Multivariable control systems

Anurag
anuragg.in@gmail.com
www.anuragg.in

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1 State space equation [1]

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$
$$y = Cx(t) \tag{1.2}$$

2 Controllability [2]

2.1 Controllable subspace

$$W_T = \int_0^T e^{At} BB' e^{A't} dt \tag{2.1}$$
$$u(t) = B' e^{A'(T-t)} z = B' e^{A'(T-t)} W_T^{-1} (x_f - e^{AT} x_0); \tag{2.2}$$

3 Observervability [2]

3.1 Preliminaries for minimal observal design

- Let $\mathcal{B} \oplus \mathcal{D} = \mathcal{X}$ and P be the projection operator on $\mathcal{D}$ along $\mathcal{B}$, then $\langle PA|PA\mathcal{B}\rangle = P\mathcal{X} = \mathcal{D}$
- If (A, B) is controllable then $\exists\ \mathcal{V}, F$ s.t. $\mathcal{B} \oplus \mathcal{V} = \mathcal{X}$ and $(A + BF)\mathcal{V} \subset \mathcal{V}$ and $\sigma[(A + BF)|\mathcal{V}] = \Lambda$ where Λ is any set of symmetric $dim(\mathcal{X}) - dim(\mathcal{B})$ eigenvalues. To find the unknowns use $\sigma[(PA + PBF_0)|\mathcal{D}] = \Lambda$, $\mathcal{V} = (P + BF_0)\mathcal{D}$ and $BF = (BF_0P - I + P)A$

3.2 Minimal observer design

1. Obtain the dual matrices A', C'
2. Find $\mathcal{D}'$ s.t. $\mathcal{C}' \oplus \mathcal{D}' = \mathcal{X}'$
3. Obtain projection matrix P on $\mathcal{D}'$ along $\mathcal{C}'$
4. Find F_0 s.t. $PA' + PC'F_0$ has desired eigenvalues Λ
5. Obtain A'-invariant subspace $\mathcal{V}' = (P + C'F_0)\mathcal{D}'$
6. $C'F = (C'F_0P - I + P)A'$

7. Obtain K using $K = -F'$
8. Find T' using insertion map V' satisfying the relation $(A - KC)'V' = V'T'$
9. Desired minimal observer which gives $Vx(t)$ s.t. $Cx(t) \oplus Vx(t) = x(t)$ is given by $\dot{z}(t) = Tz(t) + VKy(t) + VBu(t)$

3.3 Observer for bad states if $ker(C) \supset \mathcal{X}_g$

1. Find $\bar{A} : \mathcal{X}/\mathcal{X}_g \to \mathcal{X}/\mathcal{X}_g$
2. Find $\bar{C}$ using $\bar{C}P_{\mathcal{X}/\mathcal{X}_g} = C$
3. Design full observer using $\bar{A}, \bar{C}$

3.4 Observer for bad states if $ker(C) \not\supset \mathcal{X}_g$

1. Find $\mathcal{S} = ker(C) \cap \mathcal{X}_g$
2. Find $\bar{A} : \mathcal{X}/\mathcal{S} \to \mathcal{X}/\mathcal{S}$
3. Find $\bar{C}$ using $\bar{C}P_{\mathcal{X}/\mathcal{S}} = C$
4. Design full observer using $\bar{A}, \bar{C}$

3.5 Minimal detector problem

1. Find $ker(C)$
2. Take $\mathcal{K} \supset ker(C)$
3. Obtain D matrix s.t. $ker(DC) = \mathcal{K}$
4. Find largest unobservable space $\mathcal{V}$ in $\mathcal{K}$ which is $ker(DC; DCA; ...; DCA^{n-1})$
5. Check if required state to be observed is contained in $\mathcal{X}/\mathcal{V}$, if not repeat with step 2.
6. Find $\bar{A}, \bar{C}$ using $\bar{A}P = PA$ and $\bar{C}P = DC$. (Note: It is better to find the matrices in reduced subspace $\mathcal{X}/\mathcal{V}$ and then perform next step)
7. Design observer using $\bar{C}|_{\mathcal{X}/\mathcal{V}}, \bar{A}|_{\mathcal{X}/\mathcal{V}}$

4 (A, B) invariant subspaces [2]

4.1 Notation

Family of (A, B) invariant subspace in $\mathcal{X}$ is denoted by $\mathcal{I}(A, B; \mathcal{X})$.

4.2 Algorithm to find (A, B)-invariant subspace inside $\mathcal{X}$ i.e. $\mathcal{V}^*$

$$\mathcal{V}_0 = \mathcal{X} \tag{4.1}$$
$$\mathcal{V}_n = \mathcal{X} \cap A^{-1}(\mathcal{V}_{n-1} + \mathcal{B}) \tag{4.2}$$
$$\mathcal{V}_\mu = \mathcal{V}_{\mu+1} \implies \mathcal{V}^* = \mathcal{V}_\mu \tag{4.3}$$

5 Disturbance decoupling prolem [2]

Theorem 5.1. *DDP is solvable iff* $im(S) \subset \mathcal{V}^*$ *where* $\mathcal{V}^* = sup\ \mathcal{I}(A, B, ker(C))$

6 Output stabilization [2]

Theorem 6.1. *Output stabilization is solvable iff* $\mathcal{X}_b(A) \subset \langle A|\mathcal{B}\rangle + \mathcal{V}^*$ *where* $\mathcal{V}^* = \sup\ \mathcal{I}(A, B; ker(C))$

7 Controllability subspace [2]

Theorem 7.1. $\mathcal{R}$ *is a controllability subspace iff* $\mathcal{R} = \langle A + BF|\mathcal{B} \cap \mathcal{R}\rangle$

Theorem 7.2. *If* $\mathcal{V} = \langle A + BF|\mathcal{V} \cap \mathcal{B}\rangle$ *is a c.s. then* $\mathcal{V} = \langle A + BF_1|\mathcal{V} \cap \mathcal{B}\rangle\ \forall F_1 \in \mathbb{F}(\mathcal{V})$

Lemma 7.1. *If* $\mathcal{V} \in \mathcal{I}(A, B; \mathcal{X})$ $F_1, F_2 \in \mathbb{F}(\mathcal{V})$ *then* $B(F_2 - F_1)\mathcal{V} \subset \mathcal{B} \cap \mathcal{V}$

Theorem 7.3. *Let* $A_0 = (A + BF)|\mathcal{V}$ *and* $B_0 = (\mathcal{V} \cap \mathcal{B})|\mathcal{V}$*. Then,* $\langle A_0, B_0\rangle$ *is controllable.*

7.1 Largest controllability space in $\mathcal{R} \in \mathcal{I}(A, B; \mathcal{X})$

1. Let $\mathcal{R} \in \mathcal{I}(A, B; \mathcal{X})$
2. Generate $\mathcal{S}_* = \mathcal{S}_\mu = \mathcal{S}_{\mu+1}$ using $\mathcal{S}_i = (A\mathcal{S}_{i-1} + \mathcal{B}) \cap \mathcal{R}$ with $\mathcal{S}_0 = 0$

7.2 Spectrum assignability of $\mathcal{R} \in \mathcal{I}(A, B; \mathcal{X})$

Let $\mathcal{R} \in \mathcal{I}(A, B; \mathcal{X})$. Generate S_* which is the largest controllability subspace contained in $\mathcal{R}$. Then, we can only freely assign eigenvalues corresponding to subspace $\mathcal{S}_* \subset \mathcal{R}$

7.3 Disturbance decoupling problem with stability

Theorem 7.4. *Assuming* (A, B) *is controllable, DDP with stability is solvable iff* $im(S) \subset \mathcal{V}_g^*$ *where* $\mathcal{V}_g^* = \mathcal{R}^* + \mathcal{A}_{2g}$ *where* A_{2g} *is good eigenvectors of* $A_2|\mathcal{V}^*/\mathcal{R}^*$ *where* $A = \begin{bmatrix} A_1 & * & * \\ 0 & A_{2g} & 0 \\ 0 & 0 & A_{2b} \end{bmatrix}$ *in basis of* R^*, A_{2g}, A_{2b}*.*

8 Equivalent classes of systems

8.1 Controllability indices and controllability index [2, p. 121]

8.2 Canonical form [2, p. 121]

8.3 Possible c.s. and exactly one c.s. [2, p. 124]

Theorem 8.1. *Let* (A, B) *be controllable, with controllability indices* $k_1 \geq \ldots \geq k_m$*. Then the possible dimensions of these nonzero c.s. of* (A, B) *are given by the list*

$$k_m; \tag{8.1}$$

$$k_{m-1}, k_{m-1} + 1, \ldots, k_{m-1} + k_m \tag{8.2}$$

$$\vdots \tag{8.3}$$

$$k_1, k_1 + 1, \ldots, k_1 + k_2 + \cdots + k_m \tag{8.4}$$

There is exactly one c.s of dimension $r \neq 0$ if (i) $r = n$ (ii) for some $j \in 1, 2, \ldots, m-1$,

$$k_j > r = k_{j+1} + \cdots + k_m \tag{8.5}$$

9 Restricted regulator problem [2]

$$(Necessary\ condition) \quad X_b(A) \cap \mathcal{N} \subset ker(D) \tag{9.1}$$
$$X_b(A) \cap \mathcal{N} \subset \mathcal{V}; \mathcal{V} \in \mathcal{I}(A, B; ker(D) \tag{9.2}$$
$$A(\mathcal{V} \cap \mathcal{N}) \subset \mathcal{V} \tag{9.3}$$
$$X_b(A) \subset \langle A|B \rangle + \mathcal{V} \tag{9.4}$$
$$F\mathcal{N} = 0 \tag{9.5}$$

9.1 Finding maximal element of $\{\mathcal{V} \in \mathcal{I}(A, B; ker(D))\ s.t.\ A(\mathcal{V} \cap \mathcal{N}) \subset \mathcal{V}\}$

$$\mathcal{V}^M = \mathcal{V}_0 \oplus \mathcal{V}_i \tag{9.6}$$
$$\mathcal{V}_0 = sup\{\mathcal{V} : \mathcal{V} \subset ker(D) \cap \mathcal{N}, A\mathcal{V} \subset \mathcal{V}\} \tag{9.7}$$
$$\mathcal{V}_i = sup\{\mathcal{V} : \mathcal{V} \subset \mathcal{W} \cap A^{-1}(\mathcal{B} + \mathcal{V}_0 + \mathcal{V})\} \tag{9.8}$$

where W is a suitable complement of $\mathcal{N} \cap ker(D)$ in $ker(D)$

Corrolary 9.1. *If $A(\mathcal{N} \cap ker(D)) \subset ker(D)$ then RRP is solvable iff conditions for RRP are satisfied.*

10 Extended regulator problem [2]

Theorem 10.1. *ERP is solvable iff*

$$X_b(A) \cap \mathcal{N} \subset ker(D) \tag{10.1}$$
$$X_b(A) \subset \langle A|B \rangle + \mathcal{V}^* \tag{10.2}$$

Finding feedback matrix

$$\mathcal{V}_0 = A - invariant\ subspace\ contained\ in\ \mathcal{V}^* \cap \mathcal{N} \tag{10.3}$$
$$\mathcal{V}^* \cap \mathcal{N} = \mathcal{V}_0 \oplus \mathcal{V}_1 \tag{10.4}$$
$$\mathcal{V}^* = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \tag{10.5}$$
$$\mathcal{X} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \tag{10.6}$$
$$dim(\mathcal{X}_a) = dim(\mathcal{V}_1) \tag{10.7}$$
$$E : \mathcal{X}_e \to \mathcal{X}_e\ s.t.\ \ker(E) \cap \mathcal{V}_1 = 0,\ ker(E) \subset \mathcal{V}_0 \oplus \mathcal{V}_2 \tag{10.8}$$
$$\mathcal{V}_e^* = (I + E)\mathcal{V}^* \tag{10.9}$$
$$\mathbb{F}(\mathcal{V}_e^*) \ni F_e\ and\ \ker(F_e) \subset \mathcal{N} \tag{10.10}$$

11 Reference

[1] N S Nise. *Control systems engineering.* Wiley, 2004. ISBN: 9780471445777,0-471-44577-0,0-471-45243-2.

[2] W N Wonham. *Linear multivariable control : a geometric approach.* Vol. 10. Applications of Mathematics. Springer, 1985. ISBN: 0387960716,9780387960715.