SC 631: Games and Information **Autumn 2014**

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8.1 Mixed Strategies

8.1.1 Introduction

For a Z-S game matrix $A \in R^{m \times n}$, mixed strategy is defined as the probability distribution on pure strategies while the pure strategies are represented by the rows and columns.
Row player is represented by a vector $y \in R^m$, s.t. $y \geq 0$ & 1^Ty=1
Column player is represented as vector $z \in R^m$, s.t. $z \geq 0$ & 1^Tz=1
Here the vector $1 = (111....1)^T$

Now, let Y be the set of mixed strategies for row player and Z be the set of mixed strategies for column player.

8.1.2 Saddle Point

Saddle point for a Z-S game is defined as the tuple of strategies $(y^*, z^*) \in Y * Z$ s.t.

$$y^{*T} Az \leq y^{*T} Az^* \leq y^T AZ^* \forall y \in Y, \forall\ z \in Z$$

8.1.3 Security Strategy

Security Strategy for a row player is given by $y^* \in Y$ s.t.

$$\max_{z \in Z} y^{*T} Az \leq \max_{z \in Z} y^T Az \qquad \forall y \in Y$$

Similarly for a column player, the security strategy is $z^* \in Z$ s.t.

$$\min_{y \in Y} y^T Az^* \geq \min_{y \in Y} y^T Az \qquad \forall z \in Z$$

According to utility theory, a rational player should look at the expected value and hence must maximize the same to maximize his utility.
Expected value of utility for the given mixed strategy = $\sum_{i,j} a_{ij} y_i z_j = y^T Az$

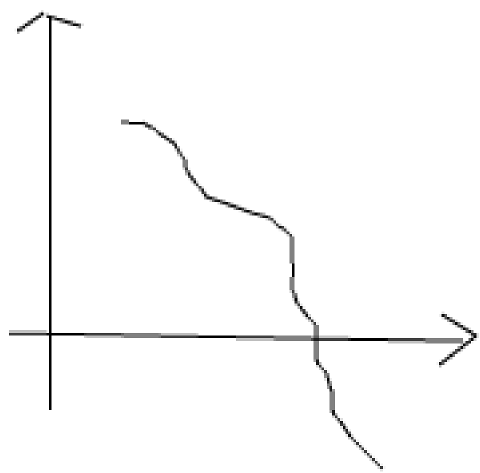
8.1.4 Solving Security Strategy Equations

$\max_{z \in Z} y^{*T} A z = \min_{y \in Y} (\max_{z \in Z} y^T A z)$

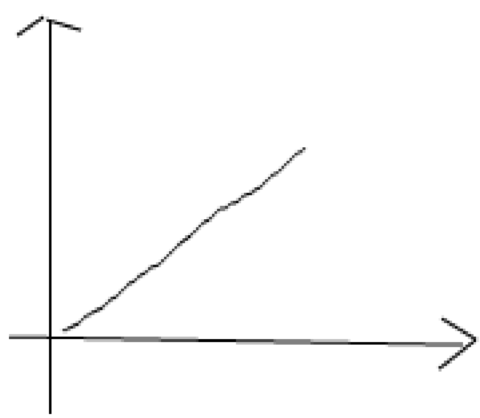
Let $\max_{z \in Z} y^T A z$ be $f(y)$.
So we need to find $\min_{y \in Y} f(y)$.

Three reasons when minimum cannot be defined:

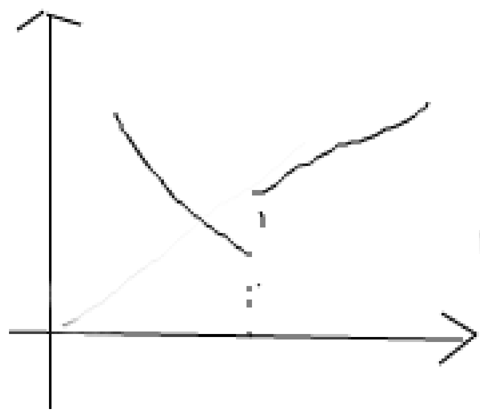
- Function decreases to $-\infty$



- Infimum is not attainable in the domain (Hole in the domain)



- Function is discontinuous (Hole in the range)



8.1.5 Weierstrass' Theorem

Theorem 8.1 *Let $f : K \to R$ be a continuous function and $K \subseteq R^n$ be a compact set. Then f attains it's infimum on K. $\Rightarrow \exists x \in K$ st. $f(x) = \inf_{z\in K} f(z)$*

Lemma 8.2 *Set $S \subseteq R^n$ is compact $\Leftrightarrow$ S is closed and bounded.*

Recall: S $\subseteq R^n$ is bounded $\Leftrightarrow \exists \infty > M > 0$ s.t. $\|x\| \leq M \quad \forall x \in S$
S $\subseteq R^n$ is closed if for any convergent sequence $x_1, x_2...$ s.t $x_i \in S$ and $x = \lim_{i\to\infty} x_i$, we have $x \in S$.

8.1.6 Existence of Mixed Strategy

We know that the mixed strategy will exist if we can obtain $\min_{y\in Y} f(y)$. Hence we only need to show that Y is compact and f is continuous. By lemma 8.2 it can be shown that Y is compact by proving that Y is closed and bounded.

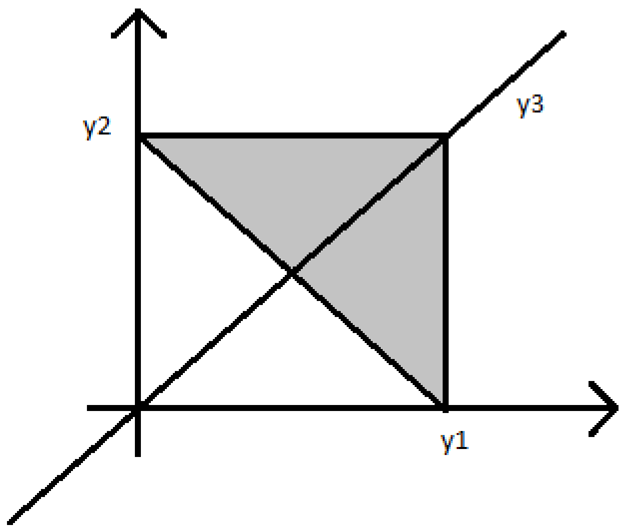
Y is bounded as $y \geq 0$ & $1^T y = 1$. Hence for any $y \in Y, y_i \leq 1$ for all $i = 1, \ldots, m$. Hence for any $y \in Y$, $\|y\| = \sqrt{y_1^2 + y_2^2 + \ldots y_m^2} \leq m$. Thus Y is bounded (we can take M above to be m)

To show closedness, consider a sequence $y^{(1)}, y^{(2)}, y^{(3)}...\overline{y} = \lim_{k\to\infty} y^{(k)}$ and $y^{(i)} \in Y \forall i$
To prove Y is closed we have to prove that $\overline{y} \in Y$: Since each $y^{(k)} \in Y$, we must have $y^{(k)} \geq 0 \Rightarrow \lim_{k\to\infty} y^{(k)} \geq 0 \Rightarrow \overline{y} \geq 0$

Again since each $y^{(k)} \in Y$, $1^T y^{(k)} = 1$

$1 = \lim_{k\to\infty} 1^T y^{(k)} = 1^T \lim_{k\to\infty} y^{(k)} = 1^T \overline{y}$

Therefore, $\overline{y} \in Y$. Hence, Y is closed and Y is also bounded. $\Rightarrow$ Y is compact.
Possible mix strategies for y_1, y_2, y_3 is shown in the figure.



8.1.7 Hogan's Paper : Theorem 7

Theorem 8.3 *Let $g : X \times Y \Rightarrow R$. If g is a continous function (on both X and Y jointly) and Y is compact then the function*
$h(x) = \max_{y \in Y} g(x, y)$ is continous.

Z is compact. (same argument as Y)
$y^T Az = \sum_{ij} a_{ij} y_i z_j$ is a polynomial in $y_1, y_2, y_3, ..., y_m$ and $z_1, z_2, z_3, ..., z_n$. Hence it is continuous in theses variables.
Therefore $f(y) = \max\limits_{z \in Z} y^T Az$ is a continuous function of y.
Therefore since Y is compact $\min\limits_{y \in Y} \max\limits_{z \in Z} y^T Az$ exists. (Weierstrass' Theorem)
$\Rightarrow \exists$ a security strategy for row players and similarly $\exists$ a security strategy for column players.

8.2 Mixed Strategies for Z-S games

Theorem 8.4 *Let A be a Z-S matrix game. Then*

1. *$\exists$ a security strategy for each player (y^*, z^*)*
2. *$\exists$ a unique security level for each player $\overline{V}_m(A) = \min_{y \in Y} \max_{z \in Z} y^T Az$*
3. *$\underline{V}(A) \leq \underline{V}_m(A) \leq \overline{V}_m(A) \leq \overline{V}(A)$*

Proof

The first two claims follow from the above proof.

We have to prove that $\underline{V}(A) \leq \underline{V}_m(A) \leq \overline{V}_m(A) \leq \overline{V}(A)$

To prove this we first prove that $\underline{V}_m(A) \leq \overline{V}_m(A)$

$\min_{\overline{y}\in Y} \overline{y}^T Az \leq y^T Az \leq \max_{\overline{z}\in Z} y^T A\overline{z}$

Put $y = y^*$ and $z = z^*$ and we get $\underline{V}_m(A) \leq \overline{V}_m(A)$

Now we prove that $\overline{V}_m(A) \leq \overline{V}(A)$

$\overline{V}(A) = \min_{y\in Y} \max_{z\in Z} y^T Az$

As $z \geq 0$ and $1^T z = 1$, hence for a fixed y, $\max_{z\in Z} y^T Az$ will be $\max_j (y^T A)_j$.

Therefore, $\overline{V}_m(A) = \min_{y\in Y} \max_j (y^T A)_j \leq \min_i \max_j a_{ij} = \overline{V}(A)$

Similarly, $\underline{V}(A) \leq \underline{V}_m(A)$.

Hence, proved.