Properties of Riemannian manifolds

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Outline

1. Tensor fields
2. Curvature
3. Euler Lagrange equations
4. Conventional robotics
5. Control on a Riemannian manifold
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1. Tensor fields
2. Curvature
3. Euler Lagrange equations
4. Conventional robotics
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Tensor fields

Definition
The $C^\infty$ sections of $T^r_s(TM)$ are called class $C^\infty(r,s)$ tensor fields on $M$ and denoted by $\Gamma^\infty(T^r_s(TM))$.

- A $C^\infty$ vector field on $M$ is a class $C^\infty(1,0)$ tensor fields on $M$.
- A $C^\infty$ covector field on $M$ is a class $C^\infty(0,1)$ tensor fields on $M$.
- A $C^\infty$ Riemannian metric on $M$ is a class $C^\infty(0,2)$ tensor field $G$ on $M$ having the property that $G(x)$ is an inner product on $T_xM$.

Associating maps with tensor fields
Given $t \in \Gamma^\infty(T^r_s(TM))$, $\alpha^1, \ldots, \alpha^r \in \Gamma^\infty(T^*M)$, and $X_1, \ldots, X_s \in \Gamma^\infty(TM)$, we define a function on $M$

$$x \rightarrow t(x) \cdot (\alpha^1(x), \ldots, \alpha^r(x), X_1(x), \ldots, X_s(x))$$

and interpret $t$ as a map

$$\Gamma^\infty(T^*M) \times \cdots \times \Gamma^\infty(T^*M) \times \Gamma^\infty(TM) \times \cdots \times \Gamma^\infty(TM) \rightarrow C^\infty(M)$$
Differentiating tensors

The Lie derivative of a tensor

- Say $\alpha \in \Gamma^\infty(T^r_s(TM))$. Then what is $\mathcal{L}_X \alpha$?
- Going the natural way, the Lie derivative of $\alpha$ must satisfy

$$\mathcal{L}_X \langle \alpha, Y \rangle = \langle \mathcal{L}_X \alpha, Y \rangle + \langle \alpha, \mathcal{L}_X Y \rangle$$

$$\Rightarrow \langle \mathcal{L}_X \alpha, Y \rangle = \mathcal{L}_X \langle \alpha, Y \rangle - \langle \alpha, \mathcal{L}_X Y \rangle$$

- Check whether the RHS is $C^\infty(M)$ linear in $Y$.

Explicit computation

$$(\mathcal{L}_X t)(\alpha^1, \ldots, \alpha^r, X_1, \ldots, X_s) = \mathcal{L}_X (t(\alpha^1, \ldots, \alpha^r, X_1, \ldots, X_s))$$

$$- \sum_{i=1}^{r} t(\alpha^1, \ldots, \mathcal{L}_X \alpha^i, \ldots, \alpha^r, X_1, \ldots, X_s) - \sum_{i=1}^{s} (\alpha^1, \ldots, \alpha^r, X_1, \ldots, \mathcal{L}_X X_j, \ldots, X_s)$$
Tensors associated with a Riemannian manifold

Definition
A tensor $T$ of order $r$ on a Riemannian manifold is a multilinear mapping

$$T : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathcal{D}(M)$$

- A tensor is a pointwise object.
- The curvature tensor

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}(M)$$

is defined as

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle \quad X, Y, Z, W \in \mathfrak{X}(M)$$

- The metric tensor

$$G : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}(M) \quad G(X, Y) = \langle X, Y \rangle$$

- The Riemannian connection is not a tensor

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}(M) \quad \nabla(X, Y, Z) = \langle \nabla_X Y, Z \rangle$$

since $\nabla$ is not linear with respect to $Y$. 
Covariantly differentiating tensors

Definition

The covariant derivative $\nabla_Z T$ of a tensor $T$ of order $r$ relative to $Z$ is a tensor of order $r$ given by

$$\nabla_Z T(Y_1, \ldots, Y_r) = Z(T(Y_1, \ldots, Y_r)) - \sum_{i=1}^{r} T(Y_1, \ldots, \nabla_Z Y_i, \ldots, Y_r)$$

Definition

The covariant differential of a tensor $T$ of order $r$ is a tensor of order $(r + 1)$ given by

$$\nabla T(Z, Y_1, \ldots, Y_r) = \nabla_Z T(Y_1, \ldots, Y_r)$$
Relating Lie differentiation and covariant differentiation

Claim

\[(\mathcal{L}_X S)(Y_1, \ldots, Y_r) = (\nabla_X S)(Y_1, \ldots, Y_r) - \nabla_{S(Y_1, \ldots, Y_r)} X + \]

\[\sum_{i=1}^{r} S(Y_1, \ldots, \nabla_{Y_i} X, \ldots, Y_r)\]

Can be proved from the two identities

\[\mathcal{L}_X (S(Y_1, \ldots, Y_r)) = \nabla_X (S(Y_1, \ldots, Y_r)) - \nabla_{S(Y_1, \ldots, Y_r)} X\]

\[\mathcal{L}_X (S(Y_1, \ldots, Y_r)) = (\mathcal{L}_X S)(Y_1, \ldots, Y_r) + \sum_{i=1}^{r} S(Y_1, \ldots, \mathcal{L}_X Y_i, \ldots, Y_r)\]
Claim

*The covariant differential of the metric tensor is the zero tensor*

\[ \nabla G(Z, X, Y) = Z \langle X, Y \rangle - \langle \nabla_Z X, Y \rangle - \langle X, \nabla_Z Y \rangle = 0 \]

- The tensor \( \nabla_Z X \) can be identified with the vector field \( \nabla_Z X \) since
  \[
  \nabla_Z X(Y) = \nabla X(Z, Y) = Z(X(Y)) - X(\nabla_Z Y) = Z \langle X, Y \rangle - \langle X, \nabla_Z Y \rangle = \langle \nabla_Z X, Y \rangle
  \]
- If \( S \) is a tensor,
  \[
  \nabla^k S = \nabla(\nabla^{k-1} S)
  \]
- From the above identity,
  \[
  \nabla^2 X(Y, Z) = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X
  \]
Contravariant tensors

- On a Riemannian manifold, to each $X \in \mathcal{X}(M)$ we associate a unique one-form $\theta_X \in \mathcal{X}^*(M)$ (dual to $\mathcal{X}$) given by

$$\theta_X(Y) = \langle X, Y \rangle \quad \forall Y \in \mathcal{X}(M)$$

- For each $x \in M$, there are isomorphisms $\mathbb{G}^\# : T^*M \to TM$ and $\mathbb{G}^\flat : TM \to T^*M$ associated with the Riemannian metric $\mathbb{G}$.

- Further,

$$d\theta_X(V, W) + \mathcal{L}_X g(V, W) = 2g(\nabla_V X, W)$$
Coordinate computation of the Lie derivative of the metric tensor -

- Take \( t = g_{ij} dx^i dx^j \) and \( Y = (Y_1, Y_2) \).

\[
\mathcal{L}_X (t(Y_1, Y_2)) = \partial_k g_{ij} X^k Y_1^i Y_2^j + g_{ij} \partial_k Y_1^i X^k Y_2^j + g_{ij} Y_1^i \partial_k Y_2^j X^k
\]

- \[
t(L_X Y_1, Y_2) + t(Y_1, L_X Y_2) = g_{ij} [\partial_k Y_1^i - \partial_m X^i Y_1^m] Y_2^j
\]

- \[
g_{ij} [\partial_k Y_2^j X^k - \partial_m X^j Y_2^m] Y_1^i
\]

- \[
\mathcal{L}_X t(Y_1, Y_2) = \mathcal{L}_X (t(Y_1, Y_2)) - [t(L_X Y_1, Y_2) + t(Y_1, L_X Y_2)]
\]
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Curvature of a Riemannian manifold

Definition
The curvature $R$ of a Riemannian manifold $M$ is a correspondence that associates to every pair $X, Y \in \mathcal{X}(M)$ a mapping $R(X, Y) : \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]}Z, \quad Z \in \mathcal{X}(M)$$

Measures the non-commutativity of the covariant derivative.

Proposition
- $R$ is bilinear in $\mathcal{X}(M) \times \mathcal{X}(M)$
- The curvature operator $R(X,Y) : \mathcal{X}(M) \to \mathcal{X}(M)$ is linear.
Curvature identities

**Proposition**

*Bianchi Identity*

\[ R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \]
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Euler-Lagrange equations on a Riemannian manifold

Preliminaries

• Action functional

\[ A_L(\gamma) = \int_a^b L(t, \gamma'(t))dt \]

• \( \gamma : [a, b] \to Q \) is a twice differentiable curve on \( M \) starting at \( q_a \) and ending at \( q_b \).

• \( \nu : J \times [a, b] \to Q \) denotes a class of such admissible curves, each curve being parametrized by \( s \in J \).

• Necessary condition for a minimum

\[ \frac{d}{ds} \bigg|_{s=0}(A_L(\nu(s, t))) = 0 \]
Facts

- Two vector fields
  \[ S_\nu(s, t) = \frac{d}{ds} \nu(s, t), \quad T_\nu(s, t) = \frac{d}{dt} \nu(s, t) \]
  Define \( X_{S_\nu} \circ \nu = S_\nu \quad X_{T_\nu} \circ \nu = T_\nu \)

- Equality of mixed partials leads to
  \[ [X_{S_\nu}, X_{T_\nu}](\nu(s, t)) = 0 \]

- \( \nabla_{S_\nu} T_\nu = \nabla_{T_\nu} S_\nu \quad (\nabla XY - \nabla Y X = [X, Y] = 0) \)

- \( \frac{d}{dt} G(S_\nu, T_\nu)(s, t) = \mathcal{L}_{T_\nu} G(S_\nu, T_\nu)(s, t) = \]
  \[ G(\nabla_{T_\nu} S_\nu, T_\nu)(s, t) + G(S_\nu, \nabla_{T_\nu} T_\nu)(s, t) \]

- \( \frac{d}{ds} G(T_\nu, T_\nu)(s, t) = \mathcal{L}_{S_\nu} G(T_\nu, T_\nu)(s, t) = 2G(\nabla_{S_\nu} T_\nu, T_\nu)(s, t) \)
Outline of the proof

\[ 0 = \frac{d}{ds} |_{s=0} (A_L(\nu(s, t))) \]
\[ = \frac{d}{ds} |_{s=0} \frac{1}{2} \int_a^b \mathcal{G}(T_\nu(s, t), T_\nu(s, t)) dt \]
\[ = \int_a^b \mathcal{G}(\nabla S_\nu T_\nu(s, t), T_\nu(s, t)) dt |_{s=0} \]
\[ = \int_a^b \mathcal{G}(\nabla T_\nu S_\nu(s, t), T_\nu(s, t)) dt |_{s=0} \]
\[ = \int_a^b \mathcal{G}(\nabla \gamma'(t) \delta \nu(t), \gamma'(t)) dt \]
\[ = \int_a^b \left[ \frac{d}{dt} \mathcal{G}(\delta \nu, \gamma'(t)) - \mathcal{G}(\delta \nu(t), \nabla \gamma'(t) \gamma'(t)) dt \right] \]

Leads to

\[ \nabla \gamma'(t) \gamma'(t) = 0 \]
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A robotic manipulator

Figure: A conventional manipulator (Courtesy: Murray, Li and Sastry)
Forward kinematics

How do the joint angles map to the position of the end-effector?

- Configuration space: Revolute joint - $S^1$, prismatic joint - $\mathbb{R}^1$
- Say $p$ revolute joints and $m$ prismatic joints,
- Forward kinematics map
  \[ f(\cdot) : S^1 \times \ldots \times S^1 \times \mathbb{R}^1 \times \ldots \times \mathbb{R}^1 = Q \rightarrow SE(3). \]
  
- $\xi_i$ is the velocity vector (translational or rotational) associated with the $i$th joint.
- For $n$ joints - the product of exponentials yields
  \[ f(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \ldots e^{\hat{\xi}_n \theta_n} f(0) \]
  a product of exponentials
- Recall the geometric interpretation - the exponential map takes the Lie algebra to the Lie group.
Kinetic energy

- **Kinetic energy** = sum of the kinetic energy of each link. For the \( i \)th link

\[
T_i(\theta, \dot{\theta}) = \frac{1}{2} (V_{bl_i}^b)^T \mathcal{M}_i (V_{bl_i}^b) = \frac{1}{2} \dot{\theta}^T [J_{bl_i}^b(\theta)^T \mathcal{M}_i J_{bl_i}^b(\theta)] \dot{\theta}
\]

- **Total kinetic energy**

\[
T(\theta, \dot{\theta}) = \sum_{i=1}^{n} \frac{1}{2} \dot{\theta}^T M(\theta) \dot{\theta}
\]

- **Manipulator inertia matrix**

\[
M(\theta) = \sum_{i=1}^{n} J_{bl_i}^b(\theta)^T \mathcal{M}_i J_{bl_i}^b(\theta)
\]
Potential energy and the Lagrangian

- **Potential energy**

\[ V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta) \]

- **Lagrangian** - \( \mathcal{L}(\theta, \dot{\theta}) = \)

\[
\frac{1}{2} \sum_{i,j=1}^{n} M_{ij}(\theta) \dot{\theta}_i \dot{\theta}_j - V(\theta)
\]

- **Equations of motion for each link**

\[
\sum_{j=1}^{n} M_{ij}(\theta) \ddot{\theta}_j + \sum_{j,k=1}^{n} \Gamma_{ij}^{k} \dot{\theta}_j \dot{\theta}_k + \frac{\partial V}{\partial \theta_i}(\theta) = \tau_i \quad i = 1, \ldots, n
\]
Interpretation of the equations of motion

• Christoffel symbols corresponding to the inertia matrix $M(\theta)$.

$$
\Gamma^k_{ij} = \frac{1}{2} \left[ \frac{\partial M_{ij}(\theta)}{\partial \theta_k} + \frac{\partial M_{ik}(\theta)}{\partial \theta_j} - \frac{\partial M_{kj}(\theta)}{\partial \theta_i} \right]
$$

arise out of the notion of a covariant derivative

• Coriolis matrix

$$
C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma^k_{ij} \dot{\theta}_k
$$

• Coriolis and centrifugal forces

$$
C(\theta, \dot{\theta}) \dot{\theta}
$$

• Equations of motion are often expressed as

$$
M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + N(\theta, \dot{\theta}) = \tau
$$
Geometric viewpoint

- The dynamic equations are described on the manifold $TQ$.
- An element of $TQ$ is a two tuple $(\theta, \dot{\theta})$.
- The vector field on $TQ$ is given by
  $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -M^{-1}(\theta)[C(\theta, \dot{\theta})\dot{\theta} + N(\theta, \dot{\theta})] \end{bmatrix}$
- The Lagrangian is a map $\mathcal{L} : TQ \to \mathbb{R}$.
- The KE is a metric on the tangent space $T_qQ$ induced by the inertia matrix $M(q)$ as
  $< \dot{\theta}_i, \dot{\theta}_j > \triangleq \frac{1}{2} \sum_{i,j=1}^{n} M_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j$
- The inertia matrix $M(q)$ is symmetric and positive definite and $\dot{M} - 2C$ is skew-symmetric
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The Riemannian framework

Basic notions

- The manifold $Q$ has a metric $G$ called the Riemannian metric that satisfies properties mentioned before.
- $G$ now defines two operators: $G^\#$ and $G^\flat$.

$$
G^\#(q) : T^*_q Q \to T_q Q \quad G^\flat(q) : T_q Q \to T^*_q Q
$$

$$
\langle v_q, w_q \rangle = [G^\flat v_q, w_q] \forall v_q, w_q \in T_q Q
$$

- The gradient and the Hessian of a function, $f : Q \to \mathbb{R}$:

$$
\text{grad} f = G^\#(df) \quad \text{Hess} f(q) \cdot (v_q, w_q) = \left\langle v_q, \nabla_{w_q} \text{grad} f \right\rangle
$$

- Please note that often in our use in problems, we interchangeably use $\text{grad} f$ and $df$. This is because we are working with the natural basis in $\mathbb{R}^n$. 

Simple mechanical systems

Equations of motion

• $\gamma(t) : [t_i, t_f] \rightarrow Q$ denotes the trajectory of the mechanical system from time $t_i$ to time $t_f$.

• Equation of motion (autonomous system)

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = -\nabla V(\gamma(t)) + G^\#(\text{Potential forces})$$

$$+ G^\#(F_{\text{diss}}(\dot{\gamma}(t)))$$

• With state-feedback control

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = -\nabla V(\gamma(t)) + G^\#(F_{\text{diss}}(\dot{\gamma}(t))) + G^\#(F(t, \dot{\gamma}(t)))$$
Coordinate computation

- Assume there are no potential forces or dissipative forces. Further, assign coordinates to $\gamma(t)$ as $x(t)$. Then, the earlier equation reduces to

$$\nabla \dot{\gamma}(t) \dot{\gamma}(t) = 0$$

(1)

- Christoffel symbols $\Gamma^k_{ij}$

$$\nabla \dot{\gamma}(t) \dot{\gamma}(t) = 0 \Rightarrow \ddot{x}^k(t) + \Gamma^k_{ij}(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t) = 0 \quad k = 1, \ldots, n.$$

- Compute the Christoffel symbols from the Riemannian metric as

$$\Gamma^k_{ij} = \frac{1}{2} G^{kl} \left( \frac{\partial G_{il}}{\partial x^j} + \frac{\partial G_{jl}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^l} \right)$$

where $G^{kl}$ stands for the inverse of $G$. 
Geodesic spray

- The geodesis spray is defined as

\[ S = v^i \frac{\partial}{\partial x^i} - \Gamma^k_{ij} v^j v^k \frac{\partial}{\partial v^i} \]
Pendulum on a cart

A schematic

Figure: A conventional manipulator
Pendulum on a cart

- Configuration space $Q = \mathbb{R} \times S^1$.
- The Lagrangian $L : TQ \rightarrow \mathbb{R}$ is given by

$$L = \frac{1}{2} \left( \alpha \dot{\theta}^2 - 2\beta \cos \theta \dot{\theta} \dot{s} + \gamma \dot{s}^2 \right) + D \cos \theta$$

where $(s, \theta) \in (\mathbb{R}, S^1)$ and $\alpha, \beta, \gamma$ are parameters dependant on $m, M, l$ and $g$.

- The Riemannian metric based on the kinetic energy is

$$\mathcal{G}(s, \theta) \left( \left[ \dot{s}, \dot{\theta} \right], \left[ \dot{s}, \dot{\theta} \right] \right) = \frac{1}{2} \left[ \begin{array}{cc} \dot{s} & \dot{\theta} \\ \dot{s} & \dot{\theta} \end{array} \right] \left[ \begin{array}{cc} \gamma & -\beta \cos \theta \\ -\beta \cos \theta & \alpha \end{array} \right] \left[ \begin{array}{c} \dot{s} \\ \dot{\theta} \end{array} \right]$$