SC 631: Games and Information	**Autumn 2014**

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6.1 Zero Sum Games

Definition A game $(\mathcal{N}, \{S_i\}_{i\in\mathcal{N}}, \{\pi_i\}_{i\in\mathcal{N}})$ with payoff functions $\pi_i : S \to R$ for each $i \in \mathcal{N}$ where $S = \prod_{i=1}^{N} S_i$ is a Zero Sum Game if

$$\sum_{i=1}^{N} \pi_i(s) = 0 \quad \forall s \in S$$

Zero Sum Games are commomly studied only for $\mathcal{N} = 2$ i.e, two player games.

Example 6.1 *For each strategy profile, the sum of utility for each player is 0.*

$(a, -a)$	$(b, -b)$
$(c, -c)$	$(d, -d)$

Zero sum matrix game is defined as a matrix $A \in \mathbb{R}^{m\times n}$ where the rows are the strategies of P_1 and columns are strategies of P_2. In zero sum game each player tries to maximize the minimum utility he can get.

Example 6.2 *This example was covered in earlier lectures. Imamura and Kenny are two generals from Japan and US respectively in a battle field. Imamuras objective is to transport the army troops to war destination. Kennys objective is to bomb the Japanese troops. Both players have two strategies, the utility function for which is given by-*

	Imamura	
Kenney	$(-2, 2)$	$(-2, 2)$
	$(-1, 1)$	$(-3, 3)$

This when represented as a zero sum game can be wriiten as-

	Kenney	
Imamura	2	1
	2	3

In the above example Imamura tries to minimize the worst possible loss. His worst possible loss is 3 and he always chooses to avoid it and play the strategy of row 1. For Kenney he tries to maximize the minimum gain he can get. His minimum gain is more when he plays the strategy of column 1.

6.2 Definitions

Security strategy (row) A strategy i^* is a **security strategy** for row player(wants to minimize the maximum loss) if

$$\max_j a_{i^*j} \leq \max_j a_{ij} \quad \forall i$$

Upper security level or security level for row player $\overline{V}(A)$ is defined as-

$$\overline{V}(A) \triangleq \max_j a_{i^*j}$$

Security strategy (col) A strategy j^* is a **security strategy** for column player(wants to maximize the minimum gain) if

$$\min_i a_{ij^*} \geq \min_i a_{ij} \quad \forall j$$

Lower security level or security level for column player $\underline{V}(A)$ is defined as-

$$\underline{V}(A) \triangleq \min_i a_{ij^*}$$

Significance: Suppose, player 1 knows that player 2 is playing his/her security strategy, then if player 1 is rational, he must play the strategy containing the security level of player 2 as that gives player 1 a maximum payoff. Since, both players play simultaneously, the players may end up with payoff's different from their security levels. But, their final payoff's will be atleast better than their respective security levels when each player plays his/her security strategy no matter what strategy other player plays.

Theorem 6.1 *Let A be Zero-Sum matrix game*

1. *∃ a unique security level for each player.*
2. *∃ at least one security strategy for each player.*
3. $\underline{V}(A) \leq \overline{V}(A)$

Proof

1. Clearly follows from the definitions. Unique levels are given by,

$$\overline{V}(A) \triangleq \max_j a_{i^*j}$$

$$\underline{V}(A) \triangleq \min_i a_{ij^*}$$

   Uniqueness follows from well ordering of real numbers over which utility function is defined.

2. A security strategy for row player is given by some i^* where

$$\max_j a_{i^*j} \leq \max_j a_{ij} \quad \forall i$$

   A security strategy for col player is given by some j^* where

$$\min_i a_{ij^*} \geq \min_i a_{ij} \quad \forall j$$

3.

$$\min_i a_{il} \leq a_{k,l} \leq \max_j a_{kj} \quad \forall k, l$$

Set $k = i^*,\ l = j^*$ to get,

$$\min_i a_{ij^*} \leq a_{i^*,j^*} \leq \max_j a_{i^*j}$$

$$\Rightarrow \min_i a_{ij^*} \leq \max_j a_{i^*j}$$

$$\Rightarrow \underline{V}(A) \leq \overline{V}(A)$$

Example 6.3 *The following is a two player zero sum game matrix. The row player P_1 has three strategies r_1, r_2, r_3 and column player P_2 has four strategies c_1, c_2, c_3, c_4. For row player, playing r_1 means, in the worst case scenario his payoff is 3. Similarly for r_2 and r_3 his worst possible payoff's are 2 and 2 respectively. The row player tries to reduce his worst payoff possible and he chooses to play either r_2 or r_3. These are the security strategies of P_1. The worst payoff possible by the security strategy is the security level of a player and in P_1's case it is 2. In case of P_2, by playing c_1 strategy, his minimum possible payoff is -2. Similarly by playing c_2, c_3, c_4, his minimum payoff's possible are -1,0,-2 respectively. Now P_2 must choose a strategy which has the largest minimum payoff. So, P_2 plays c_3 which is his security strategy. By playing his security strategy he ascertains himself of atleast a minimum payoff of 0. This is the security level of P_2. When both players play their security strategies i.e., P_1 plays r_2 and P_2 plays r_3, their payoff's are -2 and 2 respectively. Note here that P_2 gets a payoff better than his security level whereas P_1's payoff is equal to his security level.*

P_2

	1	3	3	-2
P_1	0	-1	2	1
	-2	2	0	1

$\underline{V}(A) = 0$ $\overline{V}(A) = 2$

Example 6.4 *This is also done on the same lines as the above example.*

P_2

	4	0	1
P_1	0	-1	3
	1	2	1

$\underline{V}(A) = 1$ $\overline{V}(A) = 2$

In the above examples, when each player plays their respective security strategies they finally end up with payoff's different from their security levels. But, observe that when they play their respective security strategy, it is guarenteed to fetch a payoff, atleast better than their security level.

In Example 6.3 r_2 and r_3 are the security strategies for P_1 and c_3 for P_2. Let us assume that P1 plays r_3. Then P_2(assuming rationality) must play c_2 instead of c_3 to have a higher payoff. In other words, playing their respective strategies is not a nash equilibrium in the game. Like wise, observing the game tells us that, there is no equilibrium point. The following theorem gives the condtion for the existence of nash equilibrium in zero sum games.

Definition A pair of strategies $(\bar{i}, \bar{j})$ is said to be a saddle point if

$$a_{\bar{i}j} \leq a_{\bar{i}\ \bar{j}} \leq a_{i\bar{j}} \quad \forall i, j$$

$a_{\overline{ij}}$ is said to be the saddle point value at $(\bar{i}, \bar{j})$

Example 6.3 and 6.4 have no saddle points.

Theorem 6.2 *Let A be a zero sum matrix game such that $\underline{V}(A) = \overline{V}(A)$. Then,*

1. *A has a saddle point.*
2. *Every saddle point comprises of security strategies.*
3. *$\underline{V}(A) = V(A) = \overline{V}(A)$.*

Proof

1. Given, $\underline{V}(A) = \overline{V}(A)$
   Since,
   $$a_{i^*j} \leq \max_j a_{i^*j} = \overline{V}(A) = \underline{V}(A) = \min_i a_{ij^*} \leq a_{ij^*} \quad \forall i, j$$
   Setting $i = i^*,\ j = j^*$ where, i^* and j^* are some security strategies for row and col players respectively.
   $$\overline{V}(A) = \underline{V}(A) = a_{i^*j^*}$$
   Using this in the inequality above,
   $$a_{i^*j} \leq a_{i^*j^*} \leq a_{ij^*} \quad \forall i, j$$
   $\Rightarrow (i^*, j^*)$ is a saddle point.
2. Given $\underline{V}(A) = \overline{V}(A)$, and
   $(\bar{i}, \bar{j})$ as a saddle point.
   We show $\bar{i}$ is a security strategy.
   $$a_{\bar{i}j} \leq a_{\overline{ij}} \leq a_{i\bar{j}} \quad \forall i, j$$
   $$\Rightarrow a_{\bar{i}j} \leq a_{i\bar{j}} \quad \forall i, j$$
   $$\Rightarrow a_{i\bar{j}} \leq \max_j a_{ij} \quad \forall i, j$$
   Using this in first inequality,
   $$\max_j a_{\bar{i}j} \leq a_{i\bar{j}} \leq \max_j a_{ij} \quad \forall i$$
   $\Rightarrow \bar{i}$ ia a security strategy.

3. $\underline{V}(A) = V(A) = \overline{V}(A)$ follows from the proof of 1.

   To justify this theorem we start by showing that if there exists at least one saddle-point $(\bar{i}, \bar{j})$ then 3 must hold. As $(\bar{i}, \bar{j})$ is a saddle point,

$$a_{\bar{i}j} \leq a_{\overline{ij}} \leq a_{i\bar{j}} \quad \forall i, j$$

   This can be rewritten as,

$$a_{\overline{ij}} = \min_i a_{i\bar{j}} \quad and \quad a_{\overline{ij}} = \max_j a_{\bar{i}j}$$

   Also, since $\bar{j}$ is one particular j

$$a_{\overline{ij}} = \min_i a_{i\bar{j}} \leq \max_j \min_i a_{ij} = \underline{V}(A)$$

   Also, since $\bar{i}$ is one particular i

$$\max_j a_{\bar{i}j} = \max_j a_{\bar{i}j} \geq \min_i \max_j a_{ij} = \overline{V}(A)$$

   Therefore,

$$\overline{V}(A) \leq a_{\overline{ij}} \leq \underline{V}(A)$$

   But since we saw that for every matrix A,

$$\underline{V}(A) \leq \overline{V}(A)$$

   all these inequalities are only possible if

$$\overline{V}(A) = a_{\overline{ij}} = \underline{V}(A)$$

   Therefore, the saddle point value is unique, and is denoted as $V(A)$

Significance : A saddle point includes the security strategies of both row and column player. So, if both the players are rational they must play the saddle point. The security level at saddle point is the minimum payoff a player can get in a game. A game may or may not have a saddle point. Also a game can have many saddle points. But all the saddle points will have the same security level (will be proved later).