**SC 631: Games and Information** **Autumn 2014**

# Lecture 21: October 21

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## 21.1 A QUICK RECAP-

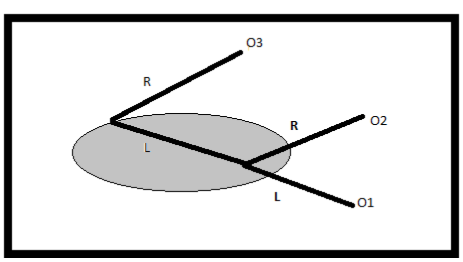
We have so far studied following two types of strategies:-
1. **Mixed Strategies** where we pick a function(strategy) randomly(akin to normal form)
2. **Behavioral Strategies** where at each information set, choose a random action.

**Definition 1** *A behavioural strategy $b_i$ is equivalent to a mixed strategy $\sigma_i$ if for any behavioural/mixed strategy combination $\sigma^{-i}$ and for all $x$,*

$$P(x; b_i, \sigma^{-i}) = P(x; \sigma_i, \sigma^{-i})$$

It is possible that for a game behavioural strategy and mixed strategy are not equivalent. This fact is very much illustrated by the following example-

### 21.1.1 Example




Here the *pure strategies* taken by player $i$ is either $\{L, R\}$. We consider $O_1, O_2, O_3$ as the 3 outcomes of the game. If he selects strategy $L$, the player forgets that he has even taken an action. And so he continues to play $L$ subsequently. It can be clearly seen that it is impossible to reach $O_2$ for any pure strategy. Also, there is no mixed strategy to reach $O_2$ as we can't have non-zero probability over something which can't be reached by pure strategy. Thus, $P(reaching\, O_2\; under\; mixed\; strategy) = 0$.

However $O_2$ can be reached with a positive probability using *behavioral strategy.*
If we consider,

$$b_i(u_i/\eta^i) = \begin{cases} L \text{ with probability } \frac{1}{2} \\ R \text{ with probability } \frac{1}{2} \end{cases}$$

Here, $P(\textit{reaching } O_2 \textit{ using behavioral strategy}) = \frac{1}{4}$.

We will avoid considering such pathological cases in rest of our study and instead focus on extensive games. Next we state a theorem that guarantees the existence of equivalent mixed strategy for every behavioural strategy.

## 21.2 THEOREM

**Theorem 21.1** *Consider an extensive form of game such that each info set of $P_i$ intersects every path from the root at most once. Then every behavioural strategy of $P_i$ has an equivalent mixed strategy.*

(In the example discussed in the beginning of this class the path from root intersects the information set twice and hence we need to avoid such cases.)

*Proof:*
Let us consider that the $P_i$ is at node $\hat{x}$ and he has to reach node $x$. Here $U_i(\hat{x} \to x)$ is action at $\hat{x}$ that leads to node $x$. Suppose $x^i_1, ..., x^i_{L^x_i}$= all nodes where $P_i$ acts.

Here, $L^x_i$ is the number of times $P_i$ acts along the path from $\hat{x} \to x$.

Suppose, $P_i$ plays a behavioral strategy $b_i$. Then, we have

$$Prob(b_i \textit{ leads to node } x) = \prod_{k=1}^{L^i_x} b_i(U_i(x^i_k \to x); \eta^i(x^i_k)) = P_i(x; b_i)$$

Here, $U_i(x^i_k \to x)$ is the action leading to node $x$ and $\eta^i(x^i_k)$ represents the information set containing node $x^i_k$.

Now suppose that player $P_i$ plays a mixed strategy $\sigma_i$.

Then,

$$P_i(x; \sigma_i) = \sum_{\gamma^i \in \Gamma^i(x)} \sigma_i(\gamma^i)$$

Let $\Gamma^i(x) \subseteq \Gamma^i$ denote a pure strategy that leads to node $x$.

$$Prob(\textit{game reaches node } x; \gamma) = \prod_i P_i(x; \sigma_i)$$

where $\gamma$ is mixed or behavioral strategy of players.

Construction of mixed strategies:
Every pure strategy is choice of action from information set. Set of pure strategies $\Gamma^i = \prod_{\eta^i \in I^i} U^i_{\eta^i}$, which is equivalent to picking one action from each information set.
Given a behavioral strategy $b_i$, consider

$$\sigma_i(\gamma^i) = \prod_{\eta^i \in I^i} b_i(\gamma^i(\eta^i); \eta^i)$$

where $b_i(\gamma^i(\eta^i);\eta^i)$ is the probability of taking $\gamma^i$ whenever we are at information set $\eta^i$.

For this to be probability, we need to have

$$\sum_{\gamma^i \in \Gamma_i} \sigma_i(\gamma^i) = 1$$

Now,

$$\begin{aligned}\sum_{\gamma^i \in \Gamma_i} \sigma_i(\gamma^i) &= \sum_{\gamma^i \in \Gamma^i} \prod_{\eta^i \in I^i} b_i(\gamma^i(\eta^i);\eta^i) \\ &= \sum_{a_1^i \in U^i_{\eta_1^i}, a_2^i \in U^i_{\eta_2^i}, \ldots, a_k^i \in U^i_{\eta_k^i}} \prod_{j=1}^{k} b_i(a_j^i, \eta_j^i) \\ &= 1\end{aligned}$$

The reason can be understood if we consider the case if $k = 2$.
Here,

$$\begin{aligned}\sum_{a_1^i \in U^i_{\eta_1^i}} \sum_{a_2^i \in U^i_{\eta_2^i}} b_i(a_1^i, \eta_1^i) \times b_i(a_2^i, \eta_2^i) &= \sum_{a_1^i \in U^i_{\eta_1^i}} b_i(a_1^i, \eta_1^i) \times \sum_{a_2^i \in U^i_{\eta_2^i}} b_i(a_2^i, \eta_2^i) \\ &= 1 \times 1 \quad [\textit{because } b_i(.) \textit{ are independent}] \\ &= 1\end{aligned}$$

We can express information set $I^i$ as union of two disjoint sets ie., $I^i = I_1^i \sqcup I_2^i$ where $I_1^i$ are the information sets where he can act to lead to $x$ and $I_2^i$ are irrelevant info sets.

$$\begin{aligned}P_i(x;\sigma_i) &= \sum_{\gamma^i \in \Gamma^i(x)} \sigma_i(\gamma^i) \\ &= \sum_{\gamma^i \in \Gamma^i(x)} \prod_{\eta^i \in I^i} b_i(\gamma^i(\eta^i);\eta^i) \\ &= \sum_{\gamma^i \in \Gamma^i(x)} [\prod_{\eta^i \in I_1^i} b_i(\gamma^i(\eta^i);\eta^i) \times \prod_{\eta^i \in I_2^i} b_i(\gamma^i(\eta^i);\eta^i)] \\ &\overset{(a)}{=} \sum_{\gamma^i \in \Gamma^i(x)} P_i(x;b_i) \times \prod_{\eta^i \in I_2^i} b_i(\gamma^i(\eta^i);\eta^i) \\ &= P_i(x;b_i) \times [\sum_{\gamma^i \in \Gamma^i(x)} \prod_{\eta^i \in I_2^i} b_i(\gamma^i(\eta^i);\eta^i)] \\ &= P_i(x,b_i) \textit{ , where, } \sum_{\gamma^i \in \Gamma^i(x)} \prod_{\eta^i \in I_2^i} b_i(\gamma^i(\eta^i);\eta^i) = 1 \textit{ (by using the same argument as we have used above)}\end{aligned}$$

$(a)$ **holds because each path intersects an information set at most once. When an information set intersects a path at most once,** $\prod_{\eta^i \in I_1^i} b_i(\gamma^i(\eta^i);\eta^i) = \prod_{k=1}^{L_x^i} b_i(U_i(x_k^i \to x);\eta^i(x_k^i))$**.** Thus, we have proved that $P_i(x,b_i) = P_i(x;\sigma_i)$ ie., the every behavioral strategy of $P_i$ has an equivalent mixed strategy. □