i}) \ > \ \pi_i(x_i, x^{-i}) \ \forall x^{-i} \in S^{-i} \ \text{ where } S^{-i} = \prod_{j=1, j\neq i}^{N} S_j \tag{2.2}$$

Therefore a rational player does not play a strictly dominated strategy.

Consider the following example:

Table 2.2: Two player game with strictly dominated strategies

	Player2			
	Strategies	Left	Middle	Right
	Upper	(4,3)	(5,1)	(6,2)
Player1	Middle	(2,1)	(8,4)	(3,6)
	Lower	(3,0)	(9,6)	(2,8)

In this example strategy Right for player 2 dominates strategy Middle for player 2. Assuming that the player is rational, player 2 is looking at the following table:

Table 2.3: Removing strictly dominated strategy

	Player2		
	Strategies	Left	Right
	Upper	(4,3)	(6,2)
Player1	Middle	(2,1)	(3,6)
	Lower	(3,0)	(2,8)

On a general level assumption is that each player is rational (Assumption 1).
Player1 can look at this new payoff table which eliminates streategy middle for player2 only if he knows that player 2 is rational.
Therefore second general assumption is that each player knows that every other player is rational (Assumption 2).
Under these assumptions strategy Upper of player1 dominates strategy Middle and Lower of player1. Therefore player 1 is looking at the following payoff table:

Table 2.4: Removing strictly dominated strategy

	Player2		
	Strategies	Left	Right
Player1	Upper	(4,3)	(6,2)

Now if player 2 knows that player 1 knows that player 2 is rational than player 2 is also looking at the above table. The general assumption in this case is that each player knows that each player knows that each player is rational(assumption 3).
In which case strategy Left dominates strategy Right. Thus we arrive at the strategies Upper for player 1 and Left for player 2 with payoff (4,3). This x=(Upper,Left) is also the nash equilibrium of the game.
Assumptions are 1,2,3 are different levels of knowledge which players can have in a game,we have used assumption 1,2,3 to reduce the 3x3 matrix to a single Nash quilibrium. We would have needed more levels of knowledge to reduce a bigger matrix ($rows, columns > 3$).

2.6 Common Knowledge

Definition A fact is common knowledge among players of a game if for any finite chain of players $i_1, i_2, ..., i_k$ the following holds that player i_1 knows the player i_2 knows that player .... that player i_k knows the fact. An example for a fact would be that all players are rational as seen in the previous example.

2.7 Elimination of dominated Strategies

Theorem 1 If a unique strategy profile is left after the iterated elimination of dominated strategies then the profile is the unique Nash Equilibrium of the game.

Theorem 2 The set of strategy profiles left after iterated elimination of strictly dominated strategies is independent of the order of elimination.