SC 631: Games and Information **Autumn 2014**

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7.1 Recap: Definitions

7.1.1 Zero Sum game

- A Zero Sum game is a game, where for each strategy profile, sum of payoffs of all the players is equal to zero. This game is commonly studied only for $N = 2$ players.
- In a zero-sum matrix game, payoffs of Row and Column players are of opposite sign for all strategy profiles.
- A Zero Sum matrix is a matrix $[a_{ij}]_{m\times n}$ such that a_{ij} represents payoff of Column player for the strategy profile (i, j).
- The Row player tries to minimise the outcome while the Column player tries to maximise the same.

7.1.2 Security Strategy:

Security strategy for Row player is a strategy $i^* \in S_{row}$ such that

$$\max_j a_{i^*j} \leq \max_j a_{ij} \quad \forall i$$

Here $\max\limits_j a_{i^*j}$ is called the upper security level and is represented by $\overline{V}(A)$.
Security strategy for Column player is a strategy $j^* \in S_{column}$ such that

$$\min_i a_{ij^*} \geq \min_i a_{ij} \quad \forall j$$

Here $\min\limits_i a_{ij^*}$ is called the lower security level and represented by $\underline{V}(A)$.

7.1.3 Saddle Point:

A strategy profile (i^*, j^*) is said to be saddle point if

$$a_{i^*j} \leq a_{i^*j^*} \leq a_{ij^*}, \quad \forall i, j$$

It turns out that this is also a Nash equilibrium.

7.2 Theorem

7.2.1 Statement:

Suppose A is a Zero Sum game matrix and $\overline{V}(A) = \underline{V}(A)$. Then

- The game has a saddle point.
- Any saddle point is a pair of security strategies and any pair of security strategies is a saddle point.
- Every saddle point has the same value.

7.2.2 Proof:

Part-1:

We are given,

$$\overline{V}(A) = \underline{V}(A)$$

Let us assume i^* to be a security strategy for Row player (there exists at least one) and j^* to be a security strategy for the Column player.

$$\overline{V}(A) = \max_j a_{i^*j} \geq a_{i^*j^*}$$

$$\underline{V}(A) = \min_i a_{ij^*} \leq a_{i^*j^*}$$

$$\implies \underline{V}(A) \leq a_{i^*j^*} \leq \overline{V}(A)$$

but,

$$\overline{V}(A) = \underline{V}(A) \implies \overline{V}(A) = \underline{V}(A) = a_{i^*j^*}.$$

Now consider the strategy profile (i^*, j^*),

$$a_{i^*j^*} \geq a_{i^*j} \quad \forall j$$

since $a_{i^*j^*} = \overline{V}(A) = \max_j a_{i^*j}$

$$a_{i^*j^*} \leq a_{ij^*} \quad \forall i$$

since $a_{i^*j^*} = \underline{V}(A) = \min_i a_{ij^*}$

Hence (i^*, j^*) is indeed a saddle point.

Therefore, the game has a saddle point and we can also see that i^* and j^* can be any security strategies for Row player and Column player respectively.
This implies any pair of security strategies (i, j) result in a saddle point a_{ij}.

Part-2:

Let (i^*, j^*) be a saddle point. Then by the definition of saddle point,

$$a_{i^*j} \leq a_{i^*j^*} \leq a_{ij^*} \qquad \forall i, j$$

$$\implies \max_j a_{i^*j} \le a_{ij^*} \quad \forall i$$
$$\le \max_j a_{ij} \quad \forall i$$
$$\implies i^* \text{ is a security strategy for Row player.}$$

Similarly for Column player

$$a_{i^*j^*} \le a_{ij^*} \quad \forall i, j$$
$$\implies \min_i a_{ij} \le a_{i^*j} \le \min_i a_{ij^*} \quad \forall j$$
$$\implies j^* \text{ is a security strategy for Column player.}$$

This implies any saddle point is a pair of security strategies. Note that, in the above proof we did not use the condition $\overline{V}$(A) = $\underline{V}$(A). We only used that (i^*, j^*) is a saddle point.

Now, combining part 1 and part 2 of theorem we get the following:

If $\overline{V}(A) = \underline{V}(A)$, then any pair of security strategies is a saddle point, and hence there exists a saddle point. Conversely, if there is a saddle point, then every saddle point is comprised of a pair of security strategies and the game satisfies $\overline{V}(A) = \underline{V}(A)$.

Part-3:

In part 1, we have proved that if (i, j) is a saddle point, then i and j are security strategies for Row and Column player respectively and also

$$\overline{V}(A) = \underline{V}(A) = a_{ij}$$

We also know that there are unique lower and upper security levels $\overline{V}(A)$, $\underline{V}(A)$.
Hence we can conclude that **value, a_{ij} of the saddle points, (i, j) has to be same**.

7.2.3 Summary:

- $\overline{V}(A) = \underline{V}(A)$ if and only if there exists a saddle point.
- Any saddle point is a pair of security strategies
- Any pair of security strategies is a saddle point if $\overline{V}(A) = \underline{V}(A)$.
- Every saddle point has the same value.
- If there exists a saddle point then every pair of security strategies constitute a saddle point.
- **However:** If $\overline{V}(A) = \underline{V}(A)$ then $a_{ij} = \overline{V}(A) = \underline{V}(A)$ does not necessarily imply that (i, j) is a saddle point, while the converse is true.

7.2.4 Examples:

Example 1
An amazing consequence of above theorem is illustrated in following game:

	D	R
D	(2,2)*	(0,1)
R	(1,0)	$(\frac{1}{2}, \frac{1}{2})$*

* - Nash equilibrium

As shown, there are two Nash equilibria in the above game namely (D,D) and (R,R). So, if Row player, to play a Nash equilibrium, picks D and if Column player, for whatever reason chooses R to play a Nash equilibrium, they land in (D,R) which is not a Nash equilibrium. Thus, although both wanted to play a Nash equilibrium, they end up playing a non-Nash equilibrium profile.

Consider what would have happened if this were a Zero Sum game. Since in a Zero Sum game, any pair of security strategies is a saddle point and consequently a Nash equilibrium, this problem would not have arisen.

Corollary:

If (i_1, j_1) and (i_2, j_2) are saddle points in a Zero Sum game then (i_1, j_2) and (i_2, j_1) are also saddle points and

$$a_{i_1 j_1} = a_{i_2 j_2} = a_{i_1 j_2} = a_{i_2 j_1}$$

Example 2

What are saddle points in the following example?

1	3	3	-2
0	-1	2	1
-2	2	0	1

One way to find saddle points is to check all entries. Alternatively, we can calculate $\overline{V}(A)$ and $\underline{V}(A)$ and check for their equality. We can see that $\underline{V}(A) = 2$ and $\overline{V}(A) = 0$.

So, $\overline{V}(A) \neq \underline{V}(A) \implies$ this game has no saddle point. What if $\overline{V}(A)$ were equal to $\underline{V}(A)$?

If $\overline{V}(A) = \underline{V}(A)$, then

- We can either find the security strategies for the both Row and Column players while calculating $\overline{V}(A)$, $\underline{V}(A)$ and then proceed from there to find all possible saddle points.

- Other method is that, for all elements in the matrix whose value is equal to $\overline{V}(A)$ ($= \underline{V}(A)$), check whether that is a saddle point.

7.3 Mixed Strategies:

7.3.1 Introduction

If we consider a Zero Sum game matrix A, a mixed strategy for a Row player is a probability distribution over the set of rows and similarly a mixed strategy for the columns player is a probability distribution over the set of columns.

Previously we were discussing pure strategies i.e., a player's probability distribution was such that only one of its elements had a non-zero probability associated with it i.e., one element has a probability of 1 and rest have 0.

7.3.2 Definition:

A mixed strategy for the Row player is a probability distribution y_i over the set of rows of $A \in R^{m \times n}$

$$y_1, y_2 \dots y_m \in R$$

such that $y_i \geq 0 \quad \forall i \in \{1, \dots m\}$ and $\sum_i y_i = 1$

where y_i is the probability of Row player playing the strategy i.

Similarly, a mixed strategy for the Column player is a probability distribution z_j over the set of columns of $A \in R^{m \times n}$,

$$z_1, z_2 \dots z_n \in R$$

such that $z_j \geq 0 \quad \forall j \in \{1, \dots n\}$ and $\sum_j z_j = 1$

where z_j is the probability of Column player playing the strategy j.

We represent the above probabilities as a column vector as shown below:

$\mathbf{y} \in R^m$, a column vector where $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{1^T y} = 1$ represents the mixed strategy of Row player.
$\mathbf{z} \in R^n$, a column vector where $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{1^T z} = 1$ represents the mixed strategy of Column player.

7.3.3 Motivation for Mixed Strategies:

We can think of mixed strategies in two ways.

One way to think of probabilities as distributing resources between different strategies. In the Kenney and Imamura example that we took in a previous class, instead of sending all his troops in a single direction, Kenney may decide to send some troops to one direction and remaining to the other. This is in a way assigning probabilities to his strategies.

For existence of saddle point we needed the condition

$$\overline{V}(A) = \underline{V}(A)$$

Where as in mixed strategies, as we will see, there always exists a saddle point. This implies we can always find a Nash equilibrium in mixed strategies.

7.3.4 Vector Representation of Probabilities

Let us define
$\mathbf{Y}$ = a set of all mixed strategies for a Row player.
$\mathbf{Z}$ = a set of all mixed strategies for a Column player.

Probability that the Row player follows strategy i and Column player follows strategy j given their mixed strategies $\mathbf{y}$ and $\mathbf{z}$ is

$$\mathbb{P}(\text{row} = i \text{ , column} = j) = y_i z_j$$

The above equation arises from the fact that selection of strategies by the players are independent of each other.This is due to our initial assumption of the game being a non-cooperative game.

In a mixed strategy game a player tries to maximise his expected payoff.

$$\text{Expected payoff for Column player} = \sum_{i,j} a_{ij} y_i z_j = \mathbf{y^T A z}$$
$$\text{Expected payoff for Row player} = \sum_{i,j} -a_{ij} y_i z_j = -\mathbf{y^T A z}$$

Naturally, expected payoff changes with choice of probability.

Row player tries to minimise $\mathbf{y^T A z}$ with an appropriate choice of $\mathbf{y}$ and Column player tries to maximise this with an appropriate choice of $\mathbf{z}$.

7.3.5 Mixed Security Strategy:

A vector $\mathbf{y}^* \in \mathbf{Y}$ is said to be a mixed security strategy for the Row player if

$$\max_{\mathbf{z}\in\mathbf{Z}} \mathbf{y^{*T} A z} \leq \max_{\mathbf{z}\in\mathbf{Z}} \mathbf{y^T A z} \qquad \forall \mathbf{y} \in \mathbf{Y}$$

A vector $\mathbf{z}^* \in \mathbf{Z}$ is said to be a mixed security strategy for the Column player if

$$\min_{\mathbf{y}\in\mathbf{Y}} \mathbf{y^T A z^*} \geq \min_{\mathbf{y}\in\mathbf{Y}} \mathbf{y^T A z} \qquad \forall \mathbf{z} \in \mathbf{Z}$$

$\overline{V}_m(A)$ represents the value obtained at the mixed security strategy for the Row player.
$\underline{V}_m(A)$ represents the value obtained at the mixed security strategy for the Column player.

7.3.6 Saddle Point for Mixed Strategies:

A saddle point in mixed strategies is $\mathbf{y}^* \in \mathbf{Y}$, $\mathbf{z}^* \in \mathbf{Z}$ such that

$$\mathbf{y^{*T} A z} \leq \mathbf{y^{*T} A z^*} \leq \mathbf{y^T A z^*}$$

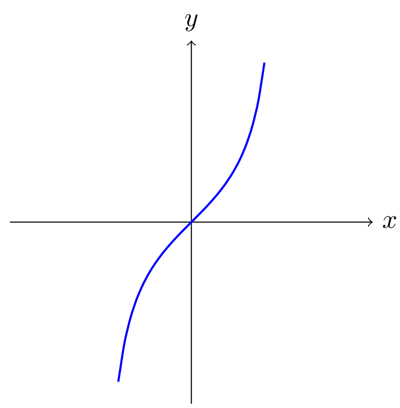
where $\mathbf{y} \in \mathbf{Y}, \mathbf{z} \in \mathbf{Z}$.

7.3.7 Finding the Security Strategies:

The following questions arise while finding the security strategies for the players, does there exist a $\mathbf{y}^*, \mathbf{z}^*$ which are the security strategies. Unlike the pure strategies where there was a finite and discrete space for determining the security strategies here there is an infinite domain from where we have find them.

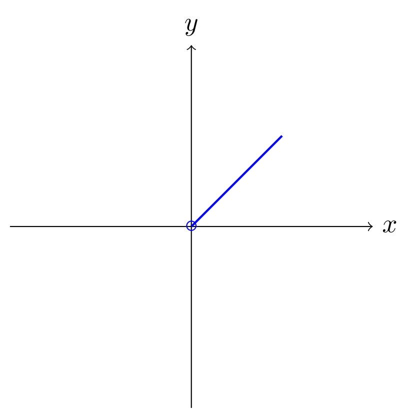
Under the following cases a function may not have a upper or lower bound:

1. If $f(x) \to +\infty(-\infty)$ as x increases.



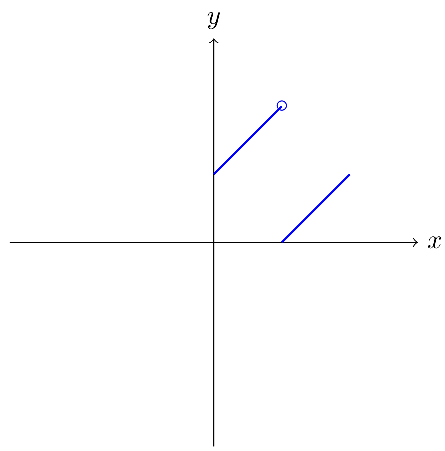
here the blue curve given by $y = \tan x \to +\infty$ as $x \to \pi/2$

2. Finding minimum of $f(x) = x$ over (0,1] where minimum is not attained for any value in domain of x.



3. When the function is discontinuous.
For example consider the problem of finding the maximum of the function $f(x)$

$$f(x) = \begin{cases} x + 1, & \text{if } x \in [0, 1) \\ x - 1, & \text{if } x \in [1, 2] \end{cases}$$



Observe here that maximum does not exist because of discontinuity of function. The hole represents the missing point (1,2).

To prove the existence of mixed strategies for the players, we first need to confirm that we don't end up in any one of the above cases.