SC 631: Games and Information	**Autumn 2014**

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23.1 Stackelberg Game - Leading and Following

Definition A stackelbeg game is a statergic game in which a player first precommits to a strategy and others respond sequentially to this commitment.

- **Leaders:** They have the power to precommit a certain stratergy
- **Followers:** They respond taking into account the leader's stratergy

It is not necessary by definition, that the leader plays before the follower, though it is so in most practical cases. However, we may have a case in which the leader plays after the follower. But he would have to precommit to a strategy.

23.2 Stackelberg Equilibrium

Definition Let Γ^1 denote the set of strategies for the leader and Γ^2 the set of strategies for the follower. Let $R_2 : \Gamma^1 \to 2^{\Gamma^2}$ be the best response function of the follower. A leader stratergy $\gamma^{1\star}$ and follower stratergy $\gamma^{2\star}$ are in Stackelberg equilibrium if the following conditions hold,

- $\max_{\gamma^2 \in R_2(\gamma^{1\star})} J^1(\gamma^{1\star}, \gamma^2) = \min_{\gamma^1 \in \Gamma^1} \max_{\gamma^2 \in R_2(\gamma^1)} J^1(\gamma^1, \gamma^2)$
- $\gamma^{2\star} \in R_2(\gamma^{1\star})$

Theorem 23.1 *In a finite game with 2 players, there is always a Stackelberg equilibrium in pure stratergies.*

Proof Assume a game where P1 is the leader and P2 is the follower. Since the game is finite, Γ^1 and Γ^2 are finite and thus the best response function R_2 for any strategy γ^1 maps to a finite non-empty set. Thus if the leader's equilibrium strategy, $\gamma^{1\star}$ exits then so does a strategy for the follower, $\gamma^{2\star} \in R_2(\gamma^{1\star})$.

Further more in finding equilibrium strrategy of the leader, we always find a minimum or maximum over finite non-empty sets. Since a extremum would always exist over a finite non-empty set, a Stackelberg equilibrium strategy (γ^1, γ^2) would always exist in a finite 2-player game.

23.2.1 Relation to Nash Equilibrium

This is not necessarily the same as Nash equilibrium. If $(\gamma^{1\star\star}, \gamma^{2\star\star})$ is Nash equilibrium then,

- $\gamma^{2\star\star} \in R_2(\gamma^{1\star\star})$
- $J^1(\gamma^{1\star\star}, \gamma^{2\star\star}) = \min_{\gamma^1 \in \Gamma^1} J^1(\gamma^1, \gamma^{2\star\star})$
- $J^2(\gamma^{1\star\star}, \gamma^{2\star\star}) = \min_{\gamma^2 \in \Gamma^2} J^1(\gamma^{1\star\star}, \gamma^2)$

There is no reason why these should be the same, or be related. However, as we will see later, if we put certain conditions on the best response functions or leader strategy formulation, we can prove that the payoff of the Stackelberg equilibrium cannot be worse than that of Nash equilibrium.

23.3 Optimistic Formulation of Stackelberg Equilibrium

Assume a 2-player game where, Γ^1 denotes the set of strategies for the leader and Γ^2 the set of strategies for the follower. Let $R_2 : \Gamma^1 \to 2^{\Gamma^2}$ be the best response function of the follower.

We say that for an *optimistic formulation* or when the leader is optimistic, the Stackelberg equilibrium in pure strategies (γ^1, γ^2) exists and is such that,

- $\min_{\gamma^2 \in R_2(\gamma^{1\star})} J^1(\gamma^{1\star}, \gamma^2) = \min_{\gamma^1 \in \Gamma^1} \min_{\gamma^2 \in R^2(\gamma^1)} J^1(\gamma^1, \gamma^2)$
- And again, $\gamma^{2\star} \in R_2(\gamma^{1\star})$

Here, we take the minimum over all the follower strategies in response to a leader strategy (as opposed to the maximum we took in the previous case). The leader is thus optimistic that of all the strategies the follower may respond with he would play the one that gives the leader the best payoff.

Theorem 23.2 *Consider a finite 2 player game within the Stackelberg model such that a Nash equilibrium exists. Let $J_S^{1\star}$ be the payoff of the leader at the Stackelberg equilibrium and $J_N^{1\star}$ that at the Nash equilibrium.*

Then if R_2 is single valued or if the leader is optimistic (for optimistic formulation of the game), then

$$J_S^{1\star} \leq J_N^{1\star}$$

Proof Let P1 be the leader and P2 the follower. Assume $(\gamma^{1\star}, \gamma^{2\star})$ to be a Stackelberg equilibrium and $(\gamma^{1\star\star}, \gamma^{2\star\star})$ to be a Nash equilibrium. Clearly we will have $\gamma^{2\star} \in R_2(\gamma^{1\star})$ and $\gamma^{2\star\star} \in R_2(\gamma^{1\star\star})$.

Assume R_2 is single-valued. Now, if $\gamma^2 \in R_2(\gamma^1)$ then $R_2(\gamma^1) = \{\gamma^2\}$.

Thus, $\max_{\gamma^2 \in R_2(\gamma^1)} J^1(\gamma^1, \gamma^2) = J^1(\gamma^1, \gamma^2)$ for $\gamma^2 \in R_2(\gamma^1)$.

From the definition of Stackelberg equilibrium,

$$J^1(\gamma^{1\star}, \gamma^{2\star}) = \min_{\gamma^1 \in \Gamma^1, \gamma^2 \in R_2(\gamma^1)} J^1(\gamma^1, \gamma^2)$$

Thus for all $\gamma^1 \in \Gamma^1$, $\gamma^2 \in R_2(\gamma^1)$,

$$\therefore J^1(\gamma^{1\star}, \gamma^{2\star}) \leq J^1(\gamma^1, \gamma^2)$$

For $\gamma^1 = \gamma^{1\star\star}$ and $\gamma^{2\star\star} \in R_2(\gamma^{1\star\star})$,

$$J^1(\gamma^{1\star}, \gamma^2) \leq J^1(\gamma^{1\star\star}, \gamma^{2\star\star})$$

$$\therefore J_S^{1\star} \leq J_N^{1\star}$$

If R_2 is not single-valued, then the leader is optimistic. Thus,

$$J_S^{1\star} = J^1(\gamma^{1\star}, \gamma^2) = \min_{\gamma^1 \in \Gamma^1} \min_{\gamma^2 \in R^2(\gamma^1)} J^1(\gamma^1, \gamma^2)$$

$$\leq \min_{\gamma^2 \in R_2(\gamma^{1\star\star})} J^1(\gamma^{1\star\star}, \gamma^2)$$

$$\leq J^1(\gamma^{1\star\star}, \gamma^{2\star\star}) = J_N^{1\star}$$

23.4 Stackelberg Game for 3 Players

Consider a 3-player game with players P1, P2 and P3 having respective strategy sets Γ^1, Γ^2, Γ^3 and respective best response functions R_1, R_2, R_3. We define the conditions for the Stackelberg equilibrium $(\gamma^{1\star}, \gamma^{2\star}, \gamma^{3\star})$.

Without loss of generality among the players, there can be three cases.

23.4.1 P1 is the leader, P2 and P3 are followers

- $\max\limits_{\substack{\gamma^2 \in R_2(\gamma^{1\star}, \gamma^3) \\ \gamma^3 \in R_3(\gamma^{1\star}, \gamma^2)}} J^1(\gamma^{1\star}, \gamma^2, \gamma^3) = \min\limits_{\gamma^1 \in \Gamma^1} \max\limits_{\substack{\gamma^2 \in R_2(\gamma^1, \gamma^3) \\ \gamma^3 \in R_3(\gamma^1, \gamma^2)}} J^1(\gamma^1, \gamma^2, \gamma^3)$
- $\gamma^{2\star} \in R_2(\gamma^{1\star}, \gamma^{3\star})$
- $\gamma^{3\star} \in R_3(\gamma^{1\star}, \gamma^{2\star})$

23.4.2 P1 and P2 are leaders, P3 is follower

- $\max\limits_{\gamma^3 \in R_3(\gamma^{1\star}, \gamma^{2\star})} J^1(\gamma^{1\star}, \gamma^{2\star}, \gamma^3) = \min\limits_{\gamma^1 \in \Gamma^1} \max\limits_{\gamma^3 \in R_3(\gamma^1, \gamma^{2\star})} J^1(\gamma^1, \gamma^{2\star}, \gamma^3)$
- $\max\limits_{\gamma^3 \in R_3(\gamma^{1\star}, \gamma^{2\star})} J^2(\gamma^{1\star}, \gamma^{2\star}, \gamma^3) = \min\limits_{\gamma^2 \in \Gamma^2} \max\limits_{\gamma^3 \in R_3(\gamma^{1\star}, \gamma^2)} J^2(\gamma^{1\star}, \gamma^2, \gamma^3)$
- $\gamma^{3\star} \in R_3(\gamma^{1\star}, \gamma^{2\star})$

23.4.3 P1 leads P2 and P2 leads P3

- $\max_{\gamma^2 \in S^2(\gamma^{1\star})} \max_{\gamma^3 \in R_3(\gamma^{1\star},\gamma^2)} J^1(\gamma^{1\star}, \gamma^2, \gamma^3) = \min_{\gamma^1 \in \Gamma^1} \max_{\gamma^2 \in S^2(\gamma^1)} \max_{\gamma^3 \in R_3(\gamma^1,\gamma^2)} J^1(\gamma^1, \gamma^2, \gamma^3)$

  where, $S^2(\gamma^1) = \left\{ \gamma^2 \in \Gamma^2 \,\middle|\, \max_{\gamma^3 \in R_3(\gamma^1,\gamma^2))} J(\gamma^1, \gamma^2, \gamma^3) \leq \max_{\gamma^3 \in R_3(\gamma^1,\gamma^{2\prime}))} J(\gamma^1, \gamma^{2\prime}, \gamma^3) \, \forall \gamma^{2\prime} \in \Gamma^2 \right\}$
- $\gamma^{2\star} \in S^2(\gamma^{1\star})$
- $\gamma^{3\star} \in R_3(\gamma^{1\star}, \gamma^{2\star})$