New Insights on Generalized Nash Games with Shared Constraints: Constrained and Variational Equilibria

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***Abstract*— We consider generalized noncooperative Nash games with "shared constraints" in which there is a common constraint that players' strategies are required to satisfy. We address two shortcomings that the associated generalized Nash equilibrium (GNE) is known to have: (a) shared constraint games usually have a large number (often a manifold) of GNEs and (b) the GNE may not be the appropriate solution concept for exogenously imposed constraints. For (a), we seek a refinement of the GNE and study the variational equilibrium (VE), defined by [1], [2], as a candidate. It is shown that the VE and GNE are equivalent in a certain degree theoretic sense. For a class of games the VE is shown to be a refinement of the GNE and under certain conditions the VE and GNE are observed to coincide. To address (b), a new concept called the constrained Nash equilibrium (CNE) is introduced. The CNE is an equilibrium of the game without the shared constraint that is feasible with respect to this constraint. Sufficient conditions for the existence of a CNE are derived and relationships with the GNE and VE are established.**

I. INTRODUCTION

This paper concerns our ongoing work on noncooperative N-player games where we assume players have continuous strategy sets that are *dependent* on the strategies of their adversaries. Such games represent a generalization of classical noncooperative N-player games which have allowed for strategic interactions between players to be expressed only through their objective functions. These games are aptly referred to as *generalized Nash games*[1] and a solution concept called the *generalized Nash equilibrium* (GNE) (see definition 1.1) is frequently applied to analyze them. The GNE is an extension of the *social equilibrium* proposed by Debreu [5] and applied by Rosen [4] and Arrow and Debreu [6], who had previously studied such games. A discussion of the GNE, its properties and a historical perspective can be found in the recent survey by Facchinei and Kanzow [7].

In this paper, we focus on properties of GNEs arising in a frequently encountered class of generalized Nash games: ones in which there is a *common* constraint that all players' strategies are required to satisfy. Such games are called generalized Nash games with *shared constraints* (a name due to Rosen [4]). Formally, let $\mathcal{N} = \{1, 2, \ldots, N\}$ be a set of players; $m_1, m_2, \ldots, m_N \geq 1$ be integers and $m = \sum m_i$. Let $U_i \subseteq \mathbb{R}^{m_i}$ be agent-specific strategy sets and let $x_i \in U_i$ represent strategies of the players. Suppose the players have objective functions $\varphi_i : \mathbb{R}^m \to \mathbb{R}$, $\forall i \in \mathcal{N}$. Let x denote the tuple $(x_1, x_2, \ldots, x_N) \in U$, where $U := \prod_{i \in \mathcal{N}} U_i$, x^{-i} denote the tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and (y_i, x^{-i}) the tuple $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_N)$. Suppose that we further require that x lies in a set $\mathbb{C} \subseteq \mathbb{R}^m$. Let $K_i(x^{-i})$ and $K(x)$ be defined as

$$K_i(x^{-i}) := \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\}, \quad \forall i \in \mathcal{N}$$
$$K(x) := \prod_{i \in \mathcal{N}} K_i(x^{-i}). \tag{1}$$

The requirement that $x \in \mathbb{C}$ is the aforesaid *shared constraint*. In a generalized Nash game with shared constraint $\mathbb{C}$, player i is assumed to solve the parameterized optimization problem

$$\mathrm{A}_i(x^{-i}) \qquad \begin{array}{ll} \underset{x_i}{\text{minimize}} & \varphi_i(x_i; x^{-i}) \\ \text{subject to} & x_i \in U_i, \\ & x_i \in K_i(x^{-i}). \end{array}$$

We denote the resulting generalized Nash game by $\mathcal{G}^g = (\mathcal{N}, \Phi, U \cap K)$, where $\Phi = \{\varphi_i\}_{i \in \mathcal{N}}$. The equilibrium of the game is defined as follows[2]

Definition 1.1 (Generalized Nash equilibrium (GNE)): A strategy tuple $x \equiv (x_1, x_2, \ldots, x_N)$ is a generalized Nash equilibrium of $\mathcal{G}^g$ if

$$x_i \in SOL(A_i(x^{-i})), \qquad \forall i \in \mathcal{N}.$$

This paper seeks to address two shortcomings that generalized Nash equilibria for games such as $\mathcal{G}^g$ are known to have.

(a) Games such as $\mathcal{G}^g$ are known to admit large number of equilibria. Unique equilibria are uncommon and in many settings, there may be manifolds of equilibria (see Facchinei and Kanzow [7], page 192)[3]. Indeed, in this context, there is less clarity as to what the appropriate solution concept for such games should be.

(b) A subtle shortcoming is the underlying assumption in the formulation of $\mathcal{G}^g$ that players *anticipate* the presence of their opponents' strategies in the coupling constraint as a part of their strategic decision making. Often the requirement that the tuple of equilibrium strategies lie in $\mathbb{C}$ is imposed exogenously and it is debatable whether players are cognizant of these constraints in their decision making. Constraints imposed on countries by international treaties such as the

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[1] While this terminology is due to Harker [3], the term "coupled constrained games," has also been used as per Rosen [4].

[2] The set SOL(A) represents the solution set of problem (A)

[3] See [8] for an example of a routing game in which *every* $x \in \mathbb{C}$ is a GNE.

Kyoto protocol is one such example (see [9], [10] for models pertaining to the implications of the Kyoto protocol). Another arises in games with market clearing constraints, where the assumption that these constraints can be naturally introduced within the strategy sets of players may not always be valid.

In this work we investigate two solution concepts in order to remedy each of the above weaknesses. We recognize **(a)** as a necessity to define a *refinement* of the GNE. A refinement of an equilibrium concept is a collection of equilibria that satisfy a certain rule, where the rule has the property that any strategy tuple satisfying it is also an equilibrium[4]. A refinement provides a way of selecting one or a few of the many equilibria that a game may have. For a refinement to be legitimate and indeed useful for a large class of problems, it is imperative that it possess the feature that every problem with a nonempty set of equilibria has a nonempty refinement and that the set of refined equilibria be significantly smaller than the set original set of equilibria (see [11], [12] for refinements of Nash equilibria). As a candidate refinement for the GNE, we study a concept called the variational equilibrium (VE) (see definition 2.1) which was defined by [7], [2]. It was shown in [1] that every VE is a GNE. Moreover, in the case where $\mathbb{C} = \{x \mid c(x) \geq 0\}$ for some C^1 concave function c, the VE is the tuple of strategies that simultaneously satisfies the KKT conditions of all A_i, $i \in \mathcal{N}$ with the further requirement that the Lagrange multipliers corresponding to $c(\cdot) \geq 0$ be equal for all players [1, Theorem 2.2]. Recall that in [4], such an equilibrium was called the *normalized equilibrium*. To study the VE as a refinement of the GNE, we investigate of every problem with a nonempty set of GNEs admits at least one VE. Our research on this question forms the content of section II. We show very under general conditions, that the GNE and VE are " equivalent" in a degree theoretic sense (Theorem 9). Moreover, for a certain class of games the VE is a refinement of the GNE. By studying the nature of the set-valued mapping K, some conditions are obtained under which the GNE is also a VE.

In section III we address **(b)**. As regards **(b)**, we contend that the appropriate solution concept would be one in which the shared constraints are exogenous to the game. We term such an equilibrium as a *constrained Nash equilibrium (CNE),* a new solution concept introduced in this work. The CNE is an equilibrium of a related game played with the same objective functions but with strategy sets devoid of the common constraint, such that the equilibrium of the related game lies in $\mathbb{C}$. Specifically, imagine that instead of A_i the i^{th} player solves

$$\mathrm{B}_i(x^{-i}) \qquad \begin{array}{ll} \underset{x_i}{\text{minimize}} & \varphi_i(x_i; x^{-i}) \\ \text{subject to} & x_i \in U_i \end{array}$$

with the further requirement that *at equilibrium*, x_i be chosen such that $x \in \mathbb{C}$. Thus the players are assumed to not anticipate the presence of the constraint that couples their decisions. Instead x is chosen so that the requirement that $x \in \mathbb{C}$ is imposed exogenously and need be met only at equilibrium. We denote this game as $\mathcal{G}^o = (\mathcal{N}, \Phi, U|\mathbb{C})$, and define the CNE as follows.

Definition 1.2: A constrained Nash equilibrium of $\mathcal{G}^o$ satisfies

$$x \in \mathbb{C} \qquad s.t. \qquad x_i \in SOL(B_i(x^{-i})), \quad \forall i \in \mathcal{N}.$$

The game *without* the outer requirement that $x \in \mathbb{C}$ is a conventional Nash game, which we shall call the *unconstrained game* $\mathcal{G}^u = (\mathcal{N}, \Phi, U)$. In section III we present conditions for existence of a CNE for $\mathcal{G}^o$. We relate the CNE to the VE and GNE and show that under certain conditions, the GNE, VE and CNE coincide.

Shortcomings **(a)** and **(b)** mentioned above are pertinent to those studying the abstract properties of the GNE, to those designing real world applications and to those devising algorithms to compute GNEs for such games. Specifically in connection with **(a)**, it must be emphasized that computating the GNE is more challenging since it involves solving a *quasi variational inequality* (see section I-A.1), on the other hand computing the VE is much more tractable. As a result, in the community of operations research and computational game theory it has been a common practice [7], [13], [14] to compute the VE to substitute the computation of the GNE. This approach presupposes the existence of a VE whenever the GNE exists. However no theoretical justification has as yet been available for this and is one that we seek to provide.

A variational approach has been adopted towards the analysis and special efforts have been made to provide arguments that are geometric rather than algebraic. Because of the generality of the assumptions made the use of Brouwer degree theory for analysis was unavoidable. In the next section we outline assumptions made and provide some background for the analysis carried out in the paper.

A. Background for the analysis

Following are the assumptions we make.

Assumption 1: For each $i \in \mathcal{N}$,

1) the objective function $\varphi_i \in C^2$ and $\varphi_i(x_i; x^{-i})$ is convex in x_i for all x^{-i},
2) the strategy set U_i is closed and convex and
3) unless otherwise mentioned, $\mathbb{C}$ is closed, convex and has a nonempty interior.

We provide below a brief background on variational inequalities and Brouwer degree. For concepts from convex analysis we refer the reader to Rockafellar [15]. The reader may choose to skip the following material and return to it as per necessity.

1) Variational inequalities: Recall problems A_i from section I. Under assumption 1, x_i is optimal for $\mathrm{A}_i(x^{-i})$ if and only if

$$\nabla_i \varphi_i(x)^T (y_i - x_i) \geq 0, \qquad \forall y_i \in U_i \cap K_i(x^{-i}).$$

[4]We consider both, the refined equilibria and the rule generating them, as the *refinement*.

Thus if x is a GNE of $\mathcal{G}^g$, the above condition must hold for all $i \in \mathcal{N}$. Define $F : \mathbb{R}^m \to \mathbb{R}^m$ as

$$F(x) = \begin{pmatrix} \nabla_1 \varphi_1(x) \\ \vdots \\ \nabla_N \varphi_N(x) \end{pmatrix}.$$

Then x is an GNE of $\mathcal{G}^g$ if and only if it solves the quasi-variational inequality (QVI) [16]

$$\begin{array}{l} \text{Find } x \in K(x) \cap U \text{ such that} \\ F(x)^T(y - x) \geq 0 \quad \forall y \in K(x) \cap U. \end{array} \qquad (\text{QVI}(K \cap U, F))$$

The equilibrium of the unconstrained game $\mathcal{G}^u$ mentioned after Definition 1.2 can likewise be characterized as the solution of a variational inequality (VI). Specifically x is an equilibrium of $\mathcal{G}^u$ if and only if x solves the VI

$$\begin{array}{l} \text{Find } x \in U \text{ such that} \\ F(x)^T(y - x) \geq 0 \quad \forall y \in U. \end{array} \qquad (\text{VI}(U, F))$$

The natural map of VI(U, F), $\mathbf{F}_U^{\text{nat}} : \mathbb{R}^m \to \mathbb{R}^m$, defined as

$$\mathbf{F}_U^{\text{nat}}(v) = v - \Pi_U(v - F(v))$$

where $\Pi_U : \mathbb{R}^m \to U$ is the Euclidean projection on U, provides an equation reformulation of the VI. x is a solution of VI(U, F) if and only if $\mathbf{F}_U^{\text{nat}}(x) = 0$. Let $\widetilde{\mathbf{F}_K^{\text{nat}}} : \text{dom}(K) \to \mathbb{R}^m$ denote a similar natural map for QVI(K, F) defined as

$$\widetilde{\mathbf{F}_K^{\text{nat}}}(v) := \mathbf{F}_{K(v)}^{\text{nat}}(v) = v - \Pi_{K(v)}(v - F(v)),\ \forall v : K(v) \neq \emptyset.$$

We have the following proposition.

Proposition 1: A vector $v \in SOL(VI(U, F))$ if and only if $\mathbf{F}_U^{\text{nat}}(v) = 0$ and $v \in SOL(QVI(K, F))$ if and only if $\widetilde{\mathbf{F}_K^{\text{nat}}}(v) = 0$.

Proof: See [16] for a proof which relies chiefly on the property of projection on closed convex sets that follows. ■

Lemma 2 ([16]): Let $D \subseteq \mathbb{R}^m$ be a closed convex set and x be a point in $\mathbb{R}^m$. Then the projection of x on D, $\Pi_D(x)$, satisfies

$$(y - \Pi_D(x))^T(\Pi_D(x) - x) \geq 0 \quad \forall y \in D.$$

Variational inequalities (VI) and quasi-variational inequalities (QVI) are problems that are more general than convex optimization problems and game-theoretic problems and serve as a useful tool for analysis. We refer the reader to [16], [2] for a thorough course on finite dimensional VIs and QVIs and [17] for an introduction to infinite dimensional VIs.

2) Brouwer degree theory: The following information on the Brouwer degree has been sourced form [18], [19], [20]. The Brouwer degree of a function is a topological concept whose value allows us to claim the existence of zeroes of the function in a specified open neighbourhood. Indeed under some regularity of the function the Brouwer degree of the function *equals* the number of zeros of the function in the neighbourhood. To make this more precise we need the following definition.

Definition 1.3 (Singular points): Let $D \subseteq \mathbb{R}^m$, and $\phi : D \to \mathbb{R}^m$ be a C^1 function. If $\det \nabla\phi(u) = 0$, then u is called a *singular point* and $\phi(u)$ is called a singular *value* of ϕ. We denote $S_\phi := \{\phi(u) : u \in \text{dom}(\phi), \det(\nabla\phi(u)) = 0\}$. Points (and values) that are not singular are called *regular.*

Let $\Omega \subset \mathbb{R}^m$ be an open bounded set and $f : \overline{\Omega} \to \mathbb{R}^m$, $f \in C$ and $p \in \mathbb{R}^m \backslash f(\partial\Omega)$. We say the Brouwer degree of f with respect to p on Ω, denoted as $\deg(f, \Omega, p)$, is *well defined* if $p \notin f(\partial\Omega)$ and it exists only for such p. Let $\mathbf{1} : \mathbb{R}^m \to \mathbb{R}^m$ denote the identity map. $\deg(f, \Omega, p)$ is an integer with the following properties.

1) (*Normalization*) $\deg(\mathbf{1}, \Omega, p) = 1$ if and only if $p \in \Omega$.
2) (*Solvability*) $\deg(f, \Omega, p) \neq 0$ then $f(x) = p$ for some $x \in \Omega$.
3) (*Homotopy invariance*) $\deg(H(\cdot, t), \Omega, p)$ is independent of $t \in [0, 1]$ for any continuous function $H : \overline{\Omega} \times [0, 1] \to \mathbb{R}^m$ and $p \in \mathbb{R}^m$ such that $p \notin \cup_{t \in [0,1]} H(\partial\Omega, t)$.
4) If $f \in C^1$ and $p \notin S_f$, then $\deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} \text{sgn} \det(\nabla f(x))$.
5) Let $g : \overline{\Omega} \to \mathbb{R}^m$, be any C^1 function such that

Using that f has compact domain $\overline{\Omega}$, it can be shown that this sum in 4 is finite [20]. Of particular importance to us is property of solvability and that of invariance under homotopy. Note that the converse of property 2 is not true. i.e. if $f(x) = p$ for some x in Ω and $\deg(f, \Omega, p)$ is well defined it does not imply that the degree not zero. The function H above is called a homotopy between $H(0, \cdot)$ and $H(1, \cdot)$ and may be interpreted as a continuous deformation between the images of $H(0, \cdot)$ and $H(1, \cdot)$.

II. Generalized Nash and Variational Equilibria

Recall from section I-A.1 that the GNE of $\mathcal{G}^g$ form the solutions of QVI$(K \cap U, F)$. A related VI is the following:

$$\begin{array}{l} \text{Find } x \in U \cap \mathbb{C} \text{ such that} \\ F(x)^T(y - x) \geq 0 \quad \forall y \in U \cap \mathbb{C}. \end{array} \qquad (\text{VI}(\mathbb{C} \cap U, F))$$

The solution of this VI is defined to be a variational equilibrium.

Definition 2.1 (Variational equilibrium (VE) [7], [2]): If x is a solution of VI$(U \cap \mathbb{C}, F)$ then x is said to be a variational equilibrium of $\mathcal{G}^g$.

In section we study the VE as a potential refinement of the GNE. We begin by recalling that every VE is a GNE [1].

Theorem 3: If x is a solution of $VI(U \cap \mathbb{C}, F)$ then x is a solution of QVI$(K \cap U, F)$.

Proof: See [1], Theorem 2.1. ■

So we consider showing that if $\mathcal{G}^g$ has atleast one GNE, it has a VE. But in fact in [8] we have counter-examples to this claim. We also ignore settings in which it is possible to claim the existence of a VE independently of the existence of a GNE (such as compact $\mathbb{C}$) and make the VE a refinement by default. The contribution of this section is in showing that in very general settings (including those above) there is a degree theoretic relation between the VE and the GNE: the degrees of the natural maps of QVI$(K \cap U, F)$ and VI$(\mathbb{C} \cap U, F)$ when well defined, are equal (Theorem 9). We also identify

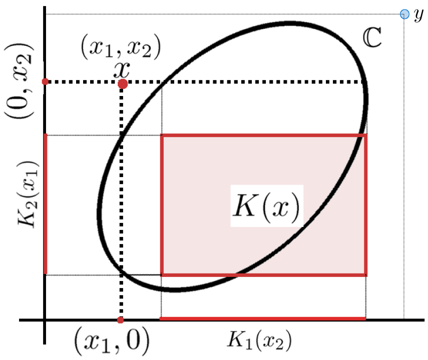

Fig. 1. Figure showing x and $K(x)$ for a convex set $\mathbb{C}$. Also shown is a point y in the top right corner with $K(y) = \emptyset$

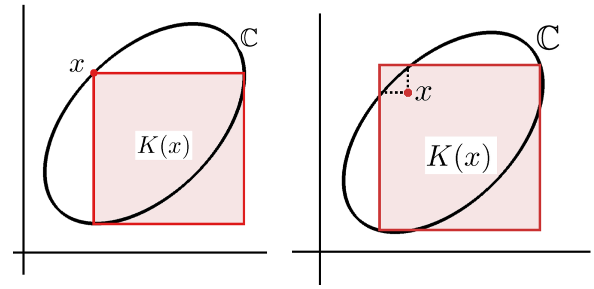

Fig. 2. Figure showing x on the boundary of $\mathbb{C}$ and in the interior of $\mathbb{C}$

conditions under which the VE is a refinement of the GNE and those under which they are equivalent (Theorem 6).

We begin our discussion with an examination of the nature of the set valued map K defined in (1).

A. *The structure of K*

The geometry of the set-valued map K is of central importance to the nature of solutions that QVI(K, F) admits. Fig 1 shows a convex set $\mathbb{C}$ and $K(x)$ for some $x \in \mathbb{R}^2$, assuming $m_1 = m_2 = 1$ and $N = 2$. By definition, $K(x)$ is a cartesian product of sets in $\mathbb{R}^{m_1} \dots \mathbb{R}^{m_N}$ for each x for which it is nonempty. Notice in Fig 1, that $K(x)$ is formed as a product, namely $K_1(x_2) \times K_2(x_1)$. Moreover, in general $\mathrm{dom}(K) := \{x \mid K(x) \neq \emptyset\}$ is not necessarily $\mathbb{R}^m$ and there may be points outside $\mathbb{C}$ whose image under at least one of the K_i's is empty. For instance in Fig 1, notice the point $y = (y_1, y_2)$ for which both $K_1(y_2)$ and $K_2(y_1)$ are empty. The following lemma adds to the observations made above.

Lemma 4:
1) Let $\mathbb{C} = \prod_{i \in \mathcal{N}} \mathbb{C}_i$, where $\mathbb{C}_i \subseteq \mathbb{R}^{m_i}$ for every $i \in \mathcal{N}$, not necessarily convex. Then $K(x) = \mathbb{C}$ for every x in $\mathbb{C}$ and is empty otherwise.
2) For any $\mathbb{C}$, x is a fixed point of K if and only if $x \in \mathbb{C}$.
3) If $\mathbb{C}$ is closed and convex, $K(x)$ is closed and convex for any $x \in \mathrm{dom}(K)$.

Proof: 1) Take any $i \in \mathcal{N}$ and consider an $x \in \mathbb{R}^m$. Note that $K_i(x^{-i}) = \{y_i \in \mathbb{R}^{m_i} \,|(y_i, x^{-i}) \in \mathbb{C}\}$

$$= \{y_i \in \mathbb{R}^{m_i} \,|y_i \in \mathbb{C}_i, x_j \in \mathbb{C}_j, j \neq i\},$$

which is nonempty if $x_j \in \mathbb{C}_j, \forall j \neq i$. Thus $K(x) \neq \emptyset$ if and only if x_j belongs to $\mathbb{C}_j$ for each j, or in other words if and only if $x \in \mathbb{C}$. Furthermore, for $x \in \mathbb{C}$, $y \in K(x)$ if and only if $y_j \in \mathbb{C}_j$ for each j, i.e. $y \in \mathbb{C}$. We conclude that $K(x) = \mathbb{C}$ if and only if $x \in \mathbb{C}$.

2) Let $x \in K(x)$ implying that $x_i \in K_i(x^{-i}) \forall \quad i \in \mathcal{N}$, and therefore $(x_i, x^{-i}) \in \mathbb{C} \quad \forall \ i \in \mathcal{N}$ and $x \in \mathbb{C}$. The converse follows by noting that $x \in \mathbb{C}$ is equivalent to $(x_i, x^{-i}) \in \mathbb{C} \quad \forall i$, i.e. $x_i \in K_i(x^{-i}) \quad \forall i$ and therefore $x \in K(x)$.

3) Let $y, z \in K(x)$. For all $i \in \mathcal{N}$, y_i and z_i belong to $K_i(x^{-i})$ respectively. But this implies that (y_i, x^{-i}) and $(z_i, x^{-i}) \in \mathbb{C}$. Since $\mathbb{C}$ is convex, we may then claim that $((\alpha y_i + (1-\alpha) z_i), x^{-i}) \in \mathbb{C}$ for each i and $\alpha \in [0, 1]$ and the convexity of $K(x)$ follows.

To show closedness, consider a sequence $\{y^k\} \subseteq K(x)$ with limit point $\bar{y}$. For each i, $\{y_i^k, x^{-i}\} \in \mathbb{C}$ and $(y_i^k, x^{-i}) \to (\bar{y}_i, x^{-i})$, which by closedness of $\mathbb{C}$ belongs to $\mathbb{C}$. It follows that $K(x)$ is closed. ■

As a consequence of the above lemma, the set of fixed points of K is nonempty when $\mathbb{C}$ is nonempty and QVI(K, F) which seeks such a fixed point as a solution is not vacuous for any such $\mathbb{C}$. The fact that $\mathbb{C}$ is the set of fixed points of K can be strengthened significantly: fixed points of K are in the interior of $\mathbb{C}$ if and only if they are in the interior of their image under K. This is illustrated below in Fig 2 and proved in the following result. Let int$(\bullet)$ and $\partial\bullet$ stand for the interior and the boundary of '$\bullet$' respectively. Recall that $\mathbb{C}$ has nonempty interior by assumption.

Lemma 5: $x \in$ int$(K(x))$ if and only if $x \in$ int$(\mathbb{C})$.

Proof: Suppose $x \in$ int$(\mathbb{C})$. Then there exists $r > 0$ such that $B(x, r)$, the open ball of radius r centered at x, lies in int$(\mathbb{C})$. For an arbitrary $y \in B(x, r)$, it is clear that (y_i, x^{-i}) is also in $B(x, r)$ and hence in $\mathbb{C}$ for all $i \in \mathcal{N}$. So y belongs to $K(x)$. Consequently, $B(x, r) \subseteq K(x)$ and $x \in$ int$(K(x))$.

For the converse, let $x \in$ int$(K(x))$. For some sufficiently small $r > 0$, let $\overline{B(x, r)}$ be a closed ball centered at x with radius r such that $\overline{B(x, r)} \subseteq K(x)$. We will show that there is a polytope $Q \subset \overline{B(x, r)}$ of dimension m such that $x \in$ int(Q) and $Q \subset \mathbb{C}$. Define a vector $e \in \mathbb{R}^m$ such that

$$e = [e(1), e(2), \dots, e(m)]^T, \quad \text{where } e(\ell) \in \{1, -1\}$$

$\forall \ell = 1, \dots, m$. We thus get 2^m distinct possible values for e, which we denote by e^j, $j = 1, \dots, 2^m$. Set

$$\widehat{y}^j := x + e^j \frac{r}{\sqrt{m}} \text{ and let } e_i^j = [e^j(m_{i-1}+1), \dots, e^j(m_i)]^T.$$

Clearly $\widehat{y}^j \in \partial B(x, r) \subseteq K(x)$ and hence $(\widehat{y}_i^j, x^{-i}) \in \mathbb{C}$ for all i, j. Finally, let

$$Q := \text{conv}\left\{(\widehat{y}_i^j, x^{-i}) : i \in \mathcal{N}, j = 1, \dots, 2^m\right\},$$

where conv$(\bullet)$ denotes the convex hull of $(\bullet)$.

Since Q is the convex hull of points in $\mathbb{C}$, $Q \subseteq \mathbb{C}$. Notice that for any $j \in \{1, \dots, 2^m\}$, there exists a unique $k \in$

$\{1,\dots 2^m\}\backslash\{j\}$ such that $\frac{1}{2}(\widehat{y}^j+\widehat{y}^k)=x$. In fact since for this k, we have

$$\left[(\widehat{y}_i^j,x^{-i})+(\widehat{y}_i^k,x^{-i})\right]/2=x$$

for any $i\in\mathcal{N}$, we conclude that x belongs to Q. Furthermore x is the center of gravity of the vertices of Q, since

$$x=\frac{1}{2^mN}\sum_{i=1}^{N}\sum_{j=1}^{2^m}(\widehat{y}_i^j,x^{-i}),$$

implying $x\in\text{int}(Q)$. But Q is a polytope of dimension m, so its interior must lie in int($\mathbb{C}$) (see [15]). We conclude that x lies in the interior of $\mathbb{C}$. ■

B. Relationship between GNE and VE

Using Lemma 5 a new relationship between GNE and VE can be established: in the interior of $\mathbb{C}$ the GNE and VE are equivalent. This is established in the theorem below.

Theorem 6: Let $x\in\text{int}(K(x))$. Then x is a GNE of $\mathcal{G}^g$ if and only if x is a VE.

Proof: Due to Theorem 3, it suffices to prove the "only if" part of the claim. Suppose $x\in\text{int}(K(x))$ is a GNE. By Lemma 5, $x\in\text{int}(\mathbb{C})$. It follows that $x\in\text{int}(K(x)\cap\mathbb{C})$ and one can construct a ball, $B(x,r)$, centered at x with sufficiently small radius r, such that $B(x,r)$ is contained in $K(x)\cap\mathbb{C}$. Since x is a GNE it solves QVI($K\cap U,F$). So it follows that

$$F(x)^T(y-x)\geq 0\qquad\forall y\in B(x,r)\cap U.\qquad(2)$$

It suffices to show that for arbitrary $z\in U\cap\mathbb{C}$, $F(x)^T(z-x)\geq 0$. To show this, define $u_t:=tz+(1-t)x\in U\cap\mathbb{C}$, for $t\in[0,1]$ and choose $\bar{t}\in(0,1]$ sufficiently small so that $u_{\bar{t}}$ lies in $B(x,r)\cap U$. Substituting for z using $u_{\bar{t}}$,

$$\begin{aligned}F(x)^T(z-x)&=F(x)^T\left(\frac{u_{\bar{t}}-(1-\bar{t})x}{\bar{t}}-x\right)\\&=(1/\bar{t})F(x)^T(u_{\bar{t}}-x)\geq 0,\end{aligned}$$

where the last inequality holds because of (2). As this is true for arbitrary $z\in U\cap\mathbb{C}$, x solves VI($U\cap\mathbb{C},F$). ■

Corollary 7: If $x\in SOL(VI(U\cap\mathbb{C},F))$ and $x\in\partial\mathbb{C}$, then $x\in SOL(QVI(K\cap U,F))$ and $x\in\partial K(x)$.

Proof: Combine Theorem 6 and 3. ■

Theorem 6 is not very surprising. If $\mathbb{C}$ is specified using a C^1 algebraic constraint $c(\cdot)\geq 0$, the hypothesis $x\in\text{int}(K(x))$ reduces to $c(x)>0$. If x is a GNE, the Lagrange multipliers corresponding to $c(\cdot)>0$ would zero for all agents. Thus x would be an equilibrium with *shared (= 0)* multiplier, which as per section I and [1, Theorem 2.2] is a VE. The question of whether there exists any such equivalence on the boundary of $\mathbb{C}$ is part of ongoing research [8]. In what follows next we present the most important contribution of this section, Theorem 9. We show that the Brouwer degrees of the natural maps of QVI($K\cap U,F$) and VI($\mathbb{C}\cap U,F$), whenever well defined, are equal. This result allows us to identity conditions under which the VE is a refinement of the GNE (Theorem 10).

Recall the definition of $\widetilde{\mathbf{F}_K^{\text{nat}}}$ from section I-A.1 and define analogously the natural map for QVI($K\cap U,F$), $\mathbf{F}_{KU}^{\text{nat}}:\text{dom}(K)\to\mathbb{R}^m$ as

$$\mathbf{F}_{KU}^{\text{nat}}(v):=\mathbf{F}_{K(v)\cap U}^{\text{nat}}(v)=v-\Pi_{K(v)\cap U}(v-F(v)).$$

Also abbrieviate $\mathbf{F}_{\mathbb{C}U}^{\text{nat}}:=\mathbf{F}_{\mathbb{C}\cap U}^{\text{nat}}$. To prove Theorem 9 we need a result from [16].

Lemma 8: Let $x\in\text{dom}(K)$ and y be any point in $\mathbb{R}^m$. Then $\phi(x,y):=\mathbf{F}_{K(x)}^{\text{nat}}(y)$ is continuous at (x,y) for all $y\in\mathbb{R}^m$ if and only if $K(\cdot)$ is continuous at x.

Proof: See Lemma 2.8.2 page 221 [16]. ■

For definitions of continuity for set valued maps see [21]. Theorem 9 follows next.

Theorem 9: Suppose $K(\cdot)$ is continuous on $\text{dom}(K)$ and Ω is an open bounded set such that $\overline{\Omega}\subseteq\text{dom}(K)$ and $0\notin\mathbf{F}_{KU}^{\text{nat}}(\partial\Omega)$. Then

$$\deg(\mathbf{F}_{KU}^{\text{nat}},\Omega,0)=\deg(\mathbf{F}_{\mathbb{C}U}^{\text{nat}},\Omega,0).$$

Proof: Note that since $0\notin\mathbf{F}_{KU}^{\text{nat}}(\partial\Omega)$, $0\notin\mathbf{F}_{\mathbb{C}U}^{\text{nat}}(\partial\Omega)$ and hence $\deg(\mathbf{F}_{\mathbb{C}U}^{\text{nat}},\Omega,0)$ is well defined. We will use the invariance of the Brouwer degree under homotopy (property 3 of from section I-A.2) to prove the claim. Define $H:[0,1]\times\text{dom}(K):\to\mathbb{R}^m$ as

$$H(\bar{t},v)=\bar{t}\mathbf{F}_{\mathbb{C}U}^{\text{nat}}+(1-\bar{t})\mathbf{F}_{KU}^{\text{nat}}\qquad\forall\bar{t}\in[0,1],v\in\text{dom}(K).$$

Since K is continuous, H is a valid homotopy between $\mathbf{F}_{KU}^{\text{nat}}$ and $\mathbf{F}_{\mathbb{C}U}^{\text{nat}}$. By property 3 of Brouwer degree it suffices to show that $0\notin H(\bar{t},\partial\Omega)$ for all $\bar{t}\in(0,1)$ to prove the result.

Assume that this is not so. i.e. assume that for some $t\in(0,1)$ and $z\in\partial\Omega$, $H(t,z)=0$. Then

$$z=tx^c+(1-t)x^k,$$

where $x^k=\Pi_{K(z)\cap U}(z-F(z))$ and $x^c=\Pi_{\mathbb{C}\cap U}(z-F(z))$. Since $x^k\in K(z)$, $(x_i^k,z^{-i})\in\mathbb{C}$ for every $i\in\mathcal{N}$, implying that the point x^a, where

$$x^a:=\frac{1}{N}\sum_{i\in\mathcal{N}}(x_i^k,z^{-i})=\frac{(N-1)}{N}z+\frac{1}{N}x^k,$$

belongs to $\mathbb{C}$. Indeed, one may verify that

$$z=\frac{N(1-t)}{N(1-t)+t}x^a+\frac{t}{N(1-t)+t}x^c,$$

which since $x^c\in\mathbb{C}$ leads us to conclude that z is also in $\mathbb{C}$. Now by property of projection Lemma 2, we get

$$\begin{aligned}&(z-x^c)^T(x^c-(z-F(z)))\geq 0\\\text{and}\quad&(z-x^k)^T(x^k-(z-F(z)))\geq 0,\end{aligned}$$

or $F(z)^T(z-x^c)\geq\|z-x^c\|^2\geq 0$ and $F(z)^T(z-x^k)\geq\|z-x^k\|^2\geq 0$. On the other hand since $z-x^c=-\frac{1-t}{t}(z-x^k)$, we have

$$-\frac{1-t}{t}F(z)^T(z-x^k)\geq 0,$$

giving $F(z)^T(z-x^k)=0$ and $z=x^k$. But this means that $\mathbf{F}_{KU}^{\text{nat}}(z)=0$, a contradiction. Hence $\deg(H(t,\cdot),\Omega,0)$ is well defined for all $t\in[0,1]$ and its value is independent of t, whence the claim follows. ■

An immediate consequence of this Theorem and property 2 of the Brouwer degree follows.

Theorem 10: Let K be continuous and x^{ref} be a GNE of $\mathcal{G}^g$. If QVI$(K \cap U, F)$ (or equivalently $\mathcal{G}^g$) has the property that there exists open bounded set Ω, $\overline{\Omega} \subseteq \text{dom}(K)$, such that

1) Ω contains x^{ref} and has no GNE of $\mathcal{G}^g$ on its boundary,
2) and $\deg(\mathbf{F}^{\text{nat}}_{KU}, \Omega, 0) \neq 0$

then $\mathcal{G}^g$ also admits a VE.

Recall the discussion on *refinements* of the GNE from section I. To confirm the VE as a refinement it imperative to establish that any game with a nonempty set of GNEs also admits a VE. The above theorem allows us to claim that for the class of generalized Nash games that satisfy (1),(2) the VE is a refinement of the GNE.

Finding sufficient conditions for (1) and (2) to hold is a part of ongoing research. Notably (2) does not follow from (1), since in general the converse of solvability property 2 of the Brouwer degree does not hold. In the case of VI's it can be shown that if, say VI(V, F) has a bounded solution set and the mapping F is pseudo-monotone then for any neighbourhood, Ω, of SOL(VI(V, F)), $\deg(\mathbf{F}^{\text{nat}}_V, \Omega, 0)$ is well defined and nonzero (see [16]). A similar condition for QVI's is not known and is one we seek in our investigations.

III. Exogenously Constrained Nash Games

We now come to the second set of contributions of this paper. Recall the exogenously constrained game $\mathcal{G}^o$ and the definition of the CNE from section I and the VI characterization of the equilibrium of the unconstrained game $\mathcal{G}^u$ from section I-A.1. We derive sufficient conditions for the existence of CNE for $\mathcal{G}^o$. Clearly x is an CNE of $\mathcal{G}^o$ if and only if $x \in SOL(VI(U, F)) \cap \mathbb{C}$. i.e. if it is an equilibrium of $\mathcal{G}^u$ and belongs to $\mathbb{C}$. Alternatively, a CNE of $\mathcal{G}^o$ is a zero of $\mathbf{F}^{\text{nat}}_U|_{\mathbb{C}}$, the restriction of $\mathbf{F}^{\text{nat}}_U$ to $\mathbb{C}$. Using this we derive our first existence result.

Theorem 11: Let $\mathbb{C}$ be any (including nonconvex) set with nonempty interior. If there exists an open bounded set $\Omega \subseteq \mathbb{C}$ and $x^{\text{ref}} \in U \cap \Omega$ such that

$$F(x)^T(x - x^{\text{ref}}) \geq 0 \qquad \forall x \in U \cap \partial\Omega, \tag{3}$$

then a CNE of $\mathcal{G}^o$ exists.

Proof: We will once again use properties 2, 3 of the Brouwer degree. Assume the contrary, i.e. suppose there is no CNE of $\mathcal{G}^o$ i.e. $\mathbf{F}^{\text{nat}}_U(v) \neq 0 \ \forall v \in U \cap \mathbb{C}$. So $\mathbf{F}^{\text{nat}}_U(v) \neq 0$ for $v \in U \cap \partial\Omega$ (and for $v \in U^c \cap \partial\Omega$) implying that $0 \notin \mathbf{F}^{\text{nat}}_U(\partial\Omega)$ and that $\deg(\mathbf{F}^{\text{nat}}_U, \Omega, 0)$ is well defined. Define the homotopy $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^m$,

$$H(t, v) = v - \Pi_U(t(v - F(v)) + (1 - t)x^{\text{ref}}),$$

for $t \in [0, 1]$ and $v \in \overline{\Omega}$. Clearly H is continuous and a valid homotopy. $H(1, v) \neq 0$ for all $v \in \partial\Omega$ and since $x^{\text{ref}} \in \Omega$, $H(0, v)$ is not zero on $\partial\Omega$. We now show that for $0 < t < 1$, $0 \notin H(t, \partial\Omega)$. Assuming the contrary and supposing that $H(\bar{t}, \bar{v}) = 0$ for some $\bar{t} \in (0, 1)$ and $\bar{v} \in \partial\Omega$ gives

$$\bar{v} - \Pi_U(\bar{t}(\bar{v} - F(\bar{v})) + (1 - \bar{t})x^{\text{ref}}) = 0.$$

So $\bar{v}$ must belong to U. By Lemma 2,

$$\langle v - \bar{v}, \bar{v} - \bar{t}(\bar{v} - F(\bar{v})) - (1 - \bar{t})x^{\text{ref}} \rangle \geq 0 \qquad \forall v \in U.$$

Putting $v = x^{\text{ref}}$ and rearranging, we get

$$(\bar{v} - x^{\text{ref}})^T F(\bar{v}) \leq -\frac{1 - \bar{t}}{\bar{t}} \|\bar{v} - x^{\text{ref}}\|^2 < 0,$$

which contradicts (3). So $0 \notin H(t, \partial\Omega)$ for all $t \in [0, 1]$. By property 3 of the Brouwer degree

$$\deg(\mathbf{F}^{\text{nat}}_U, \Omega, 0) = \deg(I - x^{\text{ref}}, \Omega, 0) = 1,$$

which by property 2 ensures that there exists $x \in \Omega$ such that $\mathbf{F}^{\text{nat}}_U(x) = 0$. Since $\Omega \subseteq \mathbb{C}$, x is a CNE. That contradicts the assumption at the begining of the proof. Hence a CNE of $\mathcal{G}^o$ exists. ■

Observe that no convexity requirement on $\mathbb{C}$ was imposed. Indeed, one of the benefits of exogenously constrained game formulation lies in handling nonconvex coupling constraints. Several corollaries of this result can be given that allow for simpler sufficiency conditions for existence.

Corollary 12: Each of the following is a sufficient condition for the existence of a CNE of $\mathcal{G}^o$

1) There exists $x^{\text{ref}} \in U \cap \text{int}(\mathbb{C})$ and $r > 0$ such that $B(x^{\text{ref}}, r)$ is included in $\mathbb{C}$ and
$$F(x)^T(x - x^{\text{ref}}) \geq 0 \qquad \forall x \in U s.t. \ \|x - x^{\text{ref}}\| = r,$$
2) There exists $x^{\text{ref}} \in U \cap \text{int}(\mathbb{C})$ such that
$$F(x)^T(x - x^{\text{ref}}) \geq 0 \qquad \forall x \in U \cap \mathbb{C},$$
3) There exists $x^{\text{ref}} \in U \cap \text{int}(\mathbb{C})$ such that the set
$$L_< = \{x \in U : F(x)^T(x - x^{\text{ref}}) < 0\},$$
is either empty or a bounded subset of int$(\mathbb{C})$.
4) $\mathbb{C}$ is bounded and $\exists \ x^{\text{ref}} \in U \cap \text{int}(\mathbb{C})$ such that
$$F(x)^T(x - x^{\text{ref}}) \geq 0 \qquad \forall \ x \in U \cap \partial\mathbb{C}.$$

Proof: Each of the above is a special case of Theorem 11 obtained by a specific choices of Ω which we indicate below
1) Take Ω in Theorem 11 as this open ball.
2) Take Ω as any ball around x^{ref} that is included in $\mathbb{C}$.
3) Since $L_<$ is bounded and contained in int$(\mathbb{C})$, there exists $V = \text{int}(V) \subset \mathbb{C}$ such that $V \supseteq L_< \cup \{x^{\text{ref}}\}$. Take Ω in Theorem 11 to be V. Clearly $\partial\Omega \cap L_< = \emptyset$, so Eq (3) holds.
4) Take $\Omega = \text{int}(\mathbb{C})$ in Theorem 11. ■

The sufficient conditions for existence of a CNE given in Theorem 11 and its corollaries are structurally similar to those for VIs, such as those in [16, Chapter 2]. The set $L_<$ is also a construct from the theory of VIs, and the boundedness of $L_<$ is often easier to check than (3). Another sufficient condition for the existence of the CNE can be obtained from a direct application of Brouwer's fixed point theorem.

Theorem 13: If there exists a compact convex subset $\widetilde{\mathbb{C}}$ of $\mathbb{C}$ such that for all $v \in \widetilde{\mathbb{C}}$, $\Pi_U(v - F(v)) \in \widetilde{\mathbb{C}}$, then a CNE of $\mathcal{G}^o$ exists.

Proof: Define the restriction, $G_U(\cdot) = \Pi_U(\cdot - F(\cdot))|_{\widetilde{\mathbb{C}}}$. By hypothesis $G_U : \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$, and is continuous. Since $\widetilde{\mathbb{C}}$ is compact and convex, Brouwer's fixed point theorem applies, whence there exists $x \in \widetilde{\mathbb{C}}$ such that $\Pi_U(x - F(x)) = x$. In other words there is a zero of $\mathbf{F}_U^{\text{nat}}$ in $\mathbb{C}$, which is the sought CNE. ■

Notice that if $\mathbb{C}$ is convex and compact, $\mathbb{C}$ is itself a candidate for $\widetilde{\mathbb{C}}$ above.

A. Relationship of CNE with GNE and VE

This section clarifies the relationship of the CNE with GNE and the VE. We first show that every CNE of $\mathcal{G}^o$ is a variational equilibrium and hence a generalized Nash equilibrium of $\mathcal{G}^g$.

Theorem 14: Consider the following three statements:

(i) x is CNE of $\mathcal{G}^o$

(ii) x is a VE of $\mathcal{G}^g$

(iii) x is GNE of $\mathcal{G}^g$.

Then it holds that (i) $\implies$ (ii) $\implies$ (iii).

Proof: Due to Theorem 3 it suffices to show that (i) $\implies$ (ii). Assume x is a CNE of $\mathcal{G}^o$. It follows that $x \in \mathbb{C} \cap U$ and is feasible for VI$(\mathbb{C} \cap U, F)$. Furthermore since

$$F(x)^T(y - x) \geq 0 \quad \forall\, y \in U,$$

it follows that holds for $y \in \mathbb{C} \cap U$ implying that x also solves VI$(\mathbb{C} \cap U, F)$. ■

In the light of the above theorem, the importance of Theorems 11 and 13 can be more fully appreciated. Theorems 11 and 13 are also provide sufficient conditions for the existence of a solution to QVI. It may be verified that both have been obtained under weaker conditions than Theorem 2.8.3 and Corollary 2.8.4 in [16]. Additionally, a CNE is an equilibrium of $\mathcal{G}^g$ that is remains an equilibrium upon removal of the coupling constraint. We end with a summarization of Theorem 6.

Theorem 15: If $x \in \text{int}(\mathbb{C})$ then statements (i), (ii) and (iii) in Theorem 14 are equivalent.

Proof: By Lemma 5 and Theorem 6 (ii) and (iii) are equivalent and by Theorem 14 (i) $\implies$ (ii). It suffices to show the claim: if x is a VE of $\mathcal{G}^g$, then x solves VI(U, F). We argue as in Theorem 6. Let $B(x, \epsilon)$ be a ball around x contained in $\mathbb{C}$. Since x solves VI$(U \cap \mathbb{C}, F)$,

$$F(x)^T(z - x) \geq 0 \qquad \forall z \in B(x, \epsilon) \cap U.$$

Consider an arbitrary $y \in U$ and for sufficiently small $t \in (0, 1]$, let $u_t := ty + (1 - t)x$ belong to the ball $B(x, \epsilon)$. So

$$F(x)^T(y - x) = (1/t)F(x)^T(u_t - x) \geq 0,$$

completing the proof. ■

IV. Conclusions and Future Work

As stated at several places in the manuscript, this is ongoing work. We have identified and addressed two shortcomings in the GNE and studied two alternatives (VE and CNE) to remedy them. These shortcomings are pertinent to questions of modelling and analyzing such games, computation of equilibria and to the theory of games in general. Our major contribution lies in showing that VE and GNE are related in the manner of Theorem 9 – a result that is both encouraging and surprising and one that warrants deeper investigation. For exogenously constrained games, we have presented existence results and clarified the relationship between the CNE, GNE and VE. Admittedly, many may disagree about the better applicability of the CNE as a solution concept rather than the GNE for exogenously constrained; we regard this as a subjective matter for which no scientific answer may be available.

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