Pontryagin Maximum Principle Under State-Action-Frequency Constraints

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Abstract

We exhibit a new Pontryagin maximum principle for discrete time optimal control problems under constraints on the frequency spectrum of the optimal control trajectories in addition to constraints on the states and the controls actions pointwise in time.

Optimal control under constrained spectrum

Consider a discrete-time control-affine system described by:

$$x_{t+1} = f_t(x_t) + g_t(x_t)\, u_t \quad \text{for } t = 0, \dots, T-1, \tag{1}$$

where the states $x_t \in \mathbb{R}^d$ and the controls $u_t \in \mathbb{R}^m$, and $(f_t)_{t=0}^{T-1}$ and $(g_t)_{t=0}^{T-1}$ are two families of smooth maps. In the context of (1), consider the following constrained optimal control problem:

$$\begin{aligned} &\underset{(u_t)_{t=0}^{T-1}}{\text{minimize}} \quad \sum_{t=0}^{T-1} c_t(x_t, u_t) \\ &\text{subject to} \begin{cases} \text{controlled dynamics (1)}, \\ x_t \in \mathbb{S}_t \quad \text{for } t = 0, \dots, T, \\ u_t \in \mathbb{U}_t \quad \text{for } t = 0, \dots, T-1, \\ F(u_0, \dots, u_{T-1}) = 0, \end{cases} \end{aligned} \tag{P}$$

with the following data:

1. $T \in \mathbb{N}^*$ is fixed;
2. $\mathbb{R}^d \times \mathbb{R}^m \ni (\xi, \mu) \longmapsto c_t(\xi, \mu) \in \mathbb{R}$ is smooth, convex in μ for each t;
3. $\mathbb{S}_t$ is a subset of $\mathbb{R}^d$ and $\mathbb{U}_t$ is a convex, compact and non-empty subset of $\mathbb{R}^m$;
4. $F : \mathbb{R}^{mT} \longrightarrow \mathbb{R}^\ell$ is a given linear map on the control trajectory $u_0, \dots, u_{T-1}$ for some $\ell \in \mathbb{N}^*$, defined as:

$$F(u_0, \dots, u_{T-1}) = \sum_{t=0}^{T-1} \widetilde{F}_t u_t = 0 \quad \text{for } \left(\widetilde{F}_t\right)_{t=0}^{T-1} \subset \mathbb{R}^{\ell \times mT}. \tag{F}$$

Remark. Band-pass constraints on the discrete Fourier transform (i.e., frequency spectrum,) of the control trajectories are expressible in the form (F). Standard versions of the Pontryagin maximum principle cater to constraints on the control actions, but do not include constraints on the control frequencies — the latter cannot be expressed as constraints on the control actions pointwise in time. The Hamiltonian in the new Pontryagin maximum principle has an additional term compared to the standard versions, and it plays a rôle in the Hamiltonian maximization condition.

Theorem

Let $\left((x_t^*)_{t=0}^T, (u_t^*)_{t=0}^{T-1}\right)$ be an optimal state-action trajectory for (P) with F as defined in (F). Define the Hamiltonian

$$\begin{aligned} &\mathbb{R} \times \mathbb{R}^\ell \times \mathbb{R}^d \times \mathbb{N} \times \mathbb{R}^d \times \mathbb{R}^m \ni (\nu, \vartheta, \zeta, s, \xi, \mu) \longmapsto \\ &\quad H^{\nu,\vartheta}(\zeta, s, \xi, \mu) := \langle \zeta, f_s(\xi) + g_s(\xi)\, \mu \rangle - \nu c_s(\xi, \mu) - \left\langle \vartheta, \widetilde{F}_s \mu \right\rangle \in \mathbb{R}. \end{aligned} \tag{2}$$

Then there exist

- *a trajectory $(\eta_t^{\mathrm{f}})_{t=0}^{T-1} \subset \mathbb{R}^d$,*
- *a sequence $(\eta_t^{\mathrm{x}})_{t=0}^{T} \subset \mathbb{R}^d$, and*
- *a pair $(\eta^{\mathrm{C}}, \widehat{\eta}^{\mathrm{u}}) \in \mathbb{R} \times \mathbb{R}^\ell$,*

satisfying the following conditions:

1. *non-negativity condition*
   $\eta^{\mathrm{C}} \in \{0, 1\}$;
2. *non-triviality condition*
   the adjoint trajectory $(\eta_t^{\mathrm{f}})_{t=0}^{T-1}$ and the pair $(\eta^{\mathrm{C}}, \widehat{\eta}^{\mathrm{u}})$ do not simultaneously vanish;
3. *state and adjoint system dynamics*
   $$\begin{aligned} x_{t+1}^* &= \frac{\partial}{\partial \zeta} H^{\eta^{\mathrm{C}},\widehat{\eta}^{\mathrm{u}}}(\eta_t^{\mathrm{f}}, t, x_t^*, u_t^*) && \textit{for } t = 0, \dots, T-1, \\ \eta_{t-1}^{\mathrm{f}} &= \frac{\partial}{\partial \xi} H^{\eta^{\mathrm{C}},\widehat{\eta}^{\mathrm{u}}}(\eta_t^{\mathrm{f}}, t, x_t^*, u_t^*) - \eta_t^{\mathrm{x}} && \textit{for } t = 1, \dots, T-1, \end{aligned}$$
   where $\eta_t^{\mathrm{x}} \in \mathbb{R}^d$ lies in the dual cone of a tent $q_t^{\mathrm{x}}(x_t^*)$ of $\mathbb{S}_t$ at x_t^*;
4. *transversality conditions*
   $$\frac{\partial}{\partial \xi} H^{\eta^{\mathrm{C}},\widehat{\eta}^{\mathrm{u}}}(\eta_0^{\mathrm{f}}, 0, x_0^*, u_0^*) - \eta_0^{\mathrm{x}} = 0 \quad \textit{and} \quad \eta_{T-1}^{\mathrm{f}} = -\eta_T^{\mathrm{x}},$$
   where η_0^{x} lies in the dual cone of a tent $q_t^{\mathrm{x}}(x_0^*)$ of $\mathbb{S}_0$ at x_0^* and η_T^{x} lies in the dual cone of a tent $q_t^{\mathrm{x}}(x_T^*)$ of $\mathbb{S}_T$ at x_T^*;
5. *Hamiltonian maximization condition*
   $$H^{\eta^{\mathrm{C}},\widehat{\eta}^{\mathrm{u}}}(\eta_t^{\mathrm{f}}, t, x_t^*, u_t^*) = \max_{\mu \in \mathbb{U}_t} H^{\eta^{\mathrm{C}},\widehat{\eta}^{\mathrm{u}}}(\eta_t^{\mathrm{f}}, t, x_t^*, \mu) \quad \textit{for } t = 0, \dots, T-1;$$
6. *frequency constraints*
   $$F(u_0^*, \dots, u_{T-1}^*) = 0.$$

Remark. The assertions (1) - (6) together constitute a well-defined two point boundary value problem with 4 giving the entire set of boundary conditions. Newton-like methods may be employed to solve this (algebraic) two point boundary value problem.

LQ optimal control problems

Define a linear time-invariant incarnation of (1):

$$x_{t+1} = Ax_t + Bu_t, \quad t = 0, \dots, T-1, \tag{3}$$

where the states $x_t \in \mathbb{R}^d$, the control actions $u_t \in \mathbb{R}^m$, and the system matrix $A \in \mathbb{R}^{d \times d}$ and the control matrix $B \in \mathbb{R}^{m \times m}$ are known. Consider the following finite horizon LQ problem for the system (3) with constraints on the frequency spectrum of the control trajectory, where the goal is to reach a specified final state $\hat{x} \in \mathbb{R}^d$ from a given initial state $x_0 = \overline{x}$.

$$\begin{aligned} &\underset{(u)_{t=0}^{T-1}}{\text{minimize}} \quad \sum_{t=0}^{T-1} \left(\frac{1}{2} \langle x_t, Qx_t \rangle + \frac{1}{2} \langle u_t, Ru_t \rangle \right) \\ &\text{subject to} \begin{cases} \text{controlled dynamics (3)}, \\ \sum_{t=0}^{T-1} \widetilde{F}_t u_t = 0, \\ x_0 = \overline{x}, \quad x_T = \hat{x}. \end{cases} \end{aligned} \tag{LQ}$$

Applying our Theorem to get first order necessary conditions of optimality of $\left((x_t^*)_{t=0}^T, (u_t^*)_{t=0}^{T-1}\right)$, we arrive at the following conditions: There exist $\eta^{\mathrm{C}} \in \{0, 1\}$, $\widehat{\eta}^{\mathrm{u}} \in \mathbb{R}^\ell$, a sequence of adjoint variables $(\eta_t^{\mathrm{f}})_{t=0}^{T-1}$, such that $\eta^{\mathrm{C}}, \widehat{\eta}^{\mathrm{u}}$, and $(\eta_t^{\mathrm{f}})_{t=0}^{T-1}$ are not simultaneously zero, and

$$\begin{cases} x_{t+1}^* = Ax_t^* + Bu_t^* & \text{for } t = 0, \dots, T-1, \\ \eta_{t-1}^{\mathrm{f}} = A^\top \eta_t^{\mathrm{f}} - \eta^{\mathrm{C}} Q x_t^* & \text{for } t = 1, \dots, T-1, \\ \eta^{\mathrm{C}} R u_t^* = B^\top \eta_t^{\mathrm{f}} - \widetilde{F}_t^\top \widehat{\eta}^{\mathrm{u}} & \text{for } t = 0, \dots, T-1, \\ \sum_{t=0}^{T-1} \widetilde{F}_t u_t^* = 0, \\ x_0^* = \overline{x}, \quad \text{and} \quad x_T^* = \hat{x}. \end{cases}$$

The adjoint variables are free at the boundary, i.e., η_0^{f} and η_{T-1}^{f} are arbitrary.

Remark. The optimal state-action trajectories $\left((x_t^*)_{t=0}^T, (u_t^*)_{t=0}^{T-1}\right)$ that satisfies the assertions of Theorem 1 with $\eta^{\mathrm{C}} = 1$ are called normal and the ones with $\eta^{\mathrm{C}} = 0$ are called abnormal extremals.

Corollary

Consider the problem (LQ). If the underlying system (A, B) in (3) is controllable, $T \geqslant d$, and the number of frequency constraints ℓ satisfies $\ell + d > mT$, then all the optimal state-action trajectories are abnormal. Conversely, all the optimal state-action trajectories are normal when the reachability matrix $\left(B \, \dots \, A^{T-1}B\right)$ and the frequency constraints matrix $\mathcal{F} D^{-1}$ have independent rows. [a]

[a] For more details, see https://arxiv.org/abs/1708.04419