SC 631: Games and Information **Autumn 2014**

Lecture 9: August 22

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9.1 Recap

9.1.1 Recap from the previous lecture

For a zero-sum matrix A

- There exists a mixed security strategy for each player
- There exist a unique mixed security level for each player

$$\underline{V}_m(A) = \max_{z\in Z}\min_{y\in Y} Y^T AZ$$

$$\overline{V}_m(A) = \min_{y\in Y}\max_{z\in Z} Y^T AZ = \min_{y\in Y}(\max_{j}(Y^T A)_j)$$

- The mixed security level of column player is not greater than that of the row player

$$\underline{V}(A) \le \underline{V}_m(A) \le \overline{V}_m(A) \le \overline{V}(A)$$

$$\min_{y\in Y} \overline{Y}^T AZ \le Y^T AZ \le \max_{z\in Z} Y^T A\overline{Z} \ \ \forall\ y,\ z$$

$$Put\ y = y^*, z = z^*$$

$$\underline{V}_m(A) \le \overline{V}_m(A)$$

9.1.2 Prerequisites

- Every linear programming problem, referred to as a primal problem, can be converted into a dual problem, which provides an upper bound to the optimal value of the primal problem. In matrix form, if we express the primal problem as:

  $$\min_x c^T x$$

  with constraints $Ax \ge b$ and $x \ge 0$, where c is a constant coefficient vector. Then the dual problem is given by

  $$\max_y b^T y$$

  with constraints $A^T y \le c$ and $y \ge 0$

- **Weak Duality Theorem**

  The optimal value of primal LP is greater than or equal to that of dual LP.

- **Strong duality theorem**

  If either primal or dual admits a solution, then optimal value of primal is equal to that of dual.

9.2 Von Neumann's Minimax Theorem

Theorem 9.1 *For a given zero-sum matrix A, if the probability vectors of players are given by y, z*

$$\min_{y\in Y}\max_{z\in Z} y^T Az = \max_{z\in Z}\min_{y\in Y} y^T Az$$

Proof:

One way to show this is to model both sides of the equation as a linear programming problems and check if they turn out to be dual of each other. Then if either of the problem have a feasible solution, then both will be equal by strong duality theorem.

Consider the left hand side of the equation given by

$$\min_{y\in Y}\max_{z\in Z} y^T Az$$

Observe that

$$\min_{y\in Y}\max_{z\in Z} y^T Az = \min_{y\in Y}\max_{j} (y^T A)_j$$

Note that $y^T A$ is linear in y. But, we cannot say that the matrix with maximum taken over j is a linear programming problem

We will try to model the above system as a linear programming problem. Since x, y are the probability vectors for player 1, 2 respectively, the sum of probabilities that a player opts for a strategy taken over all the strategies should be 1 and hence

$$1^T x = 1 \qquad \text{and} \qquad x \geq 0$$

$$1^T y = 1 \qquad \text{and} \qquad y \geq 0$$

Let

$$t = \max_j ((y^T.A)_j)$$

then

$$t \geq ((y^T.A)_j) \quad \forall j \in \{1, 2, \ldots, n\}$$
$$1^T.t \geq y^T.A$$

Or equivalently

$$\min_{1^T.t\geq y^T.A} t = \max_j\, (y^T.A)_j$$

Hence

$$\underline{V}_m(A) = \min_{y \in Y}(\min_{1^T.t \geq y^T.A} t)$$

For column player

$$\begin{aligned}\overline{V}_m(A) &= \max_{z \in Z} \min_{y \in Y} y^T.A.z \\ &= \max_{z \in Z} \min_{y \in Y} y^T(A.z)\end{aligned}$$

Let

$$w = \min_i((A.z)_i)$$

$$w \leq (A.z)_i \quad \forall i$$

$$1^T.w \leq A.z$$

$$\max_{1^T.w \leq A.x} w = \min_i (A.x)_i$$

$$\underline{V}_m(A) = \max_{z \in Z}(\max_{1^T.w \geq A.w} w)$$

Consider a model of the primal problem as given below.

$$\min_{\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}} \mathbf{c_1^T}.\mathbf{x_1} + \mathbf{c_2^T}.\mathbf{x_2} + \mathbf{c_3^T}.\mathbf{x_3}$$

$$A_{11}.\boldsymbol{x_1} + A_{12}.\boldsymbol{x_2} + A_{13}.\boldsymbol{x_3} \geq b_1$$

$$A_{21}.\boldsymbol{x_1} + A_{22}.\boldsymbol{x_2} + A_{23}.\boldsymbol{x_3} \leq b_2$$

$$A_{31}.\boldsymbol{x_1} + A_{32}.\boldsymbol{x_2} + A_{33}.\boldsymbol{x_3} = b_3$$

$$\boldsymbol{x_1} \;\geq 0,\; \boldsymbol{x_2} \leq 0,\; \text{ and } \boldsymbol{x_3} \text{ unrestricted}$$

Dual of this problem is given as :

$$\max_{\mathbf{w_1},\mathbf{w_2},\mathbf{w3}} \mathbf{b_1^T}.\mathbf{w_1} + \mathbf{b_2^T}.\mathbf{w_2} + \mathbf{b_3^T}.\mathbf{w_3}$$

$$A_{11}^T.\boldsymbol{w_1} + A_{12}^T.\boldsymbol{w_2} + A_{13}^T.\boldsymbol{w_3} \leq c_1$$

$$A_{21}^T.\boldsymbol{w_1} + A_{22}^T.\boldsymbol{w_2} + A_{23}^T.\boldsymbol{w_3} \geq c_2$$

$$A_{31}^T.\boldsymbol{w_1} + A_{32}^T.\boldsymbol{w_2} + A_{33}^T.\boldsymbol{w_3} = c_3$$

$$\boldsymbol{w_1} \;\geq 0,\; \boldsymbol{w_2} \leq 0,\; \text{ and } \boldsymbol{w_3} \text{ unrestricted}$$

Our Case

Comparing our case with the template problem, A_{12}, A_{22}, A_{32} are zero as there is no $\leq$ equation in the constraints

$$x_1 = y;\; x_3 = t;$$

$$c_1 = 0;\; c_2 = 0;\; c_3 = 1;$$

The condition is $\mathbf{1}^T.t \geq y.A^T$ (or) $\mathbf{1}^T.t - A^T.y \geq 0$ which implies

$$A_{11} = -A^T;\; A_{13} = 1;\; b_1 = 0$$

$$A_{21} = 0;\; A_{23} = 0;\; b_2 = 0$$

$$A_{31} = \mathbf{1}^T;\ A_{32} = 0;\ A_{33} = 0;\ b_3 = 1$$

Therefore the dual comes out to be

$$\max\left(\mathbf{w_3}\right)$$

subject to

$$-A.w_1 + \mathbf{1}.w_3 \leq 0$$

$$\mathbf{1}^T.w_1 = 1$$

Put $w_1 = z_1; w_3 = w$ then it becomes

$$\max_{w,z} w$$

subject to

$$-A.z + \mathbf{1}.w \leq 0$$

$$\mathbf{1}^T.z = 1$$

$$z \geq 0$$

which is nothing but the right hand side of the theorem. Hence the two sides are dual of each other.

In the last lecture, it is shown that in a zero sum game there always exists a mixed security level for each player, so both linear programs admit solutions. Hence, by strong duality their optimal values are equal. Hence proved Von Neumann's Minimax theorem.

9.2.1 Consequences of minimax theorem

For any Zero-Sum matrix game A

- $\underline{V}(A) \leq \underline{V}_m(A) = \overline{V}_m(A) \leq \overline{V}(A)$
- There is a saddle point in the mixed strategy.
- Every pair of mixed point is a strategy.
- If (y_1, z_1) and (y_2, z_2) are saddle points then (y_1, z_2) and (y_2, z_1) are also saddle points.
- All saddle points have the same value.
- Every saddle point in pure strategy is a saddle point in mixed strategy.

9.3 Some examples

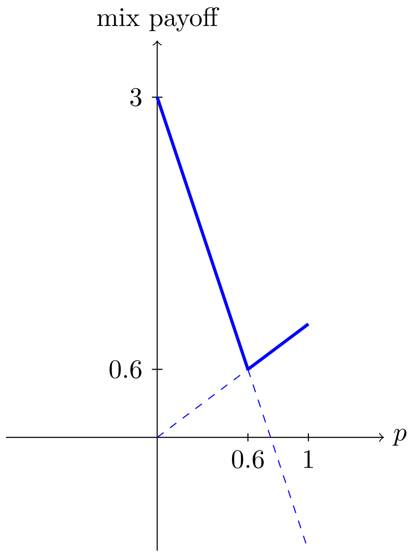
9.3.1 Example 1

Consider a 2-player zero sum game with the strategies given as

3	0
-1	1

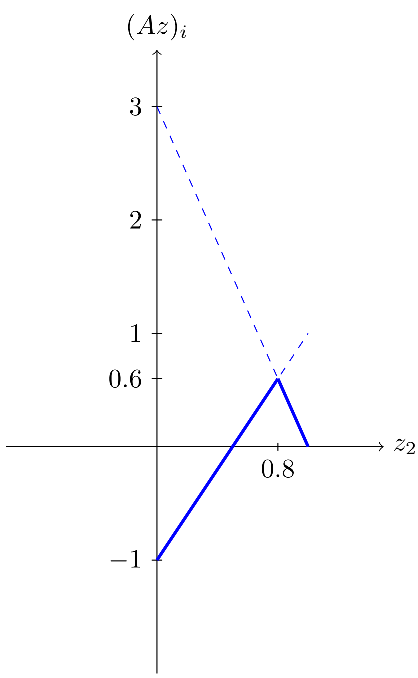
Calculate the probability for which the row and column players will get maximum payoff's for this given table. (row player will try to minimize the mix payoff and column player tries to maximize it)

For the row player: If the row player plays strategy2 with a probability of p (he plays strategy1 with a probability of $1-p$) then the mix for the column player will be $3(1-p) - 1(p) = 3 - 4p$ if he chooses strategy1 and $0(1-p) + 1(p) = p$ for strategy2. Below is the graph corresponding to these two mixes.



here the dotted blue lines are payoff $= 3 - 4p$ and payoff$=p$ $\min_p \max\{3 - 4p, p\} = 0.6$ at $p = 0.6$.This is at the point of intersection of the two lines.

Similarly for the column player : This is the graph of the mix functions for column player.



here the dashed blue lines are z_2=$2(Az)_2$+1 and z_2=-$3(Az)_1$+3 where A is the payoff matrix and z is the strategies probability distribution matrix.

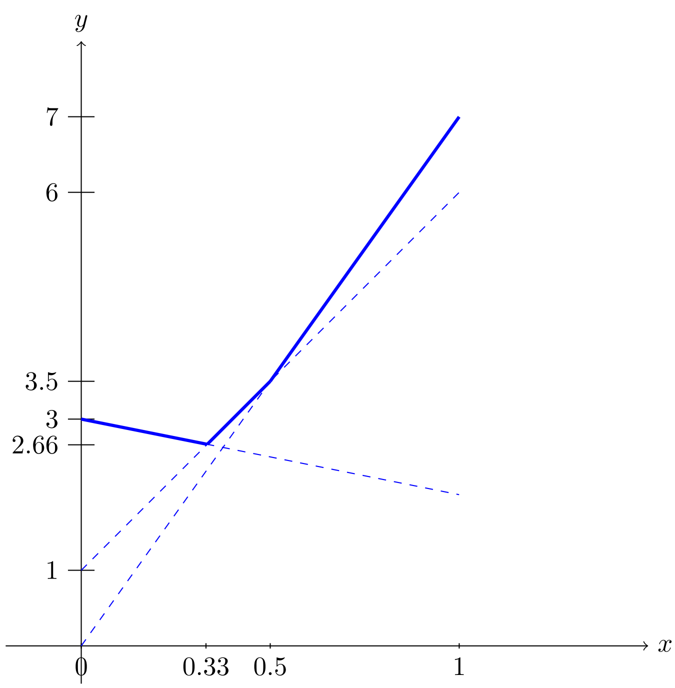
9.3.2 Example 2

For a 2-player zero sum game with the strategies given as

1	3	0
6	2	7

Lets calculate the mix for which the row and column players will get maximum payoff's for this given table. (row player will try to minimize the mix payoff and column player tries to maximize it)

For the row player: This is similar to example 1 as there are only 2 row strategies. The graph below is drawn assuming row strategy2 is played with a probabiilty p and so on the x-axis $y = p$ is drawn and x=payoff is on the y-axis.The process to find the mix payoff's is trivial.



here the dashed blue lines are y=5x+1 , y=7x and y=-x+3 The minimum of the function shown by dark line occurs when $x = 0.33$ and its value is 2.66

For the column player: Let us assume that the probability with which column player plays strategy1 be p_1 and the probabily with which he plays strategy2 be p_2, so the probability with which he plays strategy3 will be '$1 - p_1 - p_2$'.

let A be the payoff matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 6 & 2 & 7 \end{bmatrix}$$

and

$$z = \begin{bmatrix} p_1 \\ p_2 \\ 1 - p_1 - p_2 \end{bmatrix}$$

implies

$$Az = \begin{bmatrix} p_1 + 3p_2 \\ 6p_1 + 2p_2 + 7(1 - p_1 - p_2) \end{bmatrix}$$

as there are two independent variables($0 \leq p_1 + p_2 \leq 1$) are present the graph for the mix function will be in 3-dimension.The column player will try to minimize the payoff function which will occur when $p_1 + 3p_2 = 7 - p_1 - 5p_2 \implies 2p_1 + 8p_2 = 7$ and the payoff will be $((7 - 8p_2)/2) + 3p_2 = 3.5 - p_2$, to minumize this payoff set p_2=1 and so payoff=3.5