Submanifolds, Immersions and Submersions
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When is a subset \( S \) of an \( n \)-dimensional smooth manifold \( M \) called a submanifold of \( M \)? We answer this question and address a few related issues here.

Definition 0.1 Submanifold
If for each element \( p \in S \), there exists a chart \( (U_p, \phi) \) (note: \( U_p \) is an open set in \( M \) and \( \phi \) is a homeomorphism) such that \( \phi \) maps \( U_p \cap S \) to a subset of \( \mathbb{R}^k \times 0^{(n-k)} \), then \( S \) is said to be a submanifold of \( M \) of dimension \( k \).

Example 1:
Look at the unit circle \( S^1 \) as a subset of \( \mathbb{R}^2 \). Let the open sets (or the topology) on \( \mathbb{R}^2 \) be the open discs, and the mapping \( \phi \) be the identity map to \( \mathbb{R}^2 \). So
\[
\phi(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2, \phi(x, y) = (x, y)
\]
The points on the unit circle centred at 0 satisfy
\[
p = (x_p, y_p) \in S^1 \Rightarrow x_p^2 + y_p^2 = 1
\]
and get mapped to \( \mathbb{R}^2 \) through the homeomorphism \( \phi \) as
\[
\phi(x_p, y_p) = (x_p, y_p) = (x_p, \pm \sqrt{1 - x_p^2})
\]
The \( \pm \) sign in the right-most term indicates either a + or - value depending on the quadrant. Note that the coordinate mapping is a function of just one variable \( x_p \). We shall now introduce a local diffeomorphism to ”zero” one of the coordinates. Define
\[
\beta(\psi, \eta) = (\psi^2 + \eta^2 - 1, \eta) \quad [D\beta] = \begin{bmatrix} 2\psi & 2\eta \\ 0 & 1 \end{bmatrix}
\]
Notice that \( \beta \) is a diffeomorphism in a region around \((\psi_0, \eta_0)\) if and only if \( \psi_0 \neq 0 \). Around a point \( p \) on \( S^1 \) and an open set \( U_p \) containing \( p \) such that \((0, y) \notin (S^1 \cap U_p)\) (to satisfy the above Jacobian condition), we write
\[
\phi'(x_p, y_p) \triangleq \beta \circ \phi(x_p, y_p) = \beta(x_p, \pm \sqrt{1 - x_p^2}) = (0, \pm \sqrt{1 - x_p^2})
\]
Notice that the mapping \( \beta \circ \phi \) is to a one dimensional subspace \((0, \mathbb{R}^1)\) of \( \mathbb{R}^2 \).

How does one decide if a level set on a manifold is a regular submanifold?

There are a few results that could be employed. The submersion theorem and the inverse function theorem are two results that help. Before we do so, we define a few new concepts.
Immersions, submersions and critical values

Definition 0.2 (Submersion and Immersion) If \( f : X \to Y \) is smooth, then

1. If \( T_p f \) is onto for all \( p \in X \), then \( f \) is called a submersion
2. If \( T_p f \) is one-to-one for all \( p \in X \), then \( f \) is called an immersion
3. If \( f \) is an immersion and \( f \) is one-to-one, then \( f(X) \) is an immersed submanifold

Definition 0.3 (Regular point and critical point) Consider a smooth map \( f : X \to Y \)

- A point \( p \in X \) is called a critical point of \( f \) if \( T_p f \) is not onto.
- A point \( p \in X \) is called a regular point of \( f \) if \( T_p f \) is onto.
- A point \( y \in Y \) is called a critical value if \( f^{-1}(y) \) contains a critical point. Otherwise, \( y \) is called a regular value of \( f \).

The Submersion Theorem
If \( f : X^n \to Y^k \) is a smooth map, and \( y \in f(X) \subset Y \) is a regular value of \( f \), then \( f^{-1}(y) \) is a regular submanifold of \( X \) of dimension \( n - k \).

- Example: \( f(x, y) = x^2 + y^2 - 1 \). Take \( f^{-1}(0) \).
- Show that \( O(n) = \{ A \in M_n(\mathbb{R}) | A^T A = I \} \) is a submanifold of \( M_n(\mathbb{R}) \). What is its dimension?
  
  Hint: Consider a map \( f : \mathbb{R}^{n \times n} \to S^{n \times n} \) (symmetric matrices). Let \( f(A) = A^T A \) and examine the value \( I \).

Consider the sphere \( S^2 \) in \( \mathbb{R}^3 \) and consider the homeomorphism

Remark 1 For a smooth map \( f : N \to M \) of manifolds, a level set \( f^{-1}(c) \) is regular if and only if \( f \) is a submersion at every point of \( f^{-1}(c) \). (on the lines of the submersion theorem.)

Example 1 Consider the smooth map \( f : \mathbb{R} \to \mathbb{R}^2, f(t) = (t^2, t^3) \). The map is one-to-one but the Jacobian

\[
[Df](t) = \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}
\]

is not one-to-one at zero.

Is the curve a submanifold of \( \mathbb{R}^2 \)?

No. The condition to be a submanifold fails at \((0, 0)\). At all points \( p \in S \) (except \( 0 \)), one can find an opens set \( U_p \) containing \( p \) and a smooth submersion \( g \) such that \( U_p \cap S \) is a level set of \( g \).
Example 2 Consider the smooth map \( f : \mathbb{R} \to \mathbb{R}^2, f(t) = (t^2 - 1, t^3 - t) \). The map is not one-to-one \((f(-1) = f(1) = (0, 0))\) but the Jacobian

\[
[Df](t) = \begin{bmatrix}
2t \\
3t^2 - 1
\end{bmatrix}
\]

is one-to-one everywhere.

Is the curve a submanifold of \( \mathbb{R}^2 \)?
No. The condition to be a submanifold once again fails at \((0,0)\). At all points \( p \in S \) (except 0), one can find an opens set \( U_p \) containing \( p \) and a smooth submersion \( g \) such that \( U_p \cap S \) is a level set of \( g \).

Example 3 The figure-eight \( f : \mathbb{R} \to \mathbb{R}^2, f(t) = (\cos t, \sin 2t) \). The map is not one-to-one but the Jacobian

\[ Df(t) = (-\sin t, 2\cos 2t) \]

is one-to-one everywhere.

However, note that if this map is restricted to the open interval \((-\pi/2, 3\pi/2)\), then the map is one-to-one. It can also be verified that it is a submanifold of \( \mathbb{R}^2 \).

Example 4 (A spiral) Consider the smooth map \( f : \mathbb{R} \to \mathbb{R}^3, f(t) = (\sin 2\pi t, \cos 2\pi t, t) \). The map is one-to-one and the Jacobian

\[ Df(t) = (2\pi \cos 2\pi t, -2\pi \sin 2\pi t, 1) \]

is one-to-one everywhere. At all points \( p \in S \) one can find an opens set \( U_p \) containing \( p \) and a smooth submersion \( g = (g_1, g_2) : \mathbb{R}^3 \to \mathbb{R}^2 \) such that \( U_p \cap S \) is a level set of \( g \).

Definition 0.4 Embedding
A smooth map \( f : N \to M \) is called an embedding if

- it is a one-to-one immersion (recall the definition of an immersed submanifold.)
- the image \( f(N) \) with the subspace topology is homeomorphic to \( N \) under \( f \).

Example 5 The figure-eight described by the map

\[ f(t) = (\cos t, \sin 2t) \quad -\pi/2 < t < 3\pi/2 \]

is one-to-one and the Jacobian of this map

\[ Df(t) = (-\sin t, 2\cos 2t) \]
is one-to-one everywhere on the open interval \((-\pi/2, 3\pi/2)\). Hence the figure-eight is the image of a one-to-one immersion. Hence it is an immersed submanifold of \(\mathbb{R}^2\) with a topology and manifold structure induced from the open interval \((-\pi/2, 3\pi/2)\) by \(f\).

Is the figure-eight a regular submanifold of \(\mathbb{R}^2\)? No, in the subspace topology of \(\mathbb{R}^2\), the figure-eight is not even a manifold. Why? (Hint: examine the homeomorphism condition around \((0,0)\).)

The tangent bundle

Let us now prove that the tangent bundle \(TM\) of a manifold \(M\) is a manifold. We shall start with manifolds immersed in \(\mathbb{R}^n\).

**Definition 0.5** A subset \(M\) of \(\mathbb{R}^n\) is said to be a submanifold of dimension \(m\) (\(\leq n\)) of \(\mathbb{R}^n\) if for every \(p \in M\) there exists an open set \(U_p \subset \mathbb{R}^n\) containing \(p\) such that

\[
M \cap U_p = \{f^{-1}(c) : \text{where } c \in \mathbb{R}^{n-m} \text{ is a constant and } f : U_p \to \mathbb{R}^{n-m} \text{ is a smooth submersion}\}
\]

We now show, from the perspective of the above definition, that the tangent bundle \(TM\) of \(M\) is a sub manifold.

For every \((p, v_p) \in TM \subset \mathbb{R}^n \times \mathbb{R}^n\), there exists an open set \(U_p \times V\), such that

\[
TM \cap (U_p \times V) = \{\tilde{f}^{-1}(\tilde{c})\},
\]

where

\[
\tilde{f} = \begin{bmatrix} f \\ Df \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} c \\ 0 \end{bmatrix},
\]

where \(Df\) denotes the differential of \(f\) (or the Jacobian).

(Why ?)

Consider a smooth curve \(\alpha(\cdot) : I \to M\), such that \(\alpha(0) = p\). Then \(\dot{\alpha}(0) = v_p\) is a tangent vector at \(p\). Now, from \(f \circ \alpha(t) = c\) for a curve on \(M\),

\[
\frac{d}{dt}(f \circ \alpha)(t) = 0 \Rightarrow Df(p) \cdot \dot{\alpha}(0) = 0
\]

Since \(f\) is a smooth submersion of rank \((n - m)\) and it follows that \([Df(p)]^{-1}(0) = \mathbb{R}^m\).