SC 631: Games and Information **Autumn 2014**

Lecture 20: October 17

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20.1 Mixed Strategies in Dynamic Games

All strategies in dynamic games are functions of actions that the player will take at a given information set. The question then arises - How does one play a mixed strategy of such a set of strategies? Consider the following two ways in which one might play a mixed strategy :

1. At each node he chooses each of his actions with a certain probability such that the total probability at each node is 1.
2. From the set of all strategies he plays each strategy with a certain probability.

At first glance one might think that the two might be equivalent. In some cases they may be, however, it is not always necessary that if he plays every strategy with a certain probability you will get , at each node, the sum of probabilities for each action at that node to be 1.

Example 20.1

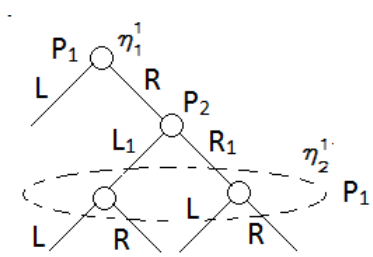

Figure 20.1: Example 1

In this game P_1 plays twice once at information set(root node) η_1^1 and then at information set η_2^1 . Let us now write all the possible strategies available to each player

$$\gamma_1^1 = \begin{cases} L & \text{if } \eta_1^1 \\ L & \text{if } \eta_2^1 \end{cases} \qquad \gamma_2^1 = \begin{cases} R & \text{if } \eta_1^1 \\ L & \text{if } \eta_2^1 \end{cases} \qquad \gamma_3^1 = \begin{cases} L & \text{if } \eta_1^1 \\ R & \text{if } \eta_2^1 \end{cases} \qquad \gamma_4^1 = \begin{cases} R & \text{if } \eta_1^1 \\ R & \text{if } \eta_2^1 \end{cases}$$

$$\gamma_1^2 = L_1 \qquad \gamma_2^2 = R_1$$

We shall now look at how the two different types of strategies play out in this game

1. The player plays each action with a certain probability
   Player 1 plays twice once at the root node and then at information set η_2^1. At the root node he plays the actions (L, R) with a probability of $(\delta, 1-\delta)$ respectively and at η_2^1 with $(\beta, 1-\beta)$ respectively.
   Player 2 plays once and has the actions (L_1, R_1) with probability $(\alpha, 1-\alpha)$ respectively.

2. The players play each strategy with a certain probability Player 1 has 4 strategies $(\gamma_1^1, \gamma_2^1, \gamma_3^1, \gamma_4^1)$ that he plays with probabilities (y_1, y_2, y_3, y_4) respectively $(y_1 + y_2 + y_3 + y_4 = 1)$.
   Whereas player 2 only has two strategies (γ_1^2, γ_2^2) which he plays with probability $(\alpha, 1-\alpha)$

If we *assume* that each of the above distributions are independent. we can derive the following relations that would equate the two cases.

$$
\begin{aligned}
y_1 &= P(\gamma_1^1) = P(L \cup L) = \delta\beta \\
y_2 &= P(\gamma_3^1) = P(L \cup R) = \delta(1-\beta) \\
y_3 &= P(\gamma_3^1) = P(R \cup L) = (1-\delta)\beta \\
y_4 &= P(\gamma_4^1) = P(R \cup R) = (1-\delta)(1-\beta)
\end{aligned}
$$

Remark 20.1 *The above relations were derived only because we assumed that the distributions were independent. If we had different assumptions we would have arrived at different relations.*

The two strategies described above are different and are the same only when the above conditions are satisfied. Note *"same"* does not mean they are equal but rather that they lead to the same payoff.

Definition 20.1 ***Behavioural strategy*** *is a function b_i that assigns a probability distribution on set $U_{\eta_i}^i$ $\forall\ \eta_i$*

$$b_i : I^i \longrightarrow Y^i$$

$$\textit{where } Y^i = \bigcup_{\eta_i} Y_{\eta_i}^i$$

$$Y_{\eta_i}^i = \textit{set of distributions on } U_{\eta_i}^i$$

Definition 20.2 ***Mixed strategy*** *σ_i is a distribution on Γ^i (the set of pure strategies)*

It can be seen in the previous example that the two strategies are *equivalent* when the relations are satisfied. Hence, we may derive an *equivalent* mixed strategy from the given behavioural strategy, *for that game.*

20.2 Equivalence of Mixed and Behavioural strategy

Definition 20.3 *A mixed strategy σ_I and a behavioural strategy b_i are equivalent if for every mixed/behavioural combination σ^{-i} and every vertex x in the game tree*

$$P(x, \sigma_i, \sigma^{-i}) = P(x, b_i, \sigma^{-i})$$

where $P()$ is the probability that we reach vertex x

Remark 20.2 *We have assumed again that the probability distributions are independent*

What the definition says is that regardless of what other players play, mixed or behavioural, for player i mixed and behavioural strategies are equivalent if the probability that the game reaches a node x is the same.

Lemma 20.1 *The probability that we reach the leaf nodes are the same. This implies that for any mixed/behavioural combination σ^{-i} if σ_i is equivalent to b_i*

$$J^i(\sigma_i; \sigma^{-i}) = J^i(b_i; \sigma^{-i})$$

$$\Rightarrow \textit{Expected payoff } = \sum_{x=leaf} J^i(x) P(x, \sigma_i, \sigma^{-i})$$

Let us ask the following questions :

1. Is there an equivalent mixed strategy for every behavioural strategy?
2. Is there an equivalent behavioural strategy for every mixed strategy?

To answer the questions let us look at a few example games

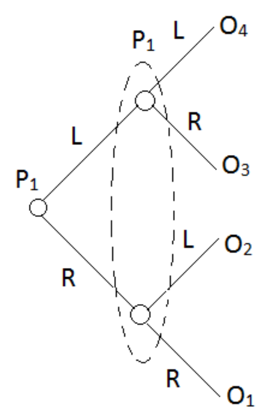

Figure 20.2: Player forgets which action he has taken

Example 20.2 *For the above game consider the mixed strategy*

$$P(L, L) = \frac{1}{2} \quad P(L, R) = 0 \quad P(R, L) = 0 \quad P(R, R) = \frac{1}{2}$$

This game represents a case when the player forgets what he has played.

Let us consider the following behavioural strategy
At information set η_1^1 he plays (L, R) with probability of $(\alpha, 1-\alpha)$ respectively and with $(\beta, 1-\beta)$ at information set η_2^1.

For the two strategies to be equivalent the probability of reaching each node must be the same in both cases. We will consider the probability to reach the leaf nodes O_1, O_2, O_3, O_4. This gives rise to the following *equations*

$$P(O_1) = (1-\alpha)(1-\beta) = \frac{1}{2}$$
$$P(O_2) = (1-\alpha)\beta = 0$$
$$P(O_3) = \alpha(1-\beta) = 0$$
$$P(O_4) = \alpha\beta = \frac{1}{2}$$

This set of linear equations does not have a solution.

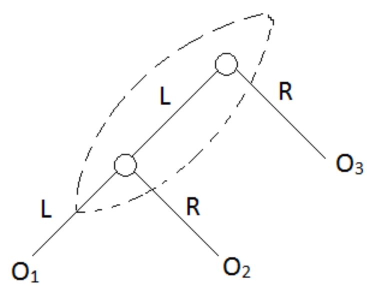

Figure 20.3: Player forgets that he played

Example 20.3 *This game violates the rules for an extended from, however, let us consider such a game. This game the player does not know if had taken an action already or not. This is different from the previous example in which the player knew that an action had been taken but was unsure of what the action was.*

The player has 2 pure strategies L and R. He will play these strategies with a certain probability. However, he can never reach O_2 in pure strategies. If he plays R he will reach O_3 and if he plays L he will reach O_1, as he is obligated to play L only.

Consider the behavioural strategy

$$P(L) = \frac{1}{2} \qquad P(R) = \frac{1}{2}$$

we can then calculate the following

$$P(O_1) = \frac{1}{4}$$
$$P(O_2) = \frac{1}{4}$$
$$P(O_3) = \frac{1}{2}$$

It can be seen that in behavioural strategy it is possible to reach node O_2 with a probability of 0.25*. This impossible in pure strategies. Mixed strategies are distributions over pure strategies, hence, cannot be achieved with any mixed strategy. Therefore there is no equivalent mixed strategy for this behavioural strategy.*

Theorem 20.2 *Consider an extensive form such that each path from the root intersects every information set at most once. Then every behavioural strategy has an equivalent mixed strategy for P_i*

Remark 20.3 *This does not say anything about the plays of the other players. They may play any strategy they choose to.*