SC 631: Games and Information **Autumn 2014**

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3.1 Simultaneous Move Games

Simultaneous move games are games in which the information about one player's move is not available to the other players i.e both players move simultaneously, or even if they do not move simultaneously, the later players are unaware of the earlier players' actions (making them effectively simultaneous).

3.1.1 Weakly Dominated Strategies

We assume the player's are maximizing their payoff's. A strategy $s_i \in S_i$ is *weakly dominated* by a strategy $t_i \in S_i$ for P_i if $U_i(t_i, s^{-i}) \geq U_i(s_i, s^{-i}), \forall s^{-i} \in S^{-i}$ and the inequality is strict for at least one s^{-i}.

In other words, a strategy s_i is weakly dominated by another strategy t_i if, regardless of what any of the other players do, the t_i earns a payoff at least as high as s_i, and, t_i earns a strictly higher payoff for some profile of the other players' strategies.

3.1.2 Strictly Dominated Strategies

A strategy $s_i \in S_i$ is strictly dominated by a strategy $t_i \in S_i$ for P_i if $U_i(t_i, s^{-i}) > U_i(s_i, s^{-i}), \forall s^{-i} \in S^{-i}$.

In other words, a strategy s_i is strictly dominated by another strategy t_i if, regardless of what any of the other players do, the strategy t_i earns a payoff higher than s_i. A strictly dominated strategy is also weakly dominated. A rational player will never choose a strictly dominated strategy. Strategies left after elimination of strictly dominated strategies are called *rationalisable strategies*.

3.1.3 Example

Let us consider situation of two generals in a battle field and model them as a simultaneous game to find the Nash equilibrium. Imamura and Kenny are two generals from Japan and US respectively. Imamura's objective is to transport the army troops to war destination. He has two strategies, either a "short" path or a "long" path. Kenny's objective is to bomb the Japanese troops. He has two options as well, the "short" path and the "long" path.

Assuming that rationality is common knowledge, we can think of the situation as a simultaneous move game with the respective utilities given as below.

Kenny \ Imamura	Short	Long
Short	(2,-2)	(2,-2)
Long	(1,-1)	(3,-3)

Both players try to maximize their respective payoff's. For Imamura "Long" is weakly dominated by "Short". So it can be eliminated. Now both players see the game as shown below.

Kenny \ Imamura	Short
Short	(2,-2)
Long	(1,-1)

Now "Long" is strongly dominated by "Short" for Kenny. Therefore, (Short, Short) $\Rightarrow (2, -2)$ is the Nash Equilibrium.

3.1.4 Theorems

Theorem 3.1. *Let $G = (\mathcal{N}, \{S_i\}_{i\in\mathcal{N}}, \{U_i\}_{i\in\mathcal{N}})$ be a game and let $\widehat{G} = (\mathcal{N}, \{\widehat{S}_i\}_{i\in\mathcal{N}}, \{U_i\}_{i\in\mathcal{N}})$ be another game such that $\widehat{S}_i \subseteq S_i \ \forall\ i \in \mathcal{N}$,i.e. $\widehat{G}$ has been obtained from G by the elimination of a few strategies. If $s^*$is an equilibrium of G and $s_i^* \in \widehat{S}_i \ \forall\ i \in \mathcal{N}$ then s^* is an equilibrium of $\widehat{G}$.*

Proof. Since s^* is an equilibrium of G, $U_i(s^*) \geq U_i(s_i, s^{-i*}),\ s_i \in S_i \ \forall\ i$. But $\widehat{S}_i \subseteq S_i$. Since $s_i^* \in \widehat{S}_i \ \forall\ i$, $s^* \in \widehat{S} = \prod_{i\in\mathcal{N}} \widehat{S}_i \Rightarrow U_i(s^*) \geq U_i(s_i, s^{-i*}),\ s_i \in \widehat{S}_i \ \forall\ i$. Therefore s^* is an equilibrium of $\widehat{G}$.

□

Theorem 3.2. *Let $\widehat{G}$ be obtained from G by elimination of a weakly dominated strategy $\widehat{s}_j$ for some player j. Then every equilibrium of $\widehat{G}$ is an equilibrium of G.*

Proof. Let $\widehat{S}_i$ be strategies of player i in game $\widehat{G}$

$$\widehat{S}_i = \begin{cases} S_i & i \neq j \\ S_j \setminus \{\widehat{s}_j\} & i = j \end{cases}$$

Let s^* be an equilibrium of $\widehat{G}$. Then

$$U_i(s^*) \geq U_i(s_i, s^{-i*}),\ \forall s_i \in \widehat{S}_i \ \forall\ i \in \mathcal{N}. \tag{3.1}$$

For $i \neq j$, removal of strategy $\widehat{s}_j$ doesn't affect their equilibrium condition $\Rightarrow U_i(s^*) \geq U_i(s_i, s^{-i*}),\ \forall s_i \in S_i \ \forall\ i \neq j$.

Now we need to show that switching to $\widehat{s}_j$ isn't beneficial. $\widehat{s}_j$ is a weakly dominated strategy. $\Rightarrow \exists\ t_j \in S_j$ such that $U_j(t_j, s^{-j}) \geq U_j(\widehat{s}_j, s^{-j}),\ \forall\ s^{-j} \in S^{-j}$
$\Rightarrow U_j(t_j, s^{-j*}) \geq U_j(\widehat{s}_j, s^{-j*})$ But $t_j \in \widehat{S}_j$ because a strategy cannot weakly dominate itself. Substituting $s_i = t_j$ and $i = j$ in (3.1) we get $U_j(s^*) \geq U_j(t_j, s^{-j*}) \geq U_j(\widehat{s}_j, s^{-j*})$ Therefore $U_i(s^*) \geq U_i(s_i, s^{-i*})\ \forall s_i \in S_i,\ \forall\ i \in \mathcal{N}$ □

Consequence: Elimination of weakly dominated strategies does not create new equilibria.

Theorem 3.3. *Let $\widehat{G}$ be obtained from G by elimination of a strictly dominated strategy $\widehat{s}_j$ for some player j. Then the set of equilibria of G is equal to the set of equilibria of $\widehat{G}$.*

Proof. Let E be the set of equilibria of G and $\widehat{E}$ be the set of equilibria of $\widehat{G}$. From Theorem 3.2 $\widehat{E} \subseteq E$ (Since every strictly dominated strategy is also weakly dominated). It suffices to show that $E \subseteq \widehat{E}$.
Let $s^* \in E$. To show that $s^* \in \widehat{E}$, we only need to show that s^* belongs to the set of strategies of $\widehat{G}$(using Theorem 3.1). As $\widehat{G}$ was derived from G by eliminating $\widehat{s_j}$, it is only necessary to show that $s_j^* \neq \widehat{s_j}$. $\widehat{s}_j$ is strictly dominated $\Rightarrow \exists\, t_j \in S_j$ such that $U_j(\widehat{s}_j, s^{-j}) < U_j(t_j, s^{-j}), \forall s^{-j} \in S^{-j}$. Replacing s^{-j} with s^{-j*}, we get $U_j(\widehat{s}_j, s^{-j*}) < U_j(t_j, s^{-j*}) \leq U_j(s_j^*, s^{-j*})$. Hence $\widehat{s}_j \neq s_j^*$. □

Corollary: If a unique rationalisable strategy profile is left at the end of elimination of strictly dominated strategies then that is the unique Nash equilibrium of the original game.