On the Consistency of Leaders' Conjectures in Hierarchical Games

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***Abstract*— In multi-leader multi-follower games, a set of *leaders* compete in a Nash game, while anticipating the equilibrium arising from a game between a set of *followers*. Conventional formulations are complicated by several concerns. First, since follower equilibria need not be unique, conjectures made by leaders regarding follower equilibria may not be *consistent* at equilibrium. When the follower equilibrium is a physical quantity to be exchanged, one is led to ask whether an equilibrium without *consistent conjectures* is even sensible. Second, these games are often irregular and nonconvex and no general sufficiency conditions for existence of equilibria are known. Third, no globally convergent algorithms for computing equilibria are known. We show that these concerns are addressed *en masse* by a *modified model* we introduce in this paper. In this model each leader makes conjectures while also requiring that his conjectures are consistent with those made by other leaders. If leader payoff functions admit a potential function, then under mild conditions, this model admits an equilibrium. At equilibrium, the conjectures of leaders are necessarily consistent, and when there is a unique follower equilibrium, the equilibria of the original model are equilibria of the new model. Preliminary empirical evidence suggests that such equilibria are also significantly easier to compute.**

I. INTRODUCTION

This paper concerns hierarchical multi-leader multi-follower games [1], where each *leader* is a Stackelberg leader with respect to all *followers*, followers play a Nash game amongst themselves taking the decisions of the leaders as given, and leaders play a Nash game with other leaders while anticipating the equilibrium of the game between followers. We make no assumptions regarding how leaders may influence followers; each follower may have in its information set [2], the decision of all leaders.

Let $\mathcal{N}$ be the set of leaders and $\mathcal{M}$ be the set of followers. Let $z_i \in \mathbb{R}^{m_i}$ be the action of follower i and $x_i \in \mathbb{R}^{n_i}$ denote the action of leader i. We denote follower i's objective function by $f_i : \mathbb{R}^{m+n} \to \mathbb{R}$, where $m = \sum_{i\in\mathcal{M}} m_i$ and $n = \sum_{i\in\mathcal{N}} n_i$, and let $C_i \subseteq \mathbb{R}^{m_i}$ denote the set of feasible strategies for follower i. Then each follower solves an optimization problem,

$$\mathrm{F}_i(z^{-i};x) \qquad \min_{z_i\in C_i} \quad f_i(z_i; z^{-i}, x),$$

parametrized by the profile of leader strategies, x. Here we have used the standard notation $z \triangleq (z_1,\dots,z_M), x \triangleq (x_1,\dots,x_N)$ and $z^{-i} \triangleq (z_1,\dots,z_{i-1},z_{i+1},\dots,z_M)$.

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Definition 1.1: Let x denote a strategy profile of the leaders. A point $z \in C \triangleq \prod_{i=1}^M C_i$. is a follower equilibrium for x if for all $i \in \mathcal{M}$, $f_i(z;x) \le f_i(z_i'; z^{-i}, x)\ \forall z_i' \in C_i$.

Suppose C_i is closed and convex for every i and for all z^{-i}, x, $f_i(\bullet; z^{-i}, x)$ is continuously differentiable and convex. Then z is a follower equilibrium for the profile of leader strategies x if and only if z solves the following parametrized variational inequality problem [3]:

$$\mathrm{VI}(G(\bullet;x),C): \quad G(z;x)^\top(z'-z)\ge 0 \qquad \forall z'\in C, \quad (1)$$

where $G(z;x) = (\nabla_{z_1} f_1(z;x),\dots,\nabla_{z_N} f_N(z;x))$.

Leader i's payoff depends on its action x_i, the tuple of actions of other leaders, denoted by x^{-i}, and the follower equilibrium denoted by z. However, given x, the follower equilibrium may not be unique, implying that the leader's payoff is not well defined unless one also specifies a particular follower equilibrium of interest. But since leaders act *prior to* the followers, and may only *anticipate* their response, the presumed follower equilibrium employed by a leader to compute its payoff is necessarily a *conjecture*.

Suppose leader i's objective is given by $\varphi_i(x_i, y_i; x^{-i})$ where y_i denotes leader i's conjecture regarding the follower equilibrium. Note – y_i is *not* the equilibrium strategy of the i^{th} follower, but is instead the profile of strategies of *all* followers, as conjectured by leader i. In the case of a single leader, disambiguation of the follower equilibrium is accomplished by adopting either an *optimistic formulation* or a *pessimistic formulation* [4]. In an *optimistic* formulation, leader i considers y_i to be the follower equilibrium most beneficial to it. Thus y_i is an action of leader i; this leader then solves the following parametrized mathematical program with equilibrium constraints:

$$\begin{array}{lll} \mathrm{L}_i(x^{-i}) & \displaystyle\min_{x_i\in X_i, y_i} & \varphi_i(x_i, y_i; x^{-i}) \\ \text{s.t.} & & y_i \in \mathrm{SOL}(G(\bullet;x),C), \end{array}$$

where $\mathrm{SOL}(G(\bullet;x),C)$ denotes the solution set of the parametrized $\mathrm{VI}(G(\bullet;x),C)$. The resulting multi-leader multi-follower game (or Stackelberg vs Stackelberg game (following e.g., [5])) is denoted as $\mathcal{E}$.[1]

Definition 1.2: An equilibrium of game $\mathcal{E}$ is a tuple $(x,y)\in\mathcal{F}$ such that $\varphi_i(x_i,y_i;x^{-i}) \le \varphi_i(x_i',y_i';x^{-i})$, for all $(x_i',y_i')\in\Omega_i(x^{-i})$, where $\Omega_i(x^{-i})$ is the set of feasible strategies for the leader's problem $\mathrm{L}_i(x^{-i})$:

$$\Omega_i(x^{-i}) \triangleq \{(x_i,y_i) | x_i\in X_i, y_i\in\mathcal{S}(x)\}, \qquad (2)$$

[1]This notation derives itself from mathematical programming where the game is called an *equilibrium problem with equilibrium constraints* [5].

$\Omega(x) \triangleq \prod_{i\in\mathcal{N}} \Omega_i(x^{-i}), X = \prod_{i\in\mathcal{N}} X_i$ where

$$\mathcal{S}(x) \triangleq \mathrm{SOL}(\mathrm{VI}(G(\cdot;x), C)), \quad \mathcal{S}^N(x) \triangleq \prod_{i=1}^{N} \mathcal{S}(x), \quad (3)$$

$$\text{and } \mathcal{F} \triangleq \{(x,y) | x \in X, y \in \mathcal{S}^N(x)\}. \quad (4)$$

It can be easily seen that $\mathcal{F} = \{(x,y)|(x,y) \in \Omega(x)\}$, the set of fixed points of Ω.

A. Challenges

What we have described above is the conventional formulation of multi-leader multi-follower games. This paper is motivated by two concerns regarding the equilibria of $\mathcal{E}$:

(i) Meaningfulness: That the follower equilibrium is only a conjecture that a leader makes gives rise to an unusual predicament. The nonuniqueness of the follower equilibrium leads to the possibility that at an equilibrium, leaders may not agree on the conjectures made regarding the follower equilibrium. In practical settings such as in power markets [6], the follower equilibrium represents a physical settlement to be implemented. If the equilibrium is a point at which leaders disagree on the settlement level, one is led to question if this equilibrium is indeed sensible.

(ii) Existence: For Stackelberg v/s Stackelberg games, no robust theory of existence of equilibria is known; in fact one can find simple examples where equilibria do not exist. To quote from Pang and Fukushima [5]: *"In practice, the multi-dominant firm problem is of greater importance; and yet, as a Nash game, the latter problem can have no equilibrium solution in the standard sense."* Pang and Fukushima [5] provide the following example with no equilibrium. The game has two leaders with objectives defined as follows:

$$\varphi_1(x_1, y_1) = \tfrac{1}{2}x_1 + y_1 \text{ and } \varphi_2(x_2, y_2) = -\tfrac{1}{2}x_2 - y_2. \quad (5)$$

The leader strategy sets are denoted by $X_1 = X_2 \triangleq [0, 1]$, while a single follower solves the following problem:

$$\min_{z\geq 0} \left(z(-1 + x_1 + x_2) + \tfrac{1}{2}z^2\right) = \max\{0, 1 - x_1 - x_2\}.$$

Mathematically, the analysis of the Stackelberg v/s Stackelberg game described above is hindered by two key difficulties which hinder the application of fixed-point theorems. First, the feasible region of each leader's problem is highly nonconvex. Note that there exist results that guarantee the convexity of the solution set of a variational inequality [3]. However, what is needed here is the convexity of the *graph* of the set-valued map $x_i \mapsto \mathrm{SOL}(\mathrm{VI}(G(\bullet, x_i; x^{-i}), C))$, a property that is hard to guarantee. The second difficulty is the lack of continuity in the map $x^{-i} \mapsto \Omega_i(x^{-i})$; solution sets of parametrized variational inequalities are rarely continuous (as set-valued maps) with respect to the parameter. As a consequence, the reaction map [2] that is obtained by the best response of a leader i as a function of x^{-i}, is neither convex-valued not upper-semicontinuous. Thus, the hypothesis of standard fixed-point theorems, such as those of Brouwer and Kakutani are not met, and existence of equilibria is difficult to guarantee. For a more thorough discussion of these issues, we refer the reader to our recent submission [7].

These analytical obstructions also lead to computational challenges. Since the problem described in Definition 1.2 is fundamentally nonconvex and irregular, it is exceedingly hard to devise reliable convergent computational schemes for it. To quote [5], *"... although the multi-leader-follower problem is a sensible mathematical model with a well-defined solution concept, its high level of complexity and technical hardship make it a computationally intractable problem."*

B. Contributions and outline

A possible resolution of the "meaningfulness" concern is through tightening the equilibrium concept itself: one may ask that in addition to Definition 1.2, an equilibrium also satisfy $y_i = y_j \ \forall i, j \in \mathcal{N}$. However, the challenge with imposing such an *ex-post* requirement is that equilibria in the sense of Definition 1.2 do not exist in the simplest of cases. In the light of this, we instead resort to modifying the model. We present a *modified model* of hierarchical competition in which each leader makes its conjecture about the follower equilibrium *while also requiring that its conjectures are consistent with the conjectures made by other leaders*. As a consequence, a number of the above concerns get addressed.

(i) Consistency of conjectures: Any equilibrium of this new game satisfies the consistency in leaders' conjectures regarding the follower equilibrium, regardless of any uniqueness requirement on the follower equilibrium.

(ii) Retention of equilibria of original formulation: Furthermore, if for each profile of leader strategies there is a unique follower equilibrium, then any equilibrium of the original game is an equilibrium of the modified game.

(iii) Existence of equilibria: The new formulation has a constraint structure called *shared constraints*. Consequently, if leaders' objectives admit a potential function, existence of equilibria can be guaranteed under fairly general assumptions. In particular, a modified formulation of the Pang and Fukushima example admits an equilibrium (Section III-C).

(iv) Computation of equilibria: Finally, equilibria of the new formulation appear to be significantly easier to compute. We show empirical evidence to support the claim that a Gauss-Siedel heuristic is well behaved on this problem.

A possible insight that can be distilled from these results is the following: it appears that the lack of consistency in the conjectures of leaders exhibited by the original formulation may be making the problem mathematically irregular, in addition to being responsible for the ambiguity of the follower equilibrium in physical settings. The enforcement of consistency of conjectures not only ensures the meaningfulness of the equilibrium, but also has a regularizing influence on both the existence and computation of equilibria. To the best of our knowledge these results are new as is this insight.

The paper is organized as follows. In Section II we cover some background and motivation for this work. In Section III we present the modified model with consistency requirements and explain its associated properties and existence results. In Section IV we present the numerical results and conclude in Section V.

II. BACKGROUND AND PRELIMINARIES

A. Examples and motivation

Multi-leader multi-follower games were first investigated by Sherali [1] and have thereafter been applied to various settings in power markets [8], [9]. We describe one such setting next. Suppose N players compete in the forward market (as leaders) and then in the spot market (as followers) with decisions given by $\{x_i\}_{i=1}^N$ and $\{z_i\}_{i=1}^N$, respectively (since the same set of players participate in both markets, $\mathcal{N} = \mathcal{M}$). In the spot market, player i solves the following parametrized problem:

$$\mathrm{S}_i(z^{-i}; x) \qquad \min_{z_i \geq 0} \; c_i z_i - p\left(\bar{z}\right)\left(z_i - x_i\right)$$

where $c_i z_i$ is the linear cost of producing z_i units in the spot-market and $p(\bar{z})$ denotes the price of electricity based on the aggregate spot sales, given by $\bar{z} \triangleq \sum_{i \in \mathcal{N}} z_i$.

Let y_i be the vector of conjectures of leader i regarding follower equilibrium, let $y_{i,j}$ represent its conjecture about the j^{th} follower's equilibrium decision (thus $y_{i,i}$ is its own anticipated spot-market decision). In the forward market, firm i's profit is given by $-p^f x_i - p(\bar{y}_i)(y_{i,i} - x_i) - c_i y_{i,i}$, where p^f denotes the forward price and $\bar{y}_i = \sum_{j \in \mathcal{N}} y_{i,j}$. Condition of no-arbitrage requires that $p^f = p(\bar{y}_i)$ and firm i's forward market problem reduces to the following:

$$\begin{array}{lll} \mathrm{L}_i(x^{-i}) & \min\limits_{x_i, y_i} & c_i y_{i,i} - p(\bar{y}_i) y_{i,i} \\ \text{s.t.} & & y_{i,j} \in \mathrm{SOL}(\mathrm{S}(y_i^{-j}, x_i, x^{-i})), \quad \forall\, j. \end{array}$$

B. Existence of equilibria

To the best of our knowledge there are no general existence results for hierarchical games with two levels of hierarchy, let alone multiple levels. On the other hand, there are numerous simple counterexamples, such as that of Pang and Fukushima [5]. Known existence results are model-specific [1], [10] and often rely on eliminating the second-level problem by explicit substitution of the solution map (which is required to be single-valued) and are dependent on the favorable structure of the underlying model to allow all of these properties.

In our recent paper [7] we present what is perhaps the first general existence result for the following class of games.

Definition 2.1 (Quasi-potential games [7]): The game $\mathcal{E}$ is referred to as a quasi-potential game if the following hold:

(i) For $i \in \mathcal{N}$, there exist functions $\phi_1(x), \ldots, \phi_N(x)$ and a function $h(x, y_i)$ such that each player i's objective $\varphi_i(\cdot)$ is given as $\varphi_i(x_i, y_i; x^{-i}) \equiv \phi_i(x) + h(x, y_i)$.

(ii) There exists a function π such that for all $i \in \mathcal{N}$, and for all $x \in X$ and $x_i' \in X_i$, we have

$$\phi_i(x_i; x^{-i}) - \phi_i(x_i'; x^{-i}) = \pi(x_i; x^{-i}) - \pi(x_i'; x^{-i}).$$

The function $\pi + h$ is called the quasi-potential function. We consider the following optimization problem,

$$\begin{array}{lll} \mathrm{P}^{\text{quasi}} & \min\limits_{x,w} & \pi(x) + h(x, w) \\ \text{s.t.} & & (x, w) \in \mathcal{F}^{\text{quasi}}, \end{array}$$

where $\mathcal{F}^{\text{quasi}} \triangleq \left\{(x, w) \in \mathbb{R}^n \mid x_i \in X_i, i \in \mathcal{N}, w \in \mathcal{S}(x)\right\}$, and relate its minimizers to equilibria of $\mathcal{E}$.

Theorem 2.1 (Existence of global equilibria of $\mathcal{E}$ [7]): Let $\mathcal{E}$ be a quasi-potential multi-leader multi-follower game. Suppose $\mathcal{F}^{\text{quasi}}$ is a nonempty set and φ_i is a continuous function for $i = 1, \ldots, N$. A minimizer of $\mathrm{P}^{\text{quasi}}$ is an equilibrium of $\mathcal{E}$. Consequently, if the minimizer of $\mathrm{P}^{\text{quasi}}$ exists (for example, if either π is coercive over $\mathcal{F}^{\text{quasi}}$ or if $\mathcal{F}^{\text{quasi}}$ is compact), then $\mathcal{E}$ admits an equilibrium.

C. Shared constraint games

A shared constraint game is a generalization of the classical Nash game. The underlying assumption in a classical Nash game is that while a player's payoff is dependent on the strategies of other players, the set of feasible strategies are independent of the strategies of other players. In contrast, the strategy set of a player in a shared constraint game is dependent on adversarial strategies with this dependence taking on a prescribed form. Let $u_i \in \mathbb{R}^{n_i}$ and u^{-i} denote a strategy of player i and the tuple of adversarial strategies, respectively. Let $K_i(u^{-i})$ denote the strategy set of player i and let the set-valued mapping K be defined as $K(u) \triangleq \prod K_i(u^{-i})$. Then K is a shared constraint if there exists a set $S \subseteq \mathbb{R}^n, n = \sum n_i$ such that

$$u_i \in K_i(u^{-i}) \quad \iff \quad (u_i, u^{-i}) \in S, \quad \forall i, \tag{6}$$

and the resulting games are called shared constraint games. Shared constraint games were introduced by Rosen [11] and are an area of active recent research (e.g., [12], [13], [14]).

III. CONSISTENCY OF CONJECTURES

In Stackelberg v/s Stackelberg games one may impose a requirement that *at equilibrium* the conjectures of the leaders be consistent: i.e., we may ask that if (x, y) is an equilibrium, then $y_i = y_j$ for all $i, j \in \mathcal{N}$. It is not immediately clear if such consistency should be an *ex-post* requirement on the game or an *ex-ante* one. As mentioned in Section I, an ex-post requirement runs into the hurdle of the lack of equilibria satisfying Definition 1.2.

We consider an *ex-ante* requirement that imposes that each player chooses its conjectures under the requirement that $y_j = y_i$ for all $i, j \in \mathcal{N}$. More specifically, we impose the requirement that $y_i = y_j$, for all $j \in \mathcal{N}$ as a part of *each* leader i's optimization problem. Imposing this requirement modifies the game; the optimization problem of leader i is no more L_i described in Section I. In the new game leader i solves the following problem.

$$\begin{array}{lll} \mathrm{L}_i^{\text{cc}}(x^{-i}, y^{-i}) & \min\limits_{x_i, y_i} & \varphi_i(x_i, y_i; x^{-i}) \\ & & x_i \in X_i, \\ \text{s.t.} & & y_i \in \mathcal{S}(x) \\ & & y_i = y_j, \quad j \in \mathcal{N}. \end{array}$$

We denote this game by $\mathcal{E}^{\text{cc}}$. Before proceeding, let us ask a basic question: why would a rational leader impose an additional consistency constraint? In other words, would the game $\mathcal{E}^{\text{cc}}$ emerge organically? While this remains unclear, what should be noted however is that in the absence of such constraints, the resulting problem might admit equilibria that are difficult to interpret if the conjectures of the follower decisions at equilibrium are inconsistent. We see this modified formulation as one that recognizes the physical nature of the follower problem and attempts to remove the ill-posedness of the problem.

Also note that the leader problems in isolation are infeasible if the decisions of other players y^{-i} are not identical. Our goal however is to examine the properties *at equilibrium*, where this cannot occur. Comparing L_i with $\mathrm{L}_i^{\rm cc}$ we see that x_i satisfies the same constraints in both problems, but y_i is constrained more in $\mathrm{L}_i^{\rm cc}$. Let $\Omega_i^{\rm cc}(x^{-i}, y^{-i})$ denote the feasible region of $\mathrm{L}_i^{\rm cc}(x^{-i}, y^{-i})$ and

$$\Omega^{\rm cc} \triangleq \prod_{i=1}^{N} \Omega_i^{\rm cc}, \quad \mathcal{F}^{\rm cc} \triangleq \{(x, y) \mid (x, y) \in \Omega^{\rm cc}(x, y)\},$$

where $\mathcal{F}^{\rm cc}$ is the set of fixed points of $\Omega^{\rm cc}$.

Definition 3.1: An equilibrium of $\mathcal{E}^{\rm cc}$ is a point $(x, y) \in \mathcal{F}^{\rm cc}$, such that for all $i \in \mathcal{N}$,

$$\varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(\bar{x}_i, \bar{y}_i; x^{-i}) \ \forall \ (\bar{x}_i, \bar{y}_i) \in \Omega_i^{\rm cc}(x^{-i}, y^{-i}).$$

A. Properties of the formulation

Having introduced the formulation, we now take note of some important properties that this formulation enjoys.

Proposition 3.1: Consider the multi-leader multi-follower game defined by $\mathcal{E}^{\rm cc}$. Then the following hold:

(i) $\Omega^{\rm cc}$ is a shared constraint mapping satisfying (6).

Furthermore, if $\mathcal{S}(\cdot)$ is single-valued, then we have:

(ii) $\mathcal{F} = \mathcal{F}^{\rm cc}$, and

(iii) Every equilibrium of $\mathcal{E}$ is an equilibrium of $\mathcal{E}^{\rm cc}$.

Proof: **(i)** It can be observed that for any i and any x^{-i}, y^{-i} such that $y_j = y_k$, $x_j \in X_j$, for all $k, j \neq i$,

$$\begin{aligned}\Omega_i^{\rm cc}(x^{-i}, y^{-i}) &= \{x_i, y_i | x_i \in X_i, y_i = y_j, \forall j \in \mathcal{N}, \\ &\qquad y_i \in \mathcal{S}(x)\}, \\ &= \{x_i, y_i \mid x_i \in X_i, y_i = y_j, y_j \in \mathcal{S}(x), \forall j \in \mathcal{N}\}.\end{aligned}$$

But, $y_j \in \mathcal{S}(x), j = 1, \ldots, N$ implies that $y \in \mathcal{S}^N(x)$. Let $\mathbf{A} \subseteq \mathbb{R}^{mN}$ be the set

$$\mathbf{A} \triangleq \{y \mid y_j = y_1, j = 2, \ldots, N\}. \tag{7}$$

It follows that

$$\begin{aligned}\Omega_i^{\rm cc}(x^{-i}, y^{-i}) &= \{x_i, y_i \mid x \in X, y \in Y, y \in \mathcal{S}^N(x), y \in \mathbf{A}\} \\ &= \{x_i, y_i \mid x \in X, (x, y) \in \mathcal{G}, y \in \mathbf{A}\},\end{aligned}$$

where $\mathcal{G} = \{(x, y) | y \in \mathcal{S}^N(x)\}$ is the graph of $\mathcal{S}^N$. Hence,

$$(x_i, y_i) \in \Omega_i^{\rm cc}(x^{-i}, y^{-i}) \iff (x, y) \in (X \times \mathbf{A}) \cap \mathcal{G}.$$

Since this holds for each $i \in \mathcal{N}$, $\Omega^{\rm cc}$ is a shared constraint of the form required by (6).

(ii) From (i) it follows that $\mathcal{F}^{\rm cc} = (X \times \mathbf{A}) \cap \mathcal{G}$, from which it follows that that $\mathcal{F}^{\rm cc} \subseteq (X \times \mathbb{R}^{mN}) \cap \mathcal{G}$. But from (4) we have that $\mathcal{F} = (X \times \mathbb{R}^{mN}) \cap \mathcal{G}$, whereby $\mathcal{F}^{\rm cc} \subseteq \mathcal{F}$. So it suffices to show that $\mathcal{F} \subseteq \mathcal{F}^{\rm cc}$. Let (x, y) be an arbitrary tuple in $\mathcal{F}$. Since $\mathcal{S}$ is single-valued, y belongs to $\mathbf{A}$ and consequently $(x, y) \in \mathcal{F}^{\rm cc}$. Hence $\mathcal{F} \subseteq \mathcal{F}^{\rm cc}$.

(iii) Let (x, y) be an equilibrium of $\mathcal{E}$. Clearly $(x, y) \in \mathcal{F}$, and so by (ii), $(x, y) \in \mathcal{F}^{\rm cc}$. Since $\Omega_i^{\rm cc}(x^{-i}, y^{-i}) \subseteq \Omega_i(x^{-i}, y^{-i})$, the result follows. ■

Of particular importance in these properties are (i) and (iii). One can easily check that the original mapping Ω is *not* a shared constraint in the sense of (6). But (i) shows that the addition of the consistency requirement on conjectures makes the resulting constraint $\Omega^{\rm cc}$ a shared constraint. This is perhaps the single most important conceptual difference between the formulations $\mathcal{E}$ and $\mathcal{E}^{\rm cc}$. Furthermore, (iii) shows that $\mathcal{E}^{\rm cc}$ captures all equilibria (if any) of $\mathcal{E}$ in the event that $\mathcal{S}(\cdot)$ is single-valued. In this case, one may say that $\mathcal{E}^{\rm cc}$ is a *weaker* formulation than $\mathcal{E}$; existence of an equilibrium of $\mathcal{E}^{\rm cc}$ is a *necessary condition* for the existence of an equilibrium of $\mathcal{E}$.

B. Existence of equilibria

In this section, we use Proposition 3.1 to present a general existence result for $\mathcal{E}^{\rm cc}$. This result relies on the concept of potential games, which is defined next.

Definition 3.2 (Potential game): The game $\mathcal{E}^{\rm cc}$ is called a potential game if the objectives $\varphi_i, i \in \mathcal{N}$ are such that there exists a function π, called potential function, such that for all $i \in \mathcal{N}$, for all $(x_i, x^{-i}) \in X, (y_i, y^{-i}) \in \mathbb{R}^{mN}$ and for all $x_i' \in X_i, y_i' \in \mathbb{R}^m$

$$\begin{aligned}&\varphi_i(x_i, y_i; x^{-i}, y^{-i}) - \varphi_i(x_i', y_i'; x^{-i}, y^{-i}) \\ &\qquad = \pi(x_i, y_i; x^{-i}, y^{-i}) - \pi(x_i', y_i'; x^{-i}, y^{-i}).\end{aligned} \tag{8}$$

Our main result relates the equilibria of $\mathcal{E}^{\rm cc}$ and the global minimizers of the potential function over $\mathcal{F}^{\rm cc}$, i.e., of the following optimization problem:

$\mathrm{P}^{\rm cc}$	$\min_{x,y}$	$\pi(x, y)$
s.t.		$(x, y) \in \mathcal{F}^{\rm cc}$.

Theorem 3.2 (Minimizers of $\mathrm{P}^{\rm cc}$ and Equilibria of $\mathcal{E}^{\rm cc}$): Suppose $\mathcal{F}^{\rm cc}$ is nonempty and $\varphi_i, i \in \mathcal{N}$ are continuous. Let $\mathcal{E}^{\rm cc}$ be a potential game with a potential function π. Then any global minimizer of π over $\mathcal{F}^{\rm cc}$ is an equilibrium of $\mathcal{E}^{\rm cc}$. Thus if the leader objectives are coercive or if $\mathcal{F}^{\rm cc}$ is compact, $\mathcal{E}^{\rm cc}$ admits an equilibrium.

Proof: Let $(x, y) \in \mathcal{F}^{\rm cc}$ be a global minimum of π over $\mathcal{F}^{\rm cc}$. Then $\pi(x, y) \leq \pi(x', y')$, for all $(x', y') \in \mathcal{F}^{\rm cc}$. Fix $i \in \mathcal{N}$ and let $(x', y') = (u_i, x^{-i}, v_i, y^{-i}) \in \mathcal{F}^{\rm cc}$. Then,

$$\pi(x_i, y_i, x^{-i}, y^{-i}) - \pi(u_i, v_i, x^{-i}, y^{-i}) \leq 0$$

$\forall \ (u_i, v_i) : (u_i, v_i, x^{-i}, y^{-i}) \in \mathcal{F}^{\rm cc}$. But, $(u_i, v_i, x^{-i}, y^{-i}) \in \mathcal{F}^{\rm cc}$ if and only if $(u_i, v_i) \in \Omega_i^{\rm cc}(x^{-i}, y^{-i})$, since $\Omega^{\rm cc}$ is a shared constraint (cf. Proposition 3.1 (i)). Using this, together with the fact that π is a potential function, we obtain that for each i $\varphi_i(x_i, y_i; x^{-i}, y^{-i}) - \varphi_i(u_i, v_i; x^{-i}, y^{-i}) \leq 0$, for all $(u_i, v_i) \in \Omega_i^{\rm cc}(x^{-i}, y^{-i})$. Since this holds for each $i \in \mathcal{N}$, we get that given (x^{-i}, y^{-i}), the point (x_i, y_i) is a best response for leader i. So (x, y) is an equilibrium of $\mathcal{E}^{\rm cc}$. If $\varphi_i, i \in \mathcal{N}$ are coercive or $\mathcal{F}^{\rm cc}$ is compact, $\mathrm{P}^{\rm cc}$ has a solution and thus $\mathcal{E}^{\rm cc}$ has an equilibrium. ■

C. Pang and Fukushima example with consistent conjectures

As an example of the effect that introducing consistency in conjectures has on the game, we consider the game of Pang and Fukushima [5]. From (5) it is obvious that this game admits a potential function given by:

$$\pi(x,y) = \varphi_1(x_1,y_1) + \varphi_2(x_2,y_2) = \tfrac{1}{2}x_1 + y_1 - \tfrac{1}{2}x_2 - y_2.$$

Therefore this game *is* a potential game with *no* equilibria.

Consider the modification in the form of $\mathcal{E}^{\rm cc}$.

$$\begin{array}{|lll|} \hline \mathrm{L}_1^{\rm cc}(x_2,y_2) \min\limits_{x_1,y_1} & \varphi_1(x_1,y_1) = \tfrac{1}{2}x_1 + y_1 & \\ \text{s.t.} & x_1 \in [0,1], y_1 = \max\{0, 1-x_1-x_2\}, & \\ & y_1 = y_2. & \\ \hline \end{array}$$

$$\begin{array}{|lll|} \hline \mathrm{L}_2^{\rm cc}(x_1,y_1) \min\limits_{x_2,y_2} & \varphi_2(x_2,y_2) = -\tfrac{1}{2}x_2 - y_2 & \\ \text{s.t.} & x_2 \in [0,1], y_2 = \max\{0, 1-x_1-x_2\}, & \\ & y_1 = y_2. & \\ \hline \end{array}$$

The set $\mathcal{F}^{\rm cc}$ is given by,

$$\mathcal{F}^{\rm cc} = \left\{(x,y) | x \in [0,1]^2, y_1 = y_2 = \max(0, 1-x_1-x_2)\right\}$$

and so the global minimizer of π over $\mathcal{F}^{\rm cc}$ is

$$\arg \min_{(x,y)\in\mathcal{F}^{\rm cc}} \tfrac{1}{2}x_1 + y_1 - \tfrac{1}{2}x_2 - y_2 = ((0,1),(0,0)),$$

To see why $((x_1,x_2),(y_1,y_2)) = ((0,1),(0,0))$ is an equilibrium of this modified game, consider the objectives of the two leaders at equilibrium. For Leader 1, $\varphi_1(0,0) = 0$ whereas for leader 2 $\varphi_2(1,0) = -\frac{1}{2}$. The value 0 is the global minimum for leader 1, and is clearly his optimal. Leader 2's strategy set at equilibrium is a singleton containing only his equilibrium strategy: y_2 is already fixed $= y_1 = 0$, and the conditions $0 = \max\{0, 1-x_2\}$ and $x_2 \in [0,1]$ imply $x_2 = 1$. We therefore see that the requirement of consistency of conjectures constrains the leaders' problems in such a way that it ensures the existence of an equilibrium.

We emphasize that enforcing consistency is not an avenue for *creating* equilibria; instead, we believe that this approach rids such games of some inherent ill-posedness, and perhaps thereby allows for the emergence of equilibria.

D. Ex-post and ex-ante consistency of conjectures

We now note certain subtle but important points about the consistency of conjectures of follower equilibria made by leaders. Throughout we assume that $\mathcal{S}$ is single-valued.

Let (x^*,y^*) be an equilibrium of the conventional formulation $\mathcal{E}$. By Proposition 3.1 (ii), (x^*,y^*) lies in $\mathcal{F}^{\rm cc}$. Consider an arbitrary point feasible point (x,y) in $\Omega(x^*,y^*)$. The key claim we want to make is that even though (x^*,y^*) satisfies consistency of y^*, a feasible point (x,y) may not (i.e. y may not belong to $\mathbf{A}$). The reason is as follows: $(x,y) \in \Omega(x^*,y^*)$ implies that for each i, decisions x_i, y_i satisfy $y_i = \mathcal{S}(x_i; x^{*,-i})$. But the terms $y_1 = \mathcal{S}(x_1, x^{*,-1}), \ldots, y_N = \mathcal{S}(x_N, x^{*,-N})$ may not all be equal. Therefore y is not necessarily in $\mathbf{A}$.

The situation is different in the formulation $\mathcal{E}^{\rm cc}$. Any point $(\bar{x},\bar{y}) \in \Omega^{\rm cc}(x^*,y^*)$ (i.e., one that is feasible for individual problems $\{\mathrm{L}_i^{\rm cc}\}_{i\in\mathcal{N}}$) necessarily satisfies the consistency requirement. This is because $\mathcal{E}^{\rm cc}$ is a more restrictive formulation: $\Omega^{\rm cc}(x^*,y^*)$ is nonempty only for those y^* that belong to $\mathbf{A}$. Furthermore any $(\bar{x},\bar{y}) \in \Omega^{\rm cc}(x^*,y^*)$ must satisfy $\bar{y} = y^*$ and thus must satisfy $\bar{y} \in \mathbf{A}$.

In summary, there is a distinction between the consistency of conjectures *at equilibrium* alone (as in $\mathcal{E}$ with single-valued $\mathcal{S}$) and the stronger requirement of consistency of conjectures *in the optimization problem* of each leader (as espoused by $\mathcal{E}^{\rm cc}$). The former is an *ex-post consistency* that holds only for the equilibrium (or "resolution") of the game; it is ensured by the single-valuedness of $\mathcal{S}$ in the conventional formulation. The latter, is an *ex-ante* consistency that is explicitly enforced as a part of the "rules" of the game.

At this juncture, it is worth reflecting on the meaning of the formulation $\mathcal{E}^{\rm cc}$. One possible view is that $\mathcal{E}^{\rm cc}$ is an *alternative model* of hierarchical competition in which leaders have stronger informational requirements; specifically, leaders are *cognizant* of the conjectures made by their adversaries. Another possible view is that the original game $\mathcal{E}$ is the only "right" formulation. An equilibrium of $\mathcal{E}^{\rm cc}$ is then a *weaker equilibrium notion* for $\mathcal{E}$ (when $\mathcal{S}$ is single-valued, cf. Proposition 3.1 (iii)).

IV. COMPUTATIONAL IMPLICATIONS

Multi-leader multi-follower games do not admit tractable sufficiency conditions which can then be collectively employed for equilibrium computation. The Gauss-Seidel scheme for computation of equilibria sequentially solves the agent problems and cycles through the agents, till no improvement can be determined [9]. Such schemes are not guaranteed to converge in theory and display markedly unpredictable behavior in practice. In this section, we demonstrate the role of the *ex-ante* consistency of conjectures in stabilizing the behavior of the Gauss-Seidel heuristic on a problem formulated in the form $\mathcal{E}^{\rm cc}$, even while the heuristic does not converge on the corresponding problem $\mathcal{E}$ formulated in the conventional way. We do this for a two-settlement spot-forward market under uncertainty studied in [9]. This game is essentially a stochastic multi-leader multi-follower game in which the leader is faced by an uncertain spot-market (follower equilibrium), captured by a set of scenario-specific linear complementarity problems [3]:

$$0 \le z^\xi \perp M^\xi z^\xi + Nf + q^\xi \ge 0, \qquad (9)$$

where $\xi \in \Xi$ is the realization of the uncertainty, f denotes the vector of forward decisions made by the leaders, Ξ denotes a sample space of finite cardinality, and M^ξ, N and q^ξ are defined in [9, Sec. 3]. The forward market participants are viewed as leaders with respect to the followers in the spot-market game. The firm problem in the forward market may be compactly stated as follows:

$$\begin{aligned} (\mathrm{L}_i(f^{-i})) \qquad & \max_{f_i,y_i} \ \mathbb{E}\left(\tfrac{1}{2}(y_i^\xi)^T Q^\xi y_i^\xi + (r^\xi)^T y_i^\xi\right) \\ & 0 \le y_i^\xi \perp M^\xi y_i^\xi + Nf + q^\xi \ge 0, \quad \forall \xi \in \Xi, \end{aligned}$$

where $y_i \triangleq \{y_i^j\}_{j=1}^{|\Xi|}$ denotes leader i's conjecture regarding the spot-market equilibrium and $Q^\xi, r^\xi, \xi \in \Xi$ are constants.

For computing an equilibrium of $\mathcal{E}^{\rm cc}$ if one uses a Gauss-Seidel heuristic, the solution is stuck at the initial values of the follower decision since the leader problems are required to satisfy the consistency requirement. To obviate this problem, we introduce an exact penalization on the consistency constraints across follower decisions and denote firm i's problem by $(\mathrm{EL}_i(f^{-i}, y^{-i}; \rho))$, defined as follows:

$$\max_{f_i, y_i} \mathbb{E}\left(\tfrac{1}{2}(y_i^\xi)^T Q^\xi y_i^\xi + (r^\xi)^T y_i^\xi\right) + p_i^E(y_i; y_{-i}, \rho)$$

$$0 \le y_i^\xi \perp M^\xi y_i^\xi + Nf + q^\xi \ge 0, \quad \forall \xi \in \Xi,$$

where $$p_i^E(y_i; y^{-i}, \rho) \triangleq \begin{cases} \rho \sum_{j=2}^N \|y_j - y_1\|_1, & i = 1 \\ \rho \|y_i - y_1\|_1, & i > 1. \end{cases}$$

Note that when ρ is small, the scheme closely resembles the standard Gauss-Seidel scheme on the *original (unmodified)* problem. When the parameter is large, the scheme is akin to a Gauss-Seidel type scheme applied on the $\mathcal{E}^{\rm cc}$ modification (but via a penalization as above).

N	$\|\Xi\|$	iter	$\frac{\|f^k - f^{k-1}\|_\infty}{(1+\|f^k\|_2)}$	inc	$\sum \|y_i^{**} - y_i^*\|_\infty$
2	5	10	1.56e-07	3.76e+04	5.96e+03
2	10	31	1.66e-04	1.03e+05	2.27e+04
2	15	31	1.66e-04	1.78e+05	4.76e+04
2	20	31	8.63e-01	1.80e+05	1.77e+05
2	25	31	1.90e-04	2.17e+05	4.69e+04
2	30	31	5.71e-04	4.40e+05	2.75e+05
2	35	31	6.47e-02	3.31e+05	8.79e+04
2	5	10	1.56e-07	3.76e+04	5.96e+03
3	5	31	2.63e-01	8.91e+04	2.47e+04
4	5	31	2.41e-03	1.10e+05	1.76e+04

TABLE I

PERFORMANCE OF EXACT PENALTY SCHEME WITH $\rho = 1\mathrm{e}{-3}$

N	$\|\Xi\|$	iter	$\frac{\|f^k - f^{k-1}\|_\infty}{(1+\|f^k\|_2)}$	inc	$\sum \|y_i^{**} - y_i^*\|_\infty$
2	5	16	3.06e-07	1.05e-05	1.40e-03
2	10	2	6.14e-07	8.76e+01	1.62e+00
2	15	12	4.15e-07	1.94e-05	1.17e-04
2	20	8	1.89e-07	6.23e-04	4.48e-04
2	25	11	7.55e-07	3.24e-05	1.01e-04
2	30	11	9.51e-07	2.64e-05	1.55e-04
2	35	4	1.27e-07	1.52e-01	1.01e-01
2	5	16	3.06e-07	1.05e-05	1.40e-03
3	5	5	6.72e-08	1.30e-02	4.91e-03
4	5	4	3.48e-07	3.73e-02	9.94e-03

TABLE II

PERFORMANCE OF EXACT PENALTY SCHEME WITH $\rho = 1\mathrm{e}3$

Tables I and II show the results obtained. In each table, we have shown `iter` (the number of cycles that the Gauss-Seidel scheme proceeds through before termination), $\frac{\|f^k - f^{k-1}\|_\infty}{(1+\|f^k\|_2)}$ (the scaled difference in forward decisions upon termination; the scheme terminates when this is small or when an iteration limit is reached), `inc`$= \sum \|y_1^* - y_i^*\|_\infty$ (the inconsistency in conjectures of follower equilibrium), and $\sum_{i=1}^N \|y_i^{**} - y_i^*\|_\infty$ (the deviation of y_i^* from y_i^{**}, the solution of the best-response problem derived from solving $\mathrm{L}_i^{\rm cc}(y^{-i,*})$). Note that if y_i^{**} differs significantly from y_i^*, it suggests that y_i^* may not be an equilibrium.

When employing a small penalty parameter, the scheme does not display convergence within a 30 iteration limit. In particular, neither the disparity in leader decisions nor the consistency in follower decisions are seen to converge to zero when ρ is small. In more detailed tests, we observed that the Gauss-Seidel scheme with small penalty parameter converged infrequently, as in Table I. However, with a larger penalty parameter, the scheme is seen to converge within 16 iterations. The Gauss-Seidel scheme converged in almost all runs, as seen in Table II. Whenever the scheme did converge, in most instances, the follower decisions were consistent.

V. CONCLUSIONS

This paper was motivated by two concerns associated with multi-leader multi-follower games: (i) leaders' conjectures about the follower equilibrium may be inconsistent and such equilibria may not be sensible in physical settings; and (ii) there are no general existence results. We presented a modified formulation where leader problems were constrained by a consistency requirement across leader-specific conjectures of follower equilibria. We showed that this resolves concerns (i) and (ii). In particular, if leader payoffs admit a potential function, the modified game admits equilibria under mild conditions. Finally, preliminary evidence suggests that equilibria appear significantly easier to compute for such a model.

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