SC 631: Games and Information **Autumn 2014**

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4.1 Common knowledge of the players

Motivation in Game Theory : Consider a scenario in which there are many players taking decisions simultaneously. The decision taken by a player would result in a payoff for the player, which is also affected by the actions of other players. Each player wants to maximize his payoff.

To tackle situations like this and come up with an appropriate decision, each player would want to know what other players know. This is where the concept of information and common knowledge arise in Game Theory.

Moreover a Nash Equilibrium for a game is affected by the information each player has and also by the common knowledge among the players.

Example of common knowledge: A fact **"A"** is mutual knowledge in a set of agents if each agent knows **"A"**, but the knowledge of any agent knowing **"A"** may not be known to any other agent in the set of agents.

Suppose each student arrives for a class knowing that the instructor will be late. The information that the instructor will be late is mutual knowledge, but each student might think that only he knows that the instructor will be late. Now, if one of the students openly says that **" instructor told me that he (instructor) will be late "** then each student knows that each student knows that instructor will be late, then each student knows that each student knows that each student knows that the instructor will be late, and so on, ad infinitum. The announcement made the mutual known fact *common knowledge* among the students.

Common Knowledge: A fact "A" is said to be a common knowledge among players of a game if all the players know "A", all the players know that all the players know "A", all the players know that all the players know that all players know "A" and so on.

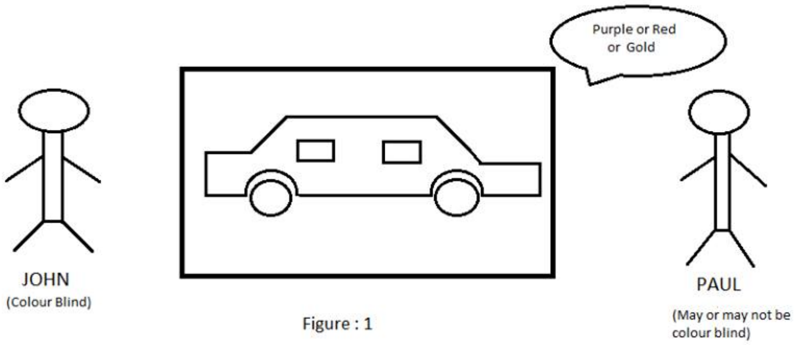

Figure : 1

Refer to Figure 1:
There are 2 players John and Paul. They both see a snapshot of a car. The colour of the car could be Red, Purple or Gold. John is colour blind and his colour blindness is known to everyone. Paul, on the other hand could be colour blind. No one other than Paul knows if he is colour blind or not. (A colour blind person can identify purple but can't distinguish between gold and red). They both see the snapshot and ask themselves what is the colour of the car?
So the parameter of interest for them is the colour of the car, which is called the states of nature. The set Y contains all the possible permutations among colours of the car, Paul being colour blind or not and John being colour blind. Y is called the set of all the states of the world.
$\Im_J$, $\Im_P$ represent the information partition of John and Paul respectively.
So we obtain :
(1) S (States of Nature)={Purple=P,Red=R,Gold=G}
(2) $\mathcal{N}$ ={John=J, Paul=P}
(3)$Y = \{P_{(J=cb,P=ncb)}, P_{(J=cb,P=cb)}, R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}\}$.
(4)$\Im_J = \{\{R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}\},\{P_{(J=cb,P=ncb)}, P_{(J=cb,P=cb)}\}\}$
(5)$\Im_P = \{\{P_{(J=cb,P=ncb)}\}, \{P_{(J=cb,P=cb)}\}, \{R_{(J=cb,P=cb)}, G_{(J=cb,P=cb)}\}, \{G_{(J=cb,P=ncb)}\}, \{R_{(J=cb,P=ncb)}\}\}$.

(6) $\chi : Y \to S$ is a function from the set of the states of the world to the set of the states of nature.

In the above notations $P_{(J=cb,P=ncb)}$ signifies the state of the world when the colour of the car is Purple, John is colour blind and Paul is not colour blind.
In the above example we obtained a partition for John and Paul. An element in the partition for **player** i contain those states of the world among which **player** i can't distinguish. For instance consider the partition for John ;
$\Im_J = \{\{R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}\},\{P_{(J=cb,P=ncb)}, P_{(J=cb,P=cb)}\}\}$. Since John doesn't know if Paul is colour blind or not and John himself can't distinguish between Red and Gold (John is colour blind and *knows* he is colour blind) hence for John $R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}$ states of the world are identical and indistinguishable. Hence these states of the world belong to the same element in the partition for John.
The function χ relates each state of the world to a state of nature. The function $\chi : Y \longrightarrow S$, in the current instance is defined as:
$\chi(P_{(J=cb,P=ncb)}) = P$
$\chi(P_{(J=cb,P=cb)}) = P$
$\chi(R_{(J=cb,P=ncb)}) = R$
$\chi(R_{(J=cb,P=cb)}) = R$
$\chi(G_{(J=cb,P=ncb)}) = G$

$\chi(G_{(J=cb,P=cb)}) = G$
Since the parameter of interest in the example is the colour of the car so when ever the state of the world has colour of the car as purple we define χ of that state of the world to be purple. Similarly for those states of the world that have colour of the car as red we define χ of those states to be red.
In the same way we can define χ for the other states of the world.

Having obtained some intuition, we now formally define the Aumann Model of incomplete information over a set of states of nature.

Definition 4.1.1 *Let S be a finite set called states of nature, an Aumann model of incomplete information over the set S comprises of:*
(1) $\mathcal{N} = \{1, 2, 3, ...N\}$ which represents the set of all players.
(2) Y; which is a finite set containing all the possible states of the world.
(3) $\Im_i$ is a partition of Y for each $i \in \mathcal{N}$.
(4) $\chi : Y \longrightarrow S$ where S is the set of states of nature. This χ function relates each state of the world to a state of nature.

Partition of a set: Let Y be a set, then a partition of Y is a collection of disjoint sets $\{Y_1, Y_2, ...Y_k\}$ such that $\bigcup_{j=1}^{k} Y_j = Y$.

Definition 4.1.2 (Event) *An* ***"event"*** *is a subset of Y. If $\omega \in Y$ is the state of the world then we say event "A" obtains (occurs) in ω if $\omega \in A$.*

Definition 4.1.3 *$F_i(\omega) \longrightarrow$ It represents those states of the world that player i can't distinguish between when $\omega \in Y$ is the true state of the world.*

Definition 4.1.4 *Let ω be the state of the world and $A \subseteq Y$ be an event we say player i knows "A" in ω if $F_i(\omega) \subseteq A$.*

Equivalently we can say that player i knows "A" in ω if $\forall \omega' \in F_i(\omega)$, $\omega' \in$ A.

- **Example 1:** Let the event be A=$\{R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}\}$ and the state of the world be $\omega = G_{(J=cb,P=cb)}$ then according to the above example
  $F_J(\omega) = \{R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}\}$
  $F_P(\omega) = \{G_{(J=cb,P=cb)}, R_{(J=cb,P=cb)}\}$
  Since $F_P(\omega) \subseteq A$ so Paul knows about the event A when the state of the world is $\omega = G_{(J=cb,P=cb)}$. Similarly $F_J(\omega) \subseteq A$ hence John also knows about event A in $\omega = G_{(J=cb,P=cb)}$.

- **Example 2:** Let the event be A1=$\{P_{(J=cb,P=ncb)}, R_{(J=cb,P=cb)}, G_{(J=cb,P=cb)}\}$ and $\omega = P_{(J=cb,P=ncb)}$ be the state of the world, then
  $F_J(\omega) = \{P_{(J=cb,P=cb)}, P_{(J=cb,P=ncb)}\}$
  $F_P(\omega) = \{P_{(J=cb,P=ncb)}\}$
  So here Paul knows about the event A1 in ω while John doesn't know about the event A1 in ω.

Definition 4.1.5 (Knowledge Operator) *Knowledge Operator is a function from set of all possible subsets of Y to set of all possible subsets of Y defined as $K_i(A) = \{\omega \in Y | F_i(\omega) \subseteq A\}$ for any event $A \subseteq Y$. Then: $K_i(A)$ represents the event that player i knows event A.*

Definition 4.1.6 (Common Knowledge) *If $\omega \in Y$ is the state of the world, than an event $A \subseteq Y$ is said to be common knowledge in ω if for any sequence $i_1, i_2, ...i_n \in \mathcal{N}$*

$$\omega \in K_{i_1}, K_{i_2}, ...K_{i_n}(A)$$

Suppose we modify the constructions of the previous example in such a way that the set of states of the world is obtained as Y=$\{P, R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}\}$.
Here P represents the state of the world when the colour of the car is purple and we don't bother if John and Paul are colour blind or not. In the modified setting partitions for John and Paul would be as:
$\Im_J = \{\{R_{(J=cb,P=cb)}, R_{(J=cb,P=ncb)}, G_{(J=cb,P=cb)}, G_{(J=cb,P=ncb)}\}, \{P\}\}$
$\Im_P = \{\{P\}, \{R_{(J=cb,P=cb)}, G_{(J=cb,P=cb)}\}, \{G_{(J=cb,P=ncb)}\}, \{R_{(J=cb,P=ncb)}\}\}$.

- **Example 3**: In the above modified example if the event is $A = \{P\}$ and the state of the world is P then $K_J(A) = \{P\}$ since $\{\omega \in Y | F_J(\omega) \subseteq A\} = \{P\}$ and $K_P(K_J(A)) = K_P(\{P\}) = \{P\} = A$.
  Hence the event $A = \{P\}$ is common knowledge when the state of the world is P.

Rationality a common knowledge: Consider a 2 player non cooperative one shot (simultaneous) game in which each player is rational and rationality is a common knowledge. Thus, in the state of the world where each player is rational, the fact that each player is rational is common knowledge. The strategy taken by a player would result in a payoff for the player, which is also affected by the strategy of other player. Each player wants to maximize his payoff.
One way of finding the Nash Equilibrium for such a game is by iteratively eliminating strictly dominated strategies and if a unique strategy is left in the end then it is the Nash Equilibrium.
This method of obtaining the Nash Equilibrium via elimination of strictly dominated strategies is valid only if rationality is a common knowledge.