SC 631: Games and Information **Autumn 2014**

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5.1 Some Definitions

An Aumann model of incomplete information over the set S comprises of

- $\mathcal{N} = \{1, \ldots, N\}$ set of players
- S = finite set of states of nature
- Y = set of states of the world, this set is assumed to be finite
- $\mathcal{F}_i$, a partition of Y for each i$\in \mathcal{N}$
- $\chi : Y \to S$ (a function relating each state of the world to a state of nature)

An **event** is a subset of Y, if $\omega \in Y$ is the state of the world then we say event A obtains in ω, if $\omega \in A$

Definition 5.1 *$F_i(\omega) \triangleq$ that element F of $\mathcal{F}_i$ s.t. $\omega \in F$. It is that particular partition of player i which contains ω.*

Definition 5.2 *Player i knows event A in state of the world ω if $F_i(\omega) \subseteq A \iff \forall\, \omega' \in F_i(\omega),\ \omega' \in A$*

Definition 5.3 *The knowledge operator $K_i(A)$ is the set of all those states of the world when player P_i knows event A. Mathematically,*

$$K_i(A) = \{\ \omega \in Y \mid F_i(\omega) \subseteq A\ \}$$

Remark 5.1 *If $\omega^* \in Y$ is the state of the world then P_i knows A in ω^* iff $\omega^* \in K_i(A)$. Mathematically,*

$$\omega^* \in K_i(A) \iff F_i(\omega^*) \subseteq A$$

Proof The proof of $\omega^* \in K_i(A) \implies F_i(\omega^*) \subseteq$ A trivially follows from the definition itself
The proof of $F_i(\omega^*) \subseteq$ A $\implies \omega^* \in K_i(A)$ is as follows :
Clearly, $\omega^* \in$ Y and since $K_i(A)$ is the set of all $\omega \in Y$ such that $F_i(\omega) \subseteq A$,
it naturally follows that $\omega^* \in K_i(A)$.

Remark 5.2 $K_j(K_i(A))$ *= Event that P_j knows that "P_i knows A"*

5.2 Important Properties

1. $K_i(A) \subseteq$ A
   If Player i knows A in some state of the world, then it is the case that some element of A actually occurred. The property is a direct consequence of the definition.

2. $A \subseteq B \implies K_i(A) \subseteq K_i(B)$
   This just says that if B occurs whenever A occurs and Player i knows A, then he knows B. Equivalently, if A implies B, then knowledge of A implies knowledge of B.

3. $\forall A \subseteq$ Y, $K_i K_i(A) = K_i(A)$
   This just means that P_i knows that P_i knows A is the same as saying that P_i knows A.

4. $K_i(A) \cap K_i(B) = K_i(A \cap B)$
   This just says that if Player i knows A and Player i knows B, then he knows both A and B

5. $K_i((K_i(A))^c) = (K_i(A))^c$
   This states that Player i knows that Player i doesn't know A is the same as Player i doesn't know A. This holds since Player i only knows A if some element of A has occurred and doesn't know anything that hasn't occurred.

These properties show that the knowledge operator that is derived from Aumann's model of incomplete information has many properties we would expect such an operator to have.

Definition 5.4 *Event A is* ***common knowledge*** *in state of the world ω if $\forall\ i_1, i_2, ...i_n \in \mathcal{N}, n < \infty$ $\omega \in K_{i_1} K_{i_2} K_{i_3} .....K_{i_n}(A)$. Note that $i_1, i_2, ...i_n$ need not be unique*

Lemma 5.1 *If A is common knowledge in ω, and $B \supseteq A$, then B is also common knowledge in ω.*

The above lemma directly follows from repeatedly applying property 2 for any $i_1, i_2, ...i_n \in \mathcal{N}, n < \infty$.

Lemma 5.2 *If A is common knowledge in ω, then $\omega \in K_i(A)$ and $F_i(\omega) \subseteq A$ for all i.*

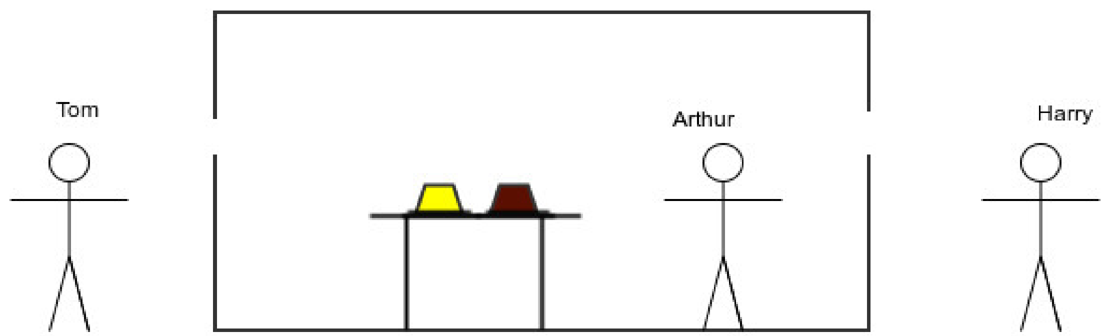
Theorem 5.3 *If A is common knowledge in ω and $\omega' \in F_i(\omega)$ for some $i \in \mathcal{N}$, then A is common knowledge in ω'.*

Proof A is common knowledge in $\omega \implies \omega \in K_i K_{i_1} K_{i_2} .....K_{i_n}(A)\ \forall i, i_1, i_2, ...i_n \in \mathcal{N}$. Suppose that $\omega' \in F_i(\omega)$. Therefore, $F_i(\omega) \subseteq K_{i_1} K_{i_2} .....K_{i_n}(A) \implies \omega' \in K_{i_1} K_{i_2} .....K_{i_n}(A)$

5.3 An Example

Consider the follwing game:

There is a room with two windows. There are two hats inside the room: Brown(B) and Yellow(Y).
There are three players in the game: Arthur(A), Tom(T) and Harry(H).



Arthur is inside the room and Tom & Harry are outside the room.
Now, Arthur wears one of the hats and knows the color of the hat that he wears.
Tom and Harry can peep through the two windows and see which hat Arthur is wearing.
For each of the two hats that Arthur chooses, there are four possible situations:

1. Both Tom and Harry peep through the window
2. Only Tom peeps through the window
3. Only Harry peeps through the window
4. None of them peeps through the window

Arthur doesn't know if anyone is peeping. Also, both Tom and Harry don't know if other person has peeped through the window

The Aumann Model of incomplete information for this problem is as follows:

1. States of nature, $S = \{B, Y\}$
   where B = Brown, Y = Yellow
2. Set of players, $\mathcal{N} = \{A, T, H\}$
   where A = Arthur, T = Tom, H = Harry
3. States of the world, $Y = \{W_{P,Q} | P \in \{B, Y\}, Q \in \{\emptyset, T, H, TH\}\}$
   where, P denotes the actual color of the hat the Arthur is wearing
   B: Brown
   Y: Yellow
   Q denotes the events of who has peeped through the window
   $\emptyset$: No one peeped through the windows
   T: Only Tom peeped through the window
   H: Only Harry peeped through the window
   TH: Both Tom and Harry peeped through the window
4. $\mathcal{F}_A = \big\{\{W_{B,Q} | Q \in \{\emptyset, T, H, TH\}\} \ \{W_{Y,Q} | Q \in \{\emptyset, T, H, TH\}\}\big\}$
   $\mathcal{F}_T = \big\{ \ \{ \ W_{B,T} \ , \ W_{B,TH} \ \} \ , \ \{ \ W_{Y,T} \ , \ W_{Y,TH} \ \} \ \{ \ W_{B,\emptyset} \ , \ W_{Y,\emptyset} \ , \ W_{Y,H} \ , \ W_{B,H} \ \} \ \big\}$
   $\mathcal{F}_H = \big\{ \ \{ \ W_{B,H} \ , \ W_{B,TH} \ \} \ , \ \{ \ W_{Y,H} \ , \ W_{Y,TH} \ \} \ \{ \ W_{B,\emptyset} \ , \ W_{Y,\emptyset} \ , \ W_{Y,T} \ , \ W_{B,T} \ \} \ \big\}$
5. Function relating each state of the world to a state of the nature $\chi : Y \to S$
   $\chi(W_{B,\bullet}) = B$
   $\chi(W_{Y,\bullet}) = Y$

Now, consider the following event

$$E = \{\text{Arthur wears Brown hat}\} = \{\ W_{B,\emptyset}\ ,\ W_{B,H}\ ,\ W_{B,T}\ ,\ W_{B,TH}\ \}$$

Then,

$K_T(\text{E}) = \{\ W_{B,T},\ W_{B,TH}\ \}$
$K_H(\text{E}) = \{\ W_{B,H},\ W_{B,TH}\ \}$
$K_T K_H(\text{E}) = \emptyset$

Hence, we can conclude that Tom doesn't know anything about what Harry knows and vice-versa.